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“NON-CANONICAL FORMULATION OF QUANTUM FIELD THEORY”

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cuyo amor es más grande
que la energía usual del vacío.

Formulación no-canónica de la teoría cuántica de campos

Resumen

En este trabajo se propone un esquema de cuantización alternativo al formalismo canónico. Esta nueva formulación no canónica aprovecha la libertad de elegir la medida de integración del operador de campo, y el álgebra entre los operadores de creación y aniquilación, para la teoría del campo escalar complejo. La medida general empleada actúa como un regulador ultravioleta, el cual permite abandonar la necesidad de usar la operación de ordenamiento normal, conduciendo a conmutadores de campo no divergentes y eliminando la divergencia ultravioleta obtenida para la energía de punto cero del tratamiento canónico. Por otro lado, el álgebra entre los operadores de creación y aniquilación es una extensión y deformación del álgebra del oscilador armónico habitual, la cual recupera la dependencia temporal en el modelo. Además, debido a la presencia del regulador ultravioleta y al abandono del álgebra del oscilador armónico, el formalismo no canónico propuesto aquí es del interés de teorías de campo con interacciones presentes, incluso con enfoques perturbativos. Hasta este momento tenemos una descripción completa del modelo en dimensiones 1+1, 1+2 y 1+3.

Esta formulación no canónica conduce a resultados teóricos interesantes, como la recuperación de la dependencia de la masa, de la dimensión del espacio, y de la coordenada temporal; la cual es opacada con la delta de Dirac obtenida en el formalismo habitual.

Non-canonical formulation of quantum field theory

Abstract

In this work, an alternative quantization scheme to the canonical formalism is proposed. This new non-canonical formulation takes advantage of the freedom to choosing the integration measure of the field operator, and the algebra between the creation and annihilation operators, for the complex scalar field theory. The general measure employed acts as an ultraviolet regulator, which allows abandoning the necessity of the normal ordering operation, leading to a non-divergent field commutators and removing the ultraviolet divergence found for the zero-point energy of the canonical treatment. On the other hand, the algebra between the creation and annihilation operators, is both an extension and a deformation of the usual harmonic oscillator algebra, which recovers the time dependence in the model. Furthermore, because of the presence of the UV regulator, and the neglecting of the harmonic oscillator algebra, the non-canonical formalism proposed here is of the interest to interacting field theories, even in perturbative approaches. Up to this moment, we have a full description for 1+1, 1+2 and 1+3 dimensions.
This non-canonical formulation leads to interesting theoretical results, just like the recovery of the dependence on mass, background dimension, and time coordinate which is obscured by the Dirac delta obtained in the usual formalism.

Publicaciones

- R. Cartas-Fuentevilla, A. Méndez-Ugalde, Weighted Lorentz invariant measures as quantum field theory regulators, *en revisión*.

Contents

Chapter 1

Introduction

The Quantum Field Theory (QFT henceforth) arose as the next step in the development of quantum mechanics by the incorporation of the foundations of the special relativity theory, in an attempt to describe phenomena that quantum mechanics can not explain. In this way, QFT treats space and time on the same footing by introducing the concept of *field*, which denotes a function evaluated in each point of the space-time; therefore, it has an infinite number of degrees of freedom.

In order to get a quantum theory, the classical fields are promoted to operators, and the commutation relations between them are established. In particular, in this work the case of the free field theory quantization is studied by applying the canonical formalism.

It is well known that in the classical textbooks of QFT, the canonical formalism for the field quantization is the most straightforward path to study the free field; it is based on the canonical relations at equal time, taken directly from the standard prescription of quantum mechanics, which are considered as the fundamental principles,

$$\begin{aligned} \left[\varphi_j(t,\vec{x}),\pi_k(t,\vec{x'})\right] &= i\delta_{jk}\delta^3(\vec{x}-\vec{x'}), \\ \left[\varphi_j(t,\vec{x}),\varphi_k(t,\vec{x'})\right] &= \left[\pi_j(t,\vec{x}),\pi_k(t,\vec{x'})\right] = 0; \end{aligned} \tag{1.1}$$

where φ_j are the field operators, π_k the momentum operators conjugate to the fields, and δ_{jk}, $\delta^3(\vec{x}-\vec{x'})$ are the Kronecker and Dirac delta respectively.

In fact, the canonical relations can be obtained by taking the short space-time distances limit in the Lorentz-invariant function

$$i\Delta(\mathbf{x}-\mathbf{y};m) \equiv \left[\varphi(\mathbf{x}),\varphi^\dagger(\mathbf{y})\right]; \tag{1.2}$$

with $\mathbf{x} = (t, \vec{x})$ in *natural* units, and whose dominant term is the Pauli-Jordan function which is in fact independent of mass. This is a suitable result since it leads to the canonical relations when the equal time limit is taken,

$$\begin{aligned} \lim_{x_0 \to y_0} \Delta(\mathbf{x}-\mathbf{y};m) &= \lim_{x_0 \to y_0} \partial_0^2 \Delta(\mathbf{x}-\mathbf{y};m) = 0, \\ \lim_{x_0 \to y_0} \partial_0 \Delta(\mathbf{x}-\mathbf{y};m) &= -\delta^3(\vec{x}-\vec{y}); \end{aligned} \tag{1.3}$$

therefore, a general postulate is proposed [1] for the quantization of interacting fields that restricts *any* non-zero contribution to causal commutators to the region

$$|\vec{x}-\vec{y}| \leq \left|x^0 - y^0\right|.$$

Nevertheless, for interacting fields, even though that rule says that the short distances behaviour is independent of the interaction, it does not hold when it appears. This is because interactions have ultra-violet divergences whose perturbative corrections to the interaction strength lead to logarithmic divergences for the equal time limit; therefore, the fundamental nature of the canonical formulation is not clear at all, and then it can be considered that the fields are not independent to each other even when they are evaluated at the same time. On the other hand, another fundamental problem of the canonical formalism is the ultraviolet divergence for the zero-point energy. This infinite energy density is the simplest UV divergence appearing in the usual formulation, and it is in some way avoided by the renormalization technique, in which the divergent term is subtracted from the total energy after the removal of an ultraviolet "cutoff"; then, leading to the justification that absolute values of energy have no physical meaning, but the energy difference is what we really measure.
However, this is not consistent with the General Relativity theory, since gravity is sensitive to energy values, not the differences [2]. In fact, it is believed that the vacuum energy density contributes to the cosmological constant Λ appearing in the Einstein's field equations.
The cosmological constant is bounded by the large scale cosmology, and this bound is reinterpreted as a bound on the vacuum energy density in QFT,

$$|\Lambda| < 10^{-56}\text{cm}^{-2} \longrightarrow |\rho_{vac}| < 10^{-29}\text{g/cm}^3 \sim 10^{-47}\text{GeV}^4.$$

The problem arises when we realize that the theoretical estimates of the vacuum energy density in QFT, within an energy range from zero up to an ultraviolet cutoff, exceed this bound by at least 40 orders of magnitude [3]; for example, in the electroweak (EW) scale

~ 100 GeV, the zero-point energy is $\rho_{vac}^{EW} \sim (100 \text{ GeV})^4$, for the Higgs field, the estimation leads to $\rho_{vac}^{Higgs} \sim -10^5 \text{GeV}^4$, and for the quantum chromo dynamics (QCD) scale, it is $\rho_{vac}^{QCD} \sim 10^{-3} - 10^{-2}$ GeV^4.

Then, it is evident that the divergent vacuum energy density in QFT is a non-trivial problem; which has further implications such as the incompatibility with the observational estimation for the cosmological constant. Despite of this fact, the QFT has had an enormous success by describing physical phenomena, and perhaps it is the reason behind the lack of motivation for searching more alternatives that fix these problems. In fact, theoretical arguments suggest the existence of a *supersymmetric* theory, in which the fermion and boson contributions to the vacuum energy are equally large, and have an opposite sign; then, they would cancel to each other. If there actually exists this *supersymmetry*, then we could understand why the vacuum energy, and hence the cosmological constant, has a small or zero value. However, these superpartners have not been observed in nature.

Even though the vacuum energy density is divergent, the vacuum physics represents a valid way of thinking about effects that are measured in reality; the Casimir effect, and the Lamb shift are examples of this [2]. Therefore, we can see that the vacuum energy problem is an interesting fact to explore.

In this work, we shall see that the integration measure by itself acts as an *ultraviolet regulator*, which removes the divergences appearing in the vacuum energy density, and also leads to a non-divergent field commutators when they are evaluated at the same space-time point, and finally leads to a finite value for the vacuum fluctuations of the field operator.

First of all, in the next chapter, a quick review of the *real* and *complex* scalar fields for the free field theory and their quantization is done. In Chapter 3, we explore the idea of a more general scheme, by taking advantage of the integration measure of the fields and the algebra between creation and annihilation operators. In Chapters 4 and 5, we abandon the harmonic oscillator algebra, and the proposals for the non-canonical quantization are exposed.

Chapter 2

A brief review of the free field theory

The objective of this chapter is to introduce the usual way in which the *real* and the *complex* scalar fields are quantized by applying the canonical formalism, remarking the weak points of this method, and motivating the study with the respective deformations.

2.1 Real scalar field

Let us begin with the relativistic action on a D+1 dimensional space

$$S = \int d^0x \int d^Dx \, \frac{1}{2}\left(\dot{\varphi}^2 - (\nabla\varphi)^2 - m^2\varphi^2\right); \tag{2.1}$$

where the convention $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, \cdots)$ for the flat space metric and the *natural units* ($c = \hbar = 1$) are employed, and $\dot{\varphi} = \partial_0\varphi(\mathbf{x})$.
The fields are given as solutions of the Klein-Gordon equation:

$$(\partial^2 + m^2)\varphi = 0; \tag{2.2}$$

with $\partial^2 \equiv \partial^\mu\partial_\mu$, and its explicit form is obtained in Fourier modes expansion [4]

$$\varphi(\mathbf{x}) = \int \frac{d^Dk}{\sqrt{(2\pi)^D}\sqrt{2\omega_k}} \left[a(\vec{k})e^{-i(\mathbf{k}\cdot\mathbf{x})} + a^*(\vec{k})e^{i(\mathbf{k}\cdot\mathbf{x})}\right]; \tag{2.3}$$

with $\mathbf{k} = (\omega_k, \vec{k})$, $k_i \in (-\infty, \infty)$, $\omega_k \equiv \sqrt{k^2 + m^2}$, and $\mathbf{k}\cdot\mathbf{x} = \omega_k t - \vec{k}\cdot\vec{x}$.
Then the first step is to promote the field to an operator by making the substitutions $\varphi \to \hat{\varphi}$,

$a \to \hat{a}$, and $a^* \to \hat{a}^\dagger$. Since the classical field is real, the correspondent field operator is Hermitian.

In this way, the field is now an operator rather than just a function; therefore, the commutation properties between them must be established. This is the heart of the canonical formalism by requiring that these commutation relations follow the canonical structure of the Hamiltonian mechanics, this is, the *equal time field commutators* are assumed to be

$$\begin{aligned}\left[\hat{\varphi}(\vec{x},t),\hat{\pi}(\vec{x'},t)\right] &= i\delta^D(\vec{x}-\vec{x'}),\\ \left[\hat{\varphi}(\vec{x},t),\hat{\varphi}(\vec{x'},t)\right] &= \left[\hat{\pi}(\vec{x},t),\hat{\pi}(\vec{x'},t)\right] = 0;\end{aligned} \tag{2.4}$$

where $\pi(\vec{x},t) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0\varphi)}$ is the canonical conjugate momentum of $\varphi(\vec{x},t)$.

With the aid of the suitable factor $\frac{1}{\sqrt{(2\pi)^D\sqrt{2\omega_k}}}$, and in analogy to the usual ladder operators that appear in the quantum harmonic oscillator, it is shown that the algebra

$$\begin{aligned}\left[\hat{a}(\vec{k}),\hat{a}^\dagger(\vec{k'})\right] &= \delta^{(D)}(\vec{k}-\vec{k'}),\\ \left[\hat{a}(\vec{k}),\hat{a}(\vec{k'})\right] = \left[\hat{a}^\dagger(\vec{k}),\hat{a}^\dagger(\vec{k'})\right] &= 0,\end{aligned} \tag{2.5}$$

must hold in favor to obtain the relations (2.4).

Nevertheless, there is no rule dictating that $\frac{1}{\sqrt{2\omega_k}}$ is the unique choice for integration measure in (2.3); as an example, some authors prefer to take $\frac{1}{2\omega_k}$ as the normalization factor, showing the Lorentz-invariance of the field explicitly; however, in order to obtain the canonical relations (2.4), the unique non-vanishing commutator must be taken as $\left[\hat{a}(\vec{k}),\hat{a}^\dagger(\vec{k'})\right] = 2\omega_k\delta^{(D)}(\vec{k}-\vec{k'})$.

In non-relativistic quantum mechanics, $\hat{a}^\dagger$ ($\hat{a}$) rises (reduces) the energy level of an eigenstate of the harmonic oscillator Hamiltonian; in the same way, in quantum field theory this operator creates (annihilates) a particle with momentum $\vec{k}$. Therefore; as the operator in quantum mechanics, $\hat{a}$ destroys the ground state (vacuum state in QFT), *i.e* $\hat{a}\left|0\right\rangle = 0$.

On the other hand, the energy of a state is given by the eigenvalues of the Hamiltonian operator, which is the operator version of the classical Hamiltonian

$$H = \int d^Dx\left(\pi^*_\varphi\pi_\varphi + \nabla\varphi^*\cdot\nabla\varphi + m^2\varphi^*\varphi\right), \tag{2.6}$$

and is given by

$$\begin{aligned}\hat{H} &= \int \frac{d^Dk}{2(2\pi)^D}\,\omega_k\,\left[\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \hat{a}(\vec{k})\hat{a}^\dagger(\vec{k})\right],\\ &= \int \frac{d^Dk}{(2\pi)^D}\,\omega_k\,\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \frac{\delta(0)}{(2\pi)^D}\int d^Dk\,\frac{1}{2}\omega_k;\end{aligned} \tag{2.7}$$

where the second equality was obtained by using the relations (2.5). Even though there is a divergence due to the presence of $\delta(0)$, it can be eliminated by confining the system within a finite box of sides of length L, and imposing periodic boundary conditions. These kinds of infinities are known as *infrared divergences*, *i.e.,* long wavelengths and low frequencies. Therefore,

$$\hat{H} = \int \frac{d^D k}{(2\pi)^D}\, \omega_k\, \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \frac{L^D}{(2\pi)^D}\int d^D k\, \frac{1}{2}\omega_k. \tag{2.8}$$

Due to the definition of the vacuum state, it is easy to compute the zero-point energy

$$H_0 = \frac{L^D}{(2\pi)^D}\int d^D k\, \frac{1}{2}\omega_k; \tag{2.9}$$

although its form is the sum of all ground state energies of the harmonic oscillators, it is evident that this energy is infinite. Unlike the previous divergence, this one appears for the high energy ranges, *i.e.,* the short wavelength and high frequencies interval, and because of this, it is called an *ultraviolet divergence*.

This is one of the most uncomfortable results of the *usual* theory, but the most accepted interpretation is that the absolute energy value lacks of physical sense; then, one should pay attention to the energy difference. In this way,

$$:\hat{H}: \equiv \hat{H} - H_0 = \int \frac{d^D k}{(2\pi)^D}\, \omega_k\, \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}); \tag{2.10}$$

where $:\hat{H}:$ is the normal ordered Hamiltonian. The operation of normal ordering appears very frequently in QFT, and it reduces to put the creation operators to the left of the annihilation ones. In this way, $:\hat{H}:|0\rangle = 0$.

Although the problem of the divergence of the zero-point energy was avoided, there is an open problem that is also avoided as a consequence; the energy density of the universe.

Observations suggest that most of the universe's energy density has the properties of a cosmological constant, which is much smaller than other scales in particle physics.

Therefore, it is an open question, why the infinite zero-point energy does not contribute to the universe's energy density, or what is canceling it. [5].

2.2 Complex scalar field

In order to generalize to a complex field description, the Lagrangian density is written as

$$\mathcal{L} = \partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi. \tag{2.11}$$

The field operator that solves the equations of motion is

$$\varphi(\mathbf{x}) = \int \frac{d^D k}{\sqrt{(2\pi)^D 2\omega_k}} \left[a(\vec{k})e^{-i(\mathbf{k}\cdot\mathbf{x})} + b^\dagger(\vec{k})e^{i(\mathbf{k}\cdot\mathbf{x})}\right]; \tag{2.12}$$

henceforth the field operator is denoted just as φ for the sake of avoiding a saturation in the notation. It is evident that the field operator is non-Hermitian, since the classical field is complex.

Such as in the real scalar field theory, the factor $\frac{1}{\sqrt{2\omega_k}}$ is taken in such a way that the algebra for the creation and annihilation operators

$$[a(\vec{k}), a^\dagger(\vec{k'})] = \delta^{(D)}(\vec{k} - \vec{k'}), \qquad [b(\vec{k'}), b^\dagger(\vec{k})] = \delta^{(D)}(\vec{k} - \vec{k'}), \tag{2.13a}$$

$$[a(\vec{k}), a(\vec{k'})] = [b(\vec{k}), b(\vec{k'})] = 0, \tag{2.13b}$$

$$[a(\vec{k}), b(\vec{k'})] = [a(\vec{k}), b^\dagger(\vec{k'})] = 0, \tag{2.13c}$$

implies the canonical relations for the fields

$$\begin{aligned} \left[\varphi(\vec{x},t), \pi_\varphi(\vec{x'},t)\right] &= i\delta^D(\vec{x} - \vec{x'}), \\ \left[\varphi(\vec{x},t), \varphi(\vec{x'},t)\right] &= \left[\pi_\varphi(\vec{x},t), \pi_\varphi(\vec{x'},t)\right] = 0. \end{aligned} \tag{2.14}$$

In view of relations (2.13), $(a, a^\dagger)$ and $(b, b^\dagger)$ are two independent sets of annihilation and creation operators and now the vacuum state is destroyed by a and b, this is, for all $\vec{k}$

$$a(\vec{k})\,|0\rangle = b(\vec{k})\,|0\rangle = 0. \tag{2.15}$$

It is easy to realise that the Lagrangian density has a global symmetry under $U(1)$, *i.e.,* it is invariant under the transformation $\varphi' = \varphi e^{i\alpha}$, with α a real global phase [6]. By Noether's theorem, this symmetry implies the existence of a conserved current,

$$J_\mu = i(\varphi^\dagger \partial_\mu \varphi - \partial_\mu \varphi^\dagger \varphi), \qquad \partial_\mu J^\mu = 0; \tag{2.16}$$

whose time component leads to a conserved charge operator given by

$$\hat{Q} \equiv \int d^D x\; J_0 = \int d^D k \left[a^\dagger(\vec{k})a(\vec{k}) - b^\dagger(\vec{k})b(\vec{k})\right] + \delta^{(D)}(0) \int d^D k. \tag{2.17}$$

The second integral is divergent; therefore it is necessary to call the normal ordering operation, and thus $: \hat{Q} :\equiv \hat{Q} - \delta^{(D)}(0) \int d^D k$, which is the charge difference between a state with particles and the vacuum state; by analogy with the energy difference $: \hat{H} :$,

$$: \hat{Q} := \int d^D k \left[a^\dagger(\vec{k})a(\vec{k}) - b^\dagger(\vec{k})b(\vec{k})\right]. \tag{2.18}$$

With this re-definition of the *charge operator* and with the aid of relations (2.13) we have

$$: \hat{Q} : a^{\dagger} |0\rangle = +a^{\dagger} |0\rangle \, , \qquad : \hat{Q} : b^{\dagger} |0\rangle = -b^{\dagger} |0\rangle \, ;$$

hence both particles carry opposite charges but they have the same mass, leading to the physical interpretation that $a^{\dagger}$ creates a particle, and $b^{\dagger}$ creates its antiparticle.

The canonical formalism leads to the well established results that were studied above, regardless of this, there are some uncomfortable consequences such as the necessity of the normal ordering operation to hide the infinite values produced by the ultraviolet divergences, or the particular choice of the integration measure and harmonic oscillator algebras.

Chapter 3

Fundamentals of the non-canonical models

The aim of this chapter is to introduce the fundamental concepts that are exploited in order to build the non-canonical quantization models presented in the subsequent chapters. We consider again the complex field theory, but keeping in mind that the canonical relations will be generalized, which implies the use of the full and a more general description of the fields and their relationship.

3.1 The field operator

As seen in the previous chapter, the Lagrangian density of the complex scalar field theory is given by

$$\mathcal{L} = \partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi; \tag{3.1}$$

whose equations of motion are the Klein-Gordon equation for both φ and $\varphi^\dagger$,

$$(\partial^2 + m^2)\varphi = 0, \qquad (\partial^2 + m^2)\varphi^\dagger = 0, \tag{3.2}$$

and with the solution [7],

$$\varphi(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^D}} \int d^{D+1}k \; \delta\left(k^2 - m^2\right) \Theta\left(k^0\right) \left(A(k)e^{-i\mathbf{k}\cdot\mathbf{x}} + B^\dagger(k)e^{i\mathbf{k}\cdot\mathbf{x}}\right), \tag{3.3}$$

and

$$\Theta(k^0) = \begin{cases} 1 & k^0 > 0, \\ 1/2 & k^0 = 0, \\ 0 & k^0 < 0, \end{cases}$$

such that $\Theta(k^0) + \Theta(-k^0) = 1$. This form of the solution is useful because the Lorentz invariance of the field is seen explicitly; since the Lorentz transformations are unitary, the space-time differential element $d^{D+1}k$ is Lorentz invariant.

In this way, the field (3.3) is invariant under orthochronous Lorentz transformations[1] by setting for the creation and annihilation operators,

$$U(\Lambda)A(k)U^{-1}(\Lambda) = A(\Lambda k), \tag{3.4}$$

and the same for the operator $B^\dagger$. Even more, the field (3.3) reduces to (2.12) by realizing that [5],

$$\int d^{D+1}k \; \delta\left(k^2 - m^2\right)\Big|_{k^0>0} = \int \frac{d^Dk}{2k^0}\Big|_{k^0=\omega_k}, \tag{3.5}$$

and defining [7],

$$a(k) = \frac{A(k)}{\sqrt{2\omega_k}}, \tag{3.6a}$$

$$b^\dagger(k) = \frac{B^\dagger(k)}{\sqrt{2\omega_k}}; \tag{3.6b}$$

of course the transformation rules for this new operators a and $b^\dagger$ also change. An explicit proof of the Lorentz invariance of the measure (3.5) is given in the Appendix A.

Then, in this work a general field operator is proposed,

$$\varphi(\mathbf{x}) = \int \frac{d^Dk}{\sqrt{(2\pi)^D \omega_k^{\frac{s}{2}}}} \left[a(\vec{k})e^{-i(\mathbf{k}\cdot\mathbf{x})} + b^\dagger(\vec{k})e^{i(\mathbf{k}\cdot\mathbf{x})}\right]; \tag{3.7}$$

where s is an integer parameter, and the following transformation rule holds,

$$U(\Lambda)\left\{\begin{array}{c} a(\vec{k}) \\ b^\dagger(\vec{k}) \end{array}\right\}U^{-1} = \frac{\omega_{\Lambda k}^{1-\frac{s}{2}}}{\omega_k^{1-\frac{s}{2}}}\left\{\begin{array}{c} a(\Lambda\vec{k}) \\ b^\dagger(\Lambda\vec{k}) \end{array}\right\}, \tag{3.8}$$

in order to ensure the Lorentz invariance of the field φ.

On the other hand, this transformation rule for the creation and annihilation operators,

[1] Under orthochronous Lorentz transformations, $\Theta(k^0)$ is explicitly invariant, since k^0 can not be changed in sign by proper Lorentz transformations.

implies that the Lorentz invariant state $U(\Lambda)\,|k\rangle = |\Lambda k\rangle$ is written as [2]

$$|k\rangle = 2\omega_k^{1-\frac{s}{2}}\,|\vec{k}\rangle = 2\omega_k^{1-\frac{s}{2}}\,a^\dagger(\vec{k})\,|0\rangle\,. \tag{3.9}$$

It is important to remind that in the usual prescription the factor $\frac{1}{\sqrt{2\omega_k}}$ is chosen suitably in favor to obtain the canonical relations for the fields by using the algebra (2.5), and in the respective chapter is also mentioned that some authors prefer to use the factor $\frac{1}{2\omega_k}$; nevertheless, with the second choice it is necessary to redefine the algebra for the creation and annihilation operators in order to achieve the requirement of the canonical structure, *i.e.*, $[a, a^\dagger] = 2\omega_k \delta^D(k - k')$, and in the case at hand, the expression

$$[a, a^\dagger] = \omega_k^{s-1}\delta^D(k - k'),$$

is required. This shows that the algebra can be chosen as suitably as long the canonical relations are required; then, it is possible to work with a general version of the field operator; keeping the Lorentz invariance by defining the correct transformation rules.
Up to this point, the theory does not change at all since it is possible to redefine at our convenience the algebra employed, but what if the canonical relations are not required?, this fact is exploited in the next section.

3.2 Field commutators

In the previous section has been seen that it is possible to build the field operator with a general integration measure; however, the algebra of the creation and annihilation operators must be redefined in order to obtain the canonical relations.
In this work, the canonical structure of the field commutators is not maintained; then, in this section, the field commutators are explored in a more general form.
Firstly, the fields are given by

$$\varphi(\mathbf{x}) = \int \frac{d^Dk}{\sqrt{(2\pi)^D}\,\omega_k^{\frac{s}{2}}}\left[a(\vec{k})e^{-i(\mathbf{k}\cdot\mathbf{x})} + b^\dagger(\vec{k})e^{i(\mathbf{k}\cdot\mathbf{x})}\right], \tag{3.10a}$$

$$\pi_\varphi(\mathbf{x}) = \partial_0(\varphi^\dagger(\mathbf{x})) = i\int \frac{d^Dk}{\sqrt{(2\pi)^D}}\omega_k^{1-\frac{s}{2}}\left[a^\dagger(\vec{k})e^{i(\mathbf{k}\cdot\mathbf{x})} - b(\vec{k})e^{-i(\mathbf{k}\cdot\mathbf{x})}\right]; \tag{3.10b}$$

[2] Note that there is a slight abuse of notation since $a^\dagger$ and $b^\dagger$ create two different particles. In fact, the Fock space state should be written with two entries $|\vec{k}_a, \vec{k}_b\rangle$.

and with the purpose of getting a shorter notation, the next functions are defined

$$u_k(\mathbf{x}) \equiv \frac{1}{\sqrt{(2\pi)^D \omega_k^{\frac{s}{2}}}} e^{-i\mathbf{k}\cdot\mathbf{x}}, \qquad u_k^*(\mathbf{x}) \equiv \frac{1}{\sqrt{(2\pi)^D \omega_k^{\frac{s}{2}}}} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Therefore, the field commutators are,

$$\begin{aligned}
[\varphi(\mathbf{x}), \varphi(\mathbf{x}')] =& \int_{k,k'} d^D k d^D k' \Big\{ \left[a(\vec{k}), b^\dagger(\vec{k'})\right] u_k(\mathbf{x}) u_{k'}^*(\mathbf{x}') \\
&+ \left[b^\dagger(\vec{k}), a(\vec{k'})\right] u_k^*(\mathbf{x}) u_{k'}(\mathbf{x}') \Big\}, && (3.11a)\\
[\pi_\varphi(\mathbf{x}), \pi_\varphi(\mathbf{x}')] =& \int_{k,k'} d^D k d^D k' \omega_k \omega_{k'} \Big\{ \left[a^\dagger(\vec{k}), b(\vec{k'})\right] u_k^*(\mathbf{x}) u_{k'}(\mathbf{x}') \\
&+ \left[b(\vec{k}), a^\dagger(\vec{k'})\right] u_k(\mathbf{x}) u_{k'}^*(\mathbf{x}') \Big\}, && (3.11b)\\
\left[\varphi, \varphi^\dagger\right] =& \int_{k,k'} d^D k d^D k' \Big\{ \left[a(\vec{k}), a^\dagger(\vec{k'})\right] u_k(\mathbf{x}) u_{k'}^*(\mathbf{x}') + \left[a(\vec{k}), b(\vec{k'})\right] u_k(\mathbf{x}) u_{k'}(\mathbf{x}') \\
&+ \left[b^\dagger(\vec{k}), a^\dagger(\vec{k'})\right] u_k^*(\mathbf{x}) u_{k'}^*(\mathbf{x}') + \left[b^\dagger(\vec{k}), b(\vec{k'})\right] u_k^*(\mathbf{x}) u_{k'}(\mathbf{x}') \Big\}, && (3.11c)\\
[\pi_\varphi, \pi_{\varphi^\dagger}] =& \int_{k,k'} d^D k d^D k' \Big\{ \left[a^\dagger(\vec{k}), a(\vec{k'})\right] u_k^*(\mathbf{x}) u_{k'}(\mathbf{x}') - \left[b(\vec{k}), a(\vec{k'})\right] u_k(\mathbf{x}) u_{k'}(\mathbf{x}') \\
&- \left[a^\dagger(\vec{k}), b^\dagger(\vec{k'})\right] u_k^*(\mathbf{x}) u_{k'}^*(\mathbf{x}') + \left[b(\vec{k}), b^\dagger(\vec{k'})\right] u_k(\mathbf{x}) u_{k'}^*(\mathbf{x}') \Big\} \omega_k \omega_{k'}, && (3.11d)\\
[\varphi, \pi_{\varphi^\dagger}] =& i \int_{k,k'} d^D k d^D k' \omega_{k'} \Big\{ - \left[b^\dagger(\vec{k}), a(\vec{k'})\right] u_k^*(\mathbf{x}) u_{k'}(\mathbf{x}') \\
&+ \left[a(\vec{k}), b^\dagger(\vec{k'})\right] u_k(\mathbf{x}) u_{k'}^*(\mathbf{x}') \Big\}, && (3.11e)\\
[\varphi, \pi_\varphi] =& i \int_{k,k'} d^D k d^D k' \omega_{k'} \Big\{ \left[a(\vec{k}), a^\dagger(\vec{k'})\right] u_k(\mathbf{x}) u_{k'}^*(\mathbf{x}') - \left[a(\vec{k}), b(\vec{k'})\right] u_k(\mathbf{x}) u_{k'}(\mathbf{x}') \\
&+ \left[b^\dagger(\vec{k}), a^\dagger(\vec{k'})\right] u_k^*(\mathbf{x}) u_{k'}^*(\mathbf{x}') - \left[b^\dagger(\vec{k}), b(\vec{k'})\right] u_k^*(\mathbf{x}) u_{k'}(\mathbf{x}') \Big\}; && (3.11f)
\end{aligned}$$

where the dependence of the fields is explicitly written only in the two first commutators, in favor to ease the reading of the expressions.
In the commutators (3.11) it was assumed that

$$[a(\vec{k}), a(\vec{k'})] = 0, \tag{3.12a}$$

$$[b(\vec{k}), b(\vec{k'})] = 0. \tag{3.12b}$$

On the other hand, we propose a deformed and extended algebra given by

$$\left[a(\vec{k}), a^\dagger(\vec{k'})\right] = \frac{\alpha}{2}\delta^D(\vec{k}-\vec{k'}), \tag{3.13a}$$
$$\left[b(\vec{k}), b^\dagger(\vec{k'})\right] = \frac{\beta}{2}\delta^D(\vec{k}-\vec{k'}), \tag{3.13b}$$
$$\left[a(\vec{k}), b(\vec{k'})\right] = i\rho\delta^D(\vec{k}+\vec{k'}), \tag{3.13c}$$
$$\left[a(\vec{k}), b^\dagger(\vec{k'})\right] = \frac{\sigma}{2}\delta^D(\vec{k}-\vec{k'}); \tag{3.13d}$$

where α and β are real constants in order to generalize the usual algebra; the constant σ is complex in general, but ρ is chosen real for later convenience. The positive sign in the Dirac delta in (3.13c) is chosen such that the commutators only depend on the distance $\Delta\vec{x} \equiv (\vec{x}-\vec{x'})$ and time t. The field commutators at equal time become

$$\left[\varphi(\vec{x},t), \varphi(\vec{x'},t)\right] = 0, \tag{3.14a}$$
$$\left[\pi_\varphi(\vec{x},t), \pi_\varphi(\vec{x'},t)\right] = 0, \tag{3.14b}$$
$$\left[\varphi(\vec{x},t), \varphi^\dagger(\vec{x'},t)\right] = \frac{1}{(2\pi)^D}\int_{-\infty}^{\infty}\frac{d^Dk}{\omega_k^s}\left\{\frac{\alpha-\beta}{2}e^{i\vec{k}\cdot\Delta\vec{x}} + 2\rho\sin(2\omega_k t)\cos\left(\vec{k}\cdot\Delta\vec{x}\right)\right\}, \tag{3.14c}$$
$$\left[\pi_\varphi(\vec{x},t), \pi_{\varphi^\dagger}(\vec{x'},t)\right] = -\frac{1}{(2\pi)^D}\int_{-\infty}^{\infty}\frac{d^Dk}{\omega_k^{s-2}}\left\{\frac{\alpha-\beta}{2}e^{-i\vec{k}\cdot\Delta\vec{x}} - 2\rho\sin(2\omega_k t)\cos\left(\vec{k}\cdot\Delta\vec{x}\right)\right\}, \tag{3.14d}$$
$$\left[\varphi(\vec{x},t), \pi_{\varphi^\dagger}(\vec{x'},t)\right] = i\frac{\sigma}{(2\pi)^D}\int_{-\infty}^{\infty}\frac{d^Dk}{\omega_k^{s-1}}\cos\left(\vec{k}\cdot\Delta\vec{x}\right), \tag{3.14e}$$
$$\left[\varphi(\vec{x},t), \pi_\varphi(\vec{x'},t)\right] = \frac{i}{(2\pi)^D}\int_{-\infty}^{\infty}\frac{d^Dk}{\omega_k^{s-1}}\left\{\frac{\alpha+\beta}{2}e^{i\vec{k}\cdot\Delta\vec{x}} - 2i\rho\cos(2\omega_k t)\cos\left(\vec{k}\cdot\Delta\vec{x}\right)\right\}. \tag{3.14f}$$

The parameter s leads to a different behavior for the usual non-vanishing commutator $[\varphi, \pi]$, and it will be explored in the next chapter; the scheme changes completely because of the appearance of the commutator $[a, b]$, since it recovers the time dependence of the field commutators; and even though the Lorentz covariance is not manifest, it will play a key role in the behaviour of the propagator outside the light cone.

It is evident that the relations (3.14) depend strongly on the choice of the measure $\frac{dk}{\omega_k^s}$, the dimension D, and the relationship between the parameters α and β, whose importance also appears in the construction of the physical observables.

3.3 Physical observables

An interesting fact is that even when the possibility of $[a, b] \neq 0$ is taken, the operators corresponding to the charge, momentum and energy, are only affected by the anticommutators $\{a, a^\dagger\}$ and $\{b, b^\dagger\}$. To see this fact, we start from the classical expressions,

$$H = \frac{1}{2}\int d^D x \left(\{\pi^*_\varphi, \pi_\varphi\} + \{\partial_i\varphi^*, \partial^i\varphi\} + m^2\{\varphi^*, \varphi\}\right), \tag{3.15a}$$

$$Q = \frac{1}{2}\int d^D x \; i(\{\varphi^\dagger, \partial_0\varphi\} - \{\partial_0\varphi^\dagger, \varphi\}), \tag{3.15b}$$

$$P_i = -\frac{i}{2}\int d^D x \; (\{\pi_\varphi, \partial_i\varphi\} + \{\pi_{\varphi^\dagger}, \partial_i\varphi^\dagger\}); \tag{3.15c}$$

where the curly brackets denote that they are *symmetrized*, *i.e.*, $\{A, B\} = AB + BA$; in this way we consider the same importance for both ordering possibilities. As it is mentioned above, after the fields are symmetrized and promoted to operators, they read,

$$\hat{H} = \int \frac{d^D k}{\omega_k^{s-2}} \left[\{a(\vec{k}), a^\dagger(\vec{k})\} + \{b(\vec{k}), b^\dagger(\vec{k})\}\right], \tag{3.16a}$$

$$\hat{Q} = \int \frac{d^D k}{\omega_k^{s-1}} \left[\{a(\vec{k}), a^\dagger(\vec{k})\} - \{b(\vec{k}), b^\dagger(\vec{k})\}\right], \tag{3.16b}$$

$$\hat{P}_i = \int d^D k \, \frac{k_i}{\omega_k^{s-1}} \left[\{a(\vec{k}), a^\dagger(\vec{k})\} + \{b(\vec{k}), b^\dagger(\vec{k})\}\right]. \tag{3.16c}$$

In order to explore the vacuum expectation values for these observables, the usual definition of the vacuum state is taken for both annihilation operators,

$$a(k)\left|0\right\rangle = 0, \tag{3.17a}$$

$$b(k)\left|0\right\rangle = 0; \tag{3.17b}$$

by expanding the anticommutators, only the first two commutators of the relations (3.13) are necessary. This leads to an infrared divergence due to the volume of integration, which can be handled by confining the system in a box with sides of length L.

After the calculations, the operators are given by,

$$\hat{H} = 2\int \frac{d^D k}{\omega_k^{s-2}} \left[a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(\vec{k})b(\vec{k})\right] + \frac{\alpha+\beta}{2}\frac{L^D}{(2\pi)^D}\int \frac{d^D k}{\omega_k^{s-2}}, \tag{3.18a}$$

$$\hat{Q} = 2\int \frac{d^D k}{\omega_k^{s-1}} \left[a^\dagger(\vec{k})a(\vec{k}) - b^\dagger(\vec{k})b(\vec{k})\right] + \frac{\alpha-\beta}{2}\frac{L^D}{(2\pi)^D}\int \frac{d^D k}{\omega_k^{s-1}}, \tag{3.18b}$$

$$\hat{P}_i = 2\int d^D k \, \frac{k_i}{\omega_k^{s-1}} \left[a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(\vec{k})b(\vec{k})\right] + \frac{\alpha+\beta}{2}\frac{L^D}{(2\pi)^D}\int \frac{d^D k}{\omega_k^{s-1}} k_i. \tag{3.18c}$$

Hence, the vacuum expectation values are determined by the power s of the measure, and the choice of α and β. In fact, the equations (3.14) and (3.18) show that the relationship between α and β is non-trivial, leading to interesting technical and conceptual differences with the usual theory. For example, the choice $\alpha = \beta$ corresponds to a neutral charge of the vacuum state, and a non-trivial vacuum energy; conversely, the choice $\alpha = -\beta$ leads to a vanishing zero-point energy, but to a charged vacuum state, and a different non-vanishing field commutators.

In the case of the momentum operator, its vacuum expectation value vanishes for every D and s because the integrand is an odd function in k_i, physically it can be seen that all spatial directions have equivalent contributions to the infinite sum.

Now that the fundamental concepts of the new scheme are established, it is the turn of the non-canonical quantization models to be studied. They are presented as particular cases of the most general case of the field commutators (3.14) obtained by the algebra (3.13).

As a final remark, it must be said that the canonical formalism can be recovered by taking $\alpha = \beta = 1$, $\rho = \sigma = 0$, and $s = 1$; of course, the intention of this work is to avoid this usual case.

Chapter 4

The simplest deformation

After establishing the fundamentals of the general scheme, this chapter takes the *simplest* deformation of the algebra for the creation and annihilation operators; fortunately, there are analytical solutions for 1+1, 2+1, and 3+1 space time dimensions.
The *simplest* deformation, refers to the choice $\sigma = \rho = 0$ in the relations (3.13), *i.e.,* the only non-vanishing commutators for the creation and annihilation operators are

$$\left[a(\vec{k}), a^\dagger(\vec{k'})\right] = \frac{\alpha}{2}\delta^D(\vec{k} - \vec{k'}), \tag{4.1a}$$

$$\left[b(\vec{k}), b^\dagger(\vec{k'})\right] = \frac{\beta}{2}\delta^D(\vec{k} - \vec{k'}). \tag{4.1b}$$

Despite of the use of the word "*simplest*", this case indeed leads to a very different and interesting behavior where the dependence on mass and background dimension is recovered. The general approach given by the power s of the measure, recovers the importance of these physical entities which are lost in the usual formalism, where the trivially obtained Dirac delta obscures their impact in the theory. In the case of the field commutators, they are obtained as a set of smooth distributions, contrasting with the simple Dirac delta of the usual theory; in the case of the energy, the ultraviolet divergences are removed and also characterized by inverse powers of the mass.

4.1 Field commutators in different dimensions

With the algebra employed in this case, the field commutators different from zero are,

$$\left[\varphi(\vec{x},t),\varphi^{\dagger}(\vec{x'},t)\right] = \frac{\alpha-\beta}{2}\frac{1}{(2\pi)^D}\int_{-\infty}^{\infty}\frac{d^Dk}{\omega_k^s}\cos(\vec{k}\cdot\Delta\vec{x}), \tag{4.2a}$$

$$\left[\pi_{\varphi}(\vec{x},t),\pi_{\varphi^{\dagger}}(\vec{x'},t)\right] = -\frac{\alpha-\beta}{2}\frac{1}{(2\pi)^D}\int_{-\infty}^{\infty}\frac{d^Dk}{\omega_k^{s-2}}\cos(\vec{k}\cdot\Delta\vec{x}), \tag{4.2b}$$

$$\left[\varphi(\vec{x},t),\pi_{\varphi}(\vec{x'},t)\right] = \frac{\alpha+\beta}{2}\frac{i}{(2\pi)^D}\int_{-\infty}^{\infty}\frac{d^Dk}{\omega_k^{s-1}}\cos(\vec{k}\cdot\Delta\vec{x}). \tag{4.2c}$$

As it is mentioned in the previous chapter, the relationship between α and β is non-trivial; hence, if $\alpha=\beta$ then the unique non-vanishing commutator is (4.2c), and the zero-point energy is in general different from zero, as we shall see in a next section. On the other hand, if the choice is $\alpha=-\beta$ then both the commutator (4.2c) and the zero-point energy will vanish, and the other two commutators (4.2a) and (4.2b), along with the charge of the vacuum state remain different from zero. This shows that the new scheme permits a variety of possibilities; nevertheless, since the idea is to do a direct comparison with the usual theory, the case when $\alpha=\beta$ is taken. In the next subsections, the unique non-vanishing commutator (4.2c) is discussed in different spatial dimensions D.

4.1.1 $D=1$

First, we consider one spatial dimension $D=1$, then the field commutator becomes

$$\left[\varphi(x,t),\pi_{\varphi}(x',t)\right] = \alpha\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{\omega_k^{s-1}}\cos(k\Delta x), \tag{4.3}$$

and the rest of them are vanishing. Fortunately there is an analytic solution for every integer $s>0$, and the choice $s=1$ corresponds to the usual Dirac delta distribution. With the aid of the integral representation of the modified Bessel function of the second kind of order ν,

$$K_{\nu}(x) = \frac{2^{\nu}}{\sqrt{\pi}x^{\nu}}\Gamma\left(\nu+\frac{1}{2}\right)\int_0^{\infty}\frac{\cos(xt)}{(t^2+1)^{\nu+\frac{1}{2}}}dt,$$

and the use of the well known Cauchy's Residue theorem, the general solution is written as

$$\left[\varphi(x,t),\pi_{\varphi}(x',t)\right] = \alpha\begin{cases} i\delta(x-x') & s=1, \\ \frac{i}{\sqrt{\pi}}\frac{|\Delta x|^{\nu-1}}{2^{\nu-1}m^{\nu-1}\Gamma(\nu-\frac{1}{2})}K_{\nu-1}(m|\Delta x|) & s=2\nu \quad \text{even}, \\ -\frac{1}{(\nu-1)!}\frac{d^{(\nu-1)}}{dk^{(\nu-1)}}\left(\frac{1}{(k+im)^{\nu}}e^{ik|\Delta x|}\right)\Big|_{k=im} & s=2\nu+1 \quad \text{odd;} \end{cases} \tag{4.4}$$

with $\nu = 1, 2, ...,$ and $|\Delta x| \geq 0$.

In Table 4.1 the explicit solutions for some values of s are described; the maximum values of the commutator, which occur when $\Delta x \to 0$, are also shown.

s	$[\varphi, \pi_\varphi]$	$\lim_{\Delta x \to 0} [\varphi, \pi_\varphi]$
1	$i\alpha\delta(\lvert\Delta x\rvert)$	$i\infty$
2	$\frac{i\alpha}{\pi} K_0(m\lvert\Delta x\rvert)$	$-i\frac{\alpha}{\pi}[\ln\left(\frac{m\lvert\Delta x\rvert}{2}\right) + \gamma]$
3	$\frac{i\alpha}{2m} e^{-m\lvert\Delta x\rvert}$	$i\frac{\alpha}{2m}$
4	$i\alpha\frac{\lvert\Delta x\rvert}{\pi m} K_1(m\lvert\Delta x\rvert)$	$i\frac{\alpha}{\pi m^2}$
5	$i\frac{\alpha}{4m^3}(1 + m\lvert\Delta x\rvert)e^{-m\lvert\Delta x\rvert}$	$i\frac{\alpha}{4m^3}$

Table 4.1: Some expressions of the field commutator in 1+1 dimensions.

where $\gamma = 0.5772...$ is the Euler-Mascheroni constant. From these solutions, it is evident that one more important difference with respect to the usual formulation is the appearance of the global factor α, which in general depends on the measure employed and the background dimension, $\alpha = \alpha(s, D)$; this ensures that the commutators have the same units for every value of s. The discussion concerning to units of commutators is given in Appendix B.

In Figure 4.1 it is seen a set of softer distributions than the Dirac delta as the value of s grows; nevertheless, as observed by comparing figures 4.1 and 4.2, the distribution of the maxima depends on the value of the mass; in fact, it is possible to show that in general, the maximum for every $s \geq 3$ is proportional to $\frac{1}{m^{s-2}}$, and as a consequence, the field commutator evaluated at the same space-time point vanishes for $m > 1$ when the limit $s \to \infty$ is taken.

Figure 4.3 shows the curves corresponding to the field commutators for different values of s as a function of the mass, and it can be seen that there are some critical points denoted by m_i where the curves cross each other. Thus Figure 4.1 represents the distribution of the maxima when $m > max\{m_i\}$, meanwhile Figure 4.2 represents those for which $0 < m < min\{m_i\}$. This shows an interesting behavior and recovers the importance of the mass scale in the theory, which is lost in the simple Dirac delta result.

4.1.2 $D = 2$

Unlike the usual formulation, the expression for the commutator in higher dimensions, is not a direct generalization as in the case of the D-dimensional Dirac delta; for example, in two spatial dimensions ($D = 2$), using polar coordinates, an integral of a zero-order Bessel

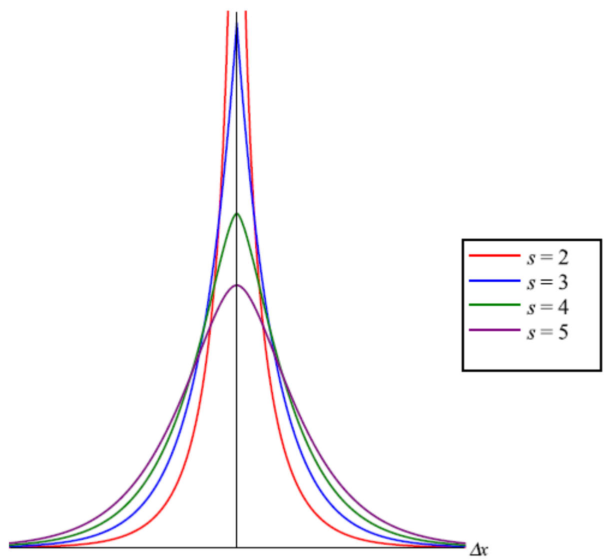


Figure 4.1: Curves corresponding to $[\varphi, \pi_\varphi]$ as a function of Δx for different values of s and with $m = 1$.

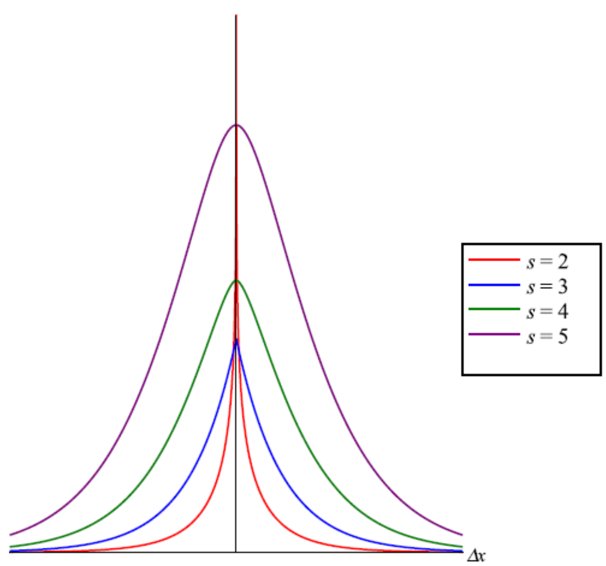


Figure 4.2: Curves corresponding to $[\varphi, \pi_\varphi]$ as function of Δx for different values of s and with $m = 0.5$.

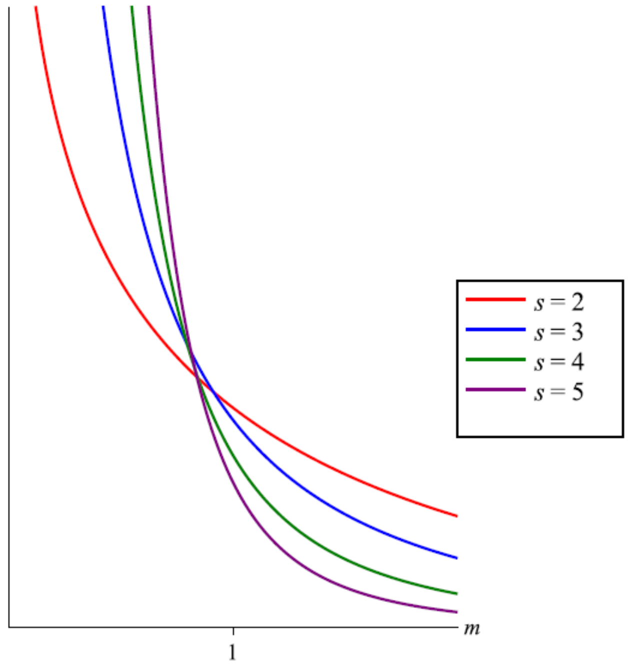


Figure 4.3: Curves corresponding to $[\varphi, \pi_\varphi]$ as a function of m and fixed $\Delta x << 1$.

function of the first kind appears,

$$\left[\varphi(\vec{x}, t), \pi_\varphi(\vec{x'}, t)\right] = i\frac{\alpha}{2\pi}\int_0^\infty dk \frac{k}{(\sqrt{k^2+m^2})^{s-1}} J_0(k|\Delta\vec{x}|); \tag{4.5}$$

where $k = \sqrt{k_1^2 + k_2^2}$, and whose solution is known [8] for some values of s, and leads to a similar behaviour than the one dimensional case in terms of $e^{-m|\Delta\vec{x}|}$ and modified Bessel function of the second kind depending on the value of s. Table 4.2 shows some explicit solutions of the integral (4.5).

s	$[\varphi, \pi_\varphi]$	$\lim_{\lvert\Delta\vec{x}\rvert\to 0}[\varphi, \pi_\varphi]$
1	$i\frac{\alpha}{2\pi}\frac{\delta(\lvert\Delta\vec{x}\rvert)}{\lvert\Delta\vec{x}\rvert}$	$i\infty$
2	$\frac{i}{2\pi}\frac{\alpha}{\lvert\Delta\vec{x}\rvert}e^{-m\lvert\Delta\vec{x}\rvert}$	$\frac{i}{2\pi}\frac{\alpha}{\lvert\Delta\vec{x}\rvert}$
3	$i\frac{\alpha}{2\pi}K_0(m\lvert\Delta\vec{x}\rvert)$	$-\frac{i\alpha}{2\pi}[\ln\left(\frac{m\lvert\Delta\vec{x}\rvert}{2}\right)+\gamma]$
4	$i\frac{\alpha}{2\pi m}e^{-m\lvert\Delta\vec{x}\rvert}$	$\frac{i}{2\pi}\frac{\alpha}{m}$
5	$i\alpha\frac{\lvert\Delta\vec{x}\rvert}{4\pi m}K_1(m\lvert\Delta\vec{x}\rvert)$	$\frac{i}{4\pi}\frac{\alpha}{m^2}$

Table 4.2: Some expressions of the field commutator in 2+1 dimensions.

The Dirac delta appearing for $s = 1$ is in fact the two dimensional delta written in polar coordinates, thus this case corresponds to the usual $(2+1)$-dimensional formulation.
The expressions for higher values of s can be computed by using the recurrence relation

$$f(s+2, m, |\Delta\vec{x}|) = \frac{1}{m(1-s)} \frac{\partial f(s, m, |\Delta\vec{x}|)}{\partial m}, \qquad s \geq 2, \tag{4.6}$$

with $f(s, m, |\Delta\vec{x}|) = \int_0^\infty \frac{dk \; k J_0(k|\Delta\vec{x}|)}{\left(\sqrt{k^2+m^2}\right)^{s-1}}$. For example, one can verify that the expression of the commutator for $s = 4$ is obtained by using (4.6) with the expression for $s = 2$; in the same way, in order to obtain the commutator for $s = 6$, the relation (4.6) must be applied to $f(4, m, |\Delta\vec{x}|)$, and so on.
Despite the relation (4.6) holds for every $s \geq 2$, the expressions for odd values of s can be computed directly thanks to the integral

$$\int_0^\infty \frac{J_0(bk)\; k}{\left(\sqrt{k^2+m^2}\right)^{s-1}} dx = \frac{b^{\frac{s-3}{2}}}{(2m)^{\frac{s-3}{2}} \Gamma(\frac{s-1}{2})} K_{\frac{s-3}{2}}(mb), \qquad \text{for } s \text{ odd.} \tag{4.7}$$

The curves of these solutions are not shown since they are the same of the one dimensional case, differing just by the multiplicative factor and the weight s associated.

4.1.3 $D = 3$

In three spatial dimensions ($D = 3$) the commutator in spherical coordinates is written as

$$\left[\varphi(\vec{x}, t), \pi_\varphi(\vec{x'}, t)\right] = i\frac{\alpha}{2\pi^2|\Delta\vec{x}|} \int_0^\infty dk \frac{k \sin(k|\Delta\vec{x}|)}{\left(\sqrt{k^2+m^2}\right)^{s-1}}; \tag{4.8}$$

where $k = \sqrt{k_1^2 + k_2^2 + k_3^2}$, and some solutions are [8],

s	$[\varphi, \pi_\varphi]$	$\lim_{\vert\Delta\vec{x}\vert\to 0} [\varphi, \pi_\varphi]$
1	$i\frac{\alpha}{2\pi}\frac{\delta(\vert\Delta\vec{x}\vert)}{\vert\Delta\vec{x}\vert^2}$	$i\infty$
2	$i\frac{m\alpha}{2\pi^2\vert\Delta\vec{x}\vert}K_1(m\vert\Delta\vec{x}\vert)$	$\frac{i}{2\pi^2}\frac{\alpha}{\vert\Delta\vec{x}\vert^2}$
3	$i\frac{\alpha}{4\pi\vert\Delta\vec{x}\vert}e^{-m\vert\Delta\vec{x}\vert}$	$\frac{i}{4\pi}\frac{\alpha}{\vert\Delta\vec{x}\vert}$
4	$i\frac{\alpha}{2\pi^2}K_0(m\vert\Delta\vec{x}\vert)$	$-\frac{i\alpha}{2\pi^2}[\ln\left(\frac{m\vert\Delta\vec{x}\vert}{2}\right)+\gamma]$
5	$i\frac{\alpha}{8\pi m}e^{-m\vert\Delta\vec{x}\vert}$	$\frac{i}{8\pi}\frac{\alpha}{m}$
6	$\frac{i\alpha}{6\pi^2}\frac{\vert\Delta\vec{x}\vert}{m}K_1(m\vert\Delta\vec{x}\vert)$	$\frac{i}{6\pi^2}\frac{\alpha}{m^2}$

Table 4.3: Some expressions of the field commutator in 3+1 dimensions.

where the case $s = 1$, corresponds to the three dimensional Dirac delta written in spherical coordinates. General expressions to calculate the result of the integrals for every s are given in the Appendix C.

By exploring the Tables 4.1, 4.2, and 4.3 it is seen that the main difference between the three, two and one dimensional cases is that the convergence for the field commutator as $\Delta x \to 0$ appears at the orders $s = 5$, $s = 4$, and $s = 3$ respectively, showing that not only the mass scale is important but also the dimension of the background, unlike the usual theory ($s = 1$) where neither the mass scale nor background dimension plays an important role since the field commutator is only described by the Dirac delta $\delta^D(\vec{x} - \vec{x'})$.

4.2 The vacuum energy

In the previous chapter, the Hamiltonian operator is found to be,

$$\hat{H} = 2 \int \frac{d^D k}{\omega_k^{s-2}} \left[a^\dagger(\vec{k}) a(\vec{k}) + b^\dagger(\vec{k}) b(\vec{k}) \right] + \frac{\alpha + \beta}{2} \frac{L^D}{(2\pi)^D} \int \frac{d^D k}{\omega_k^{s-2}}; \tag{4.9}$$

with the usual definition of the vacuum along with the choice $\alpha = \beta$, the zero-point energy is given by

$$\hat{H} |0\rangle = \frac{\alpha L^D}{(2\pi)^D} \int d^D k \, \frac{1}{\left(\sqrt{|\vec{k}|^2 + m^2} \right)^{s-2}} |0\rangle \equiv \frac{\alpha L^D}{(2\pi)^D} H_0 |0\rangle ; \qquad a|0\rangle = b|0\rangle = 0. \tag{4.10}$$

In Table 4.4 some solutions for H_0 are shown.

s	$H_{0,D=1}$	$H_{0,D=2}$	$H_{0,D=3}$
1	∞	∞	∞
2	∞	∞	∞
3	∞ as $ln(\frac{k}{m})$	∞	∞
4	$\frac{\pi}{m}$	∞ as $ln(\frac{\lvert\vec{k}\rvert}{m})$	∞
5	$\frac{2}{m^2}$	$\frac{2\pi}{m}$	∞ as $ln(\frac{\lvert\vec{k}\rvert}{m})$
6	$\frac{\pi}{2}\frac{1}{m^3}$	$\frac{\pi}{m^2}$	$\frac{\pi^2}{m}$

Table 4.4: Some values of H_0 in 1+1, 2+1, and 3+1 dimensions.

The blocks with logarithmic divergence were simplified with the approximation $\frac{m}{k} << 1$ as $k \to \infty$ after the integration.

In contrast with the well-established result in the usual formulation ($s = 1$) where the ground state energy of all the oscillators contribute to the infinite energy density, in this scheme arises a whole family of non-divergent energies for the vacuum state, showing again the importance of the background dimension and the measure employed. Note also that the vacuum energy can not to be fixed to zero within the present approach, unless one choice the constraint $\alpha = -\beta$.
The symmetrization of $\hat{Q}$ and $\hat{P}_i$, leads to the vacuum expectation values

$$\hat{Q}\left|0\right\rangle = \frac{\alpha-\beta}{2}\frac{L^D}{(2\pi)^D}\int\frac{d^Dk}{\omega_k^{s-1}}\left|0\right\rangle, \tag{4.11a}$$

$$\hat{P}_i\left|0\right\rangle = \frac{\alpha+\beta}{2}\frac{L^D}{(2\pi)^D}\int\frac{d^Dk}{\omega_k^{s-1}}k_i\left|0\right\rangle; \tag{4.11b}$$

note that, independently on the dimension D and on the choice of parameters α and β, $\hat{P}_i$ vanish trivially, due to the odd character of k_i.
Despite the infrared divergence due to the volume of integration is still handled by confining the system in a finite box, it is possible to construct a model in which the ultraviolet divergences found in the usual vacuum energy, and vacuum charge are removed.
This might be the most striking fact since ultraviolet divergences appearing in the canonical formalism are *avoided* by appealing to the normal ordering operation, which is unnecessary in this new scheme by choosing the convenient value of s. In light of this, the power s functions as an *ultraviolet regulator*.
Even though the renormalized zero-point energy in the usual formulation vanishes, the energy of the excited state $a^\dagger(\vec{k})\left|0\right\rangle$ is ω_k, which is UV divergent; therefore, in the perturbative approach to interactive field theories [9], it represents a correction which is also an ultraviolet divergence. By construction, in this new scheme, the UV regulator s removes this divergence, leading to the energy of the one-particle excited states,

$$\hat{H}a^\dagger(\vec{k})\left|0\right\rangle = \alpha\left(\frac{1}{\omega_k^{s-2}}+\frac{L^D}{(2\pi)^D}\int\frac{d^Dk'}{\omega_{k'}^{s-2}}\right)a^\dagger(\vec{k})\left|0\right\rangle, \tag{4.12a}$$

$$\hat{H}b^\dagger(\vec{k})\left|0\right\rangle = \beta\left(\frac{1}{\omega_k^{s-2}}+\frac{L^D}{(2\pi)^D}\int\frac{d^Dk'}{\omega_{k'}^{s-2}}\right)b^\dagger(\vec{k})\left|0\right\rangle; \tag{4.12b}$$

where $[a, b^\dagger] = 0$ was employed, as it is taken in the *simplest deformation* case. This is another consequence of the power s of the measure, whose most striking implication is the removal of the simplest UV divergences appearing in the canonical formalism.

4.3 Vacuum expectation values of the field operators

As in the usual theory, the definition of the vacuum state leads to a vanishing vacuum expectation value of the field operator $\langle 0|\,\varphi(\mathbf{x})\,|0\rangle$. Nevertheless, in many cases, it is better to look at the norm of the object rather than the object itself, such as in quantum mechanics where the wave function is meaningless, but its squared norm is interpreted as the probability density of the quantum mechanical system. Because of this, and with the fact that the two ways of ordering are equally considered, the expectation value of the norm is written as,

$$\begin{aligned}\langle 0|\,\frac{1}{2}\left(\varphi(\mathbf{x})\varphi^\dagger(\mathbf{x})+\varphi^\dagger(\mathbf{x})\varphi(\mathbf{x})\right)|0\rangle &= \frac{\alpha+\beta}{4(2\pi)^D}\int\frac{d^Dk}{\omega_k^s}\\ &\equiv \frac{\alpha+\beta}{4(2\pi)^D}\langle|\varphi|^2\rangle. \end{aligned} \tag{4.13}$$

Some solutions of $\langle|\varphi|^2\rangle$ are shown in Table 4.5:

	$\langle\lvert\varphi\rvert^2\rangle$		
s	$D=1$	$D=2$	$D=3$
1	∞ as $ln(\frac{k}{m})$	∞	∞
2	$\frac{\pi}{m}$	∞ as $ln(\frac{k}{m})$	∞
3	$\frac{2}{m^2}$	$\frac{2\pi}{m}$	∞ as $ln(\frac{k}{m})$
4	$\frac{\pi}{2}\frac{1}{m^3}$	$\frac{\pi}{m^2}$	$\frac{\pi^2}{m}$
5	$\frac{4}{3}\frac{1}{m^4}$	$\frac{2}{3}\frac{\pi}{m^3}$	$\frac{4}{3}\frac{\pi}{m^2}$
6	$\frac{3}{8}\frac{\pi}{m^5}$	$\frac{\pi}{2}\frac{1}{m^4}$	$\frac{\pi^2}{4}\frac{1}{m^3}$

Table 4.5: Solutions of $\langle|\varphi|^2\rangle$ in 1+1, 2+1, and 3+1 dimensions.

Therefore, the expectation value of the norm is also determined by the mass, and it presents sensibility to the background dimension and the measure employed.

As final remarks of this chapter, we can see that the convergence of the norm of the field operator has a further implication; if we define $|\varphi\rangle$ as the state formed by the action of the field operator on the vacuum state, *i.e.,*

$$|\varphi\rangle \equiv \varphi\,|0\rangle\,,$$

then the states $|\varphi\rangle$, and $|\varphi^\dagger\rangle$ are well defined; unlike in the usual theory, in which it is necessary to smear the distributions over the space in order to form well defined operators [5],

$$|\varphi\rangle = \int\frac{d^Dk}{(2\pi)^3}e^{-i\vec{k}\cdot\vec{x}}\varphi(\vec{k})|\vec{k}\rangle; \tag{4.14}$$

with $\varphi(\vec{k}) = \exp\left(-\vec{k}^{\,2}/2m^2\right)$.

On the other hand, note that according to the results obtained in this chapter, the more stringent version of a non-commutative quantum field theory can be obtained with the restriction $\alpha \neq \pm\beta$.

Chapter 5

The time dependent field commutators

After studying the consequences of the *simplest* deformation, this section deals with the very interesting case in which $[a, b] \neq 0$, recovering the time dependence in the theory and giving a more general field commutation relations; furthermore, the case $[a, b] \neq 0$ is specially interesting for theories with interactions.

In the basic reference on dissipative quantum field theory [10], the formulation is based on the dynamics for a damped harmonic oscillator; in this way, the commutation relations are inherited from the physics of a harmonic oscillator. Such commutation relations, suffer from the non-preservation by the time evolution of the canonical structure due to damping terms; therefore, it leads to the necessity of introducing fluctuating forces in favor to preserve the canonical structure.

On the other hand, some authors [11] realized that a structure based on hypercomplex numbers underlies the dissipative dynamics, reveling an internal non-compact symmetry that drives the dynamics. Then, a bifurcation emerges with respect to the usual formulation based on elliptic complex numbers, by modeling theories with dissipation with a new scheme, which also cures some pathologies associated with the existence of many unitarity inequivalent representations, and the mentioned non-preservation of the field commutators due to the presence of damping factors. This scheme is dependent on a deformed harmonic oscillator algebra, which permits to relinquish to the conventional description, and where the commutator $[a, b]$ is in fact fundamental. In this section the behavior of the commutators (3.14) is studied, giving a general expression for the field commutators where all the constants (α, β, σ and ρ) are different from zero.

5.1 The 1+1 field theory

Unlike the previous chapter, the integrals associated with the commutator $[a, b]$ are more difficult to solve, and therefore, in this work, the solution in one spatial dimension $D = 1$ is presented. Nevertheless, the 1+1 dimensional case should not be underestimated, since many interesting physical theories have exact solutions just in one spatial dimension [12]; for example, the exact solution of Quantum Electrodynamics in 1+1 dimensions proposed by Schwinger. Therefore, the study of two-dimensional models plays a key role for testing ideas in quantum field theory, which are widely applicable in other theories, like string theory or statistical mechanics. Since 1+1 dimensional QFT shows an interesting structure, it is not trivial to study from this new formalism the case of the free field in two space-time dimensions.

Then, for $D = 1$ the field commutators at the equal time are

$$\big[\varphi(x,t), \varphi(x',t)\big] = 0, \tag{5.1a}$$

$$\big[\pi_\varphi(x,t), \pi_\varphi(x',t)\big] = 0, \tag{5.1b}$$

$$\Big[\varphi(x,t), \varphi^\dagger(x',t)\Big] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{\omega_k^s}\Big\{\frac{\alpha-\beta}{2}e^{ik\Delta x} + 2\rho\sin(2\omega_k t)\cos\left(k\Delta x\right)\Big\}, \tag{5.1c}$$

$$\big[\pi_\varphi(x,t), \pi_{\varphi^\dagger}(x',t)\big] = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{\omega_k^{s-2}}\Big\{\frac{\alpha-\beta}{2}e^{-ik\Delta x} - 2\rho\sin(2\omega_k t)\cos\left(k\Delta x\right)\Big\}, \tag{5.1d}$$

$$\big[\varphi(x,t), \pi_{\varphi^\dagger}(x',t)\big] = i\frac{\sigma}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{\omega_k^{s-1}}\cos(k\Delta x), \tag{5.1e}$$

$$\big[\varphi(x,t), \pi_\varphi(x',t)\big] = \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{\omega_k^{s-1}}\Big\{\frac{\alpha+\beta}{2}e^{ik\Delta x} - 2i\rho\cos(2\omega_k t)\cos\left(k\Delta x\right)\Big\}; \tag{5.1f}$$

where $[a, b] = i\rho\delta(k + k')$.

For the case $\rho \neq 0$, the convergence of the integrals (5.1) depends on the power s, and the first choice to ensure convergence in all the integrals with $m \neq 0$ is $s = 3$. In fact, the solution for the field commutators with $s = 3$ is split in two intervals that correspond to the inside and outside regions of the light cone, as it is presented in a next section; thus, the non-vanishing commutators are written as

$$\left[\varphi, \varphi^{\dagger}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{dk}{(k^2+m^2)^{\frac{3}{2}}}\left\{\frac{\alpha-\beta}{2}e^{ik\Delta x} + 2\rho\sin\left(2t\sqrt{k^2+m^2}\right)\cos\left(k\Delta x\right)\right\}, \tag{5.2a}$$

$$\left[\pi_{\varphi}, \pi_{\varphi^{\dagger}}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{dk}{(k^2+m^2)^{\frac{1}{2}}}\left\{\frac{\beta-\alpha}{2}e^{-ik\Delta x} + 2\rho\sin\left(2t\sqrt{k^2+m^2}\right)\cos\left(k\Delta x\right)\right\}, \tag{5.2b}$$

$$\left[\varphi, \pi_{\varphi^{\dagger}}\right] = i\frac{\sigma}{2\pi}\int_{-\infty}^{\infty} \frac{dk}{k^2+m^2}\cos(k\Delta x), \tag{5.2c}$$

$$\left[\varphi, \pi_{\varphi}\right] = \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{dk}{k^2+m^2}\left\{\frac{\alpha+\beta}{2}e^{ik\Delta x} - 2i\rho\cos\left(2t\sqrt{k^2+m^2}\right)\cos\left(k\Delta x\right)\right\}. \tag{5.2d}$$

It was mentioned that the finding of the solution of the integrals (5.2) is not a trivial work; in fact, there is no complete analytic solution in the literature for them, thus in this case, computational software support was necessary. In addition, such as in the previous chapter, we consider that $\alpha = \beta$; in this case the units of α, σ, and ρ are $[E]/[T]$, as it is explained in Appendix B.

First, the change of variable $k \to 2k$ is made in the integrals depending on the time coordinate to maintain the dependence on t instead of $2t$, and the integrals appearing in [8] are considered,

$$\int_0^{\infty} dx\, \frac{\sin\left[b\left(a^2+x^2\right)^{1/2}\right]}{\left(a^2+x^2\right)^{1/2}}\cos(xy) = \begin{cases} \frac{1}{2}\pi J_0\left[a\left(b^2-y^2\right)^{1/2}\right] & 0 \leq y < b, \\ 0 & b < y < \infty \\ & \text{or} \quad b = y = 0, \\ \frac{1}{4}\pi & b = y \neq 0, \end{cases} \tag{5.3a}$$

$$\int_0^{\infty} dx\, \frac{\sin\left[b\left(a^2+x^2\right)^{1/2}\right]}{\left(a^2+x^2\right)^{3/2}}\cos(xy) = \begin{cases} \frac{\pi}{8}F(b,y) & 0 < y < b, \\ \frac{\pi}{2}\frac{b}{a}e^{-ay} & b \leq y < \infty. \end{cases} \tag{5.3b}$$

In the relations (5.2) and with the assumption that $\alpha = \beta$, the sign of Δx is irrelevant since it appears only in cosine functions; therefore it is removed the absolute value indication in order to simplify the notation, thus the reader should keep in mind that henceforth $|\Delta x| \equiv \Delta x$.

The field commutators then become

$$\left[\varphi(x,t),\varphi^{\dagger}(x',t)\right]=\rho\begin{cases}F(t,\Delta x) & 0<\frac{\Delta x}{2}<t,\\ \frac{2t}{m}e^{-m\Delta x} & t\leq\frac{\Delta x}{2}<\infty,\end{cases}\tag{5.4a}$$

$$\left[\varphi(x,t),\pi_{\varphi}(x',t)\right]=i\frac{\alpha}{2m}e^{-m\Delta x}+\rho\begin{cases}\frac{1}{2}\frac{\partial}{\partial t}F(t,\Delta x) & 0<\frac{\Delta x}{2}<t,\\ \frac{1}{m}e^{-m\Delta x} & t\leq\frac{\Delta x}{2}<\infty,\end{cases}\tag{5.4b}$$

$$\left[\pi_{\varphi}(x,t),\pi_{\varphi^{\dagger}}(x',t)\right]=\rho\begin{cases}J_0\left[2m\left(t^2-(\frac{\Delta x}{2})^2\right)^{1/2}\right] & 0\leq\frac{\Delta x}{2}<t,\\ 0 & t<\dfrac{\Delta x}{2}<\infty \\ & \text{or}\quad t=\dfrac{\Delta x}{2}=0,\\ \frac{1}{2} & t=\frac{\Delta x}{2}\neq 0,\end{cases}\tag{5.4c}$$

$$\left[\varphi(x,t),\pi_{\varphi^{\dagger}}(x',t)\right]=i\frac{\sigma}{2m}e^{-m\Delta x};\tag{5.4d}$$

with $F(t,\Delta x)=\frac{8}{\pi}\int_0^{\infty}\frac{dk}{(k^2+(2m)^2)^{\frac{3}{2}}}\sin\left(t\sqrt{k^2+(2m)^2}\right)\cos\left(k\frac{\Delta x}{2}\right)$, and Δx a positive quantity. Note that in equations (5.4), the contributions of $[a(k),a^{\dagger}(k')]$ and $[b(k),b^{\dagger}(k')]$, modulated by the constant α, are just the same such as those ones obtained in the previous chapter.

As it is explored in the next subsection, an important difference with the usual formulation is that $F(t,\Delta x)$ has finite values for $\Delta x=0$; thus, the field commutators evaluated at the same space-time point will not diverge, and in general they are described by smoother distributions than the usual Dirac delta, leading to the idea that there are no natural limits between both formulations.

In the relations (5.4), the solution for the field commutators with a contribution of ρ is split in two regions, $0<\frac{\Delta x}{2}<t$ and $t<\frac{\Delta x}{2}<\infty$ which corresponds to the inside and outside regions of the light cone respectively as it is studied in a next section. Furthermore, the contribution in the outside region is proportional to $e^{-m\Delta x}$; this fact is another interesting difference with the usual theory, since, in the canonical formulation, the exponential decay of the free propagator appears outside the light cone [4], and in this non-canonical formalism the leak outside the light cone is recovered at the level of the field commutators.

5.2 The $F(t, \Delta x)$ function

In order to analyse the function $F(t, \Delta x)$, the contributions to the field commutators that are independent of time are temporarily turned off by taking $\alpha = \sigma = 0$, and $\rho = 1$. Then the unique commutators involving time dependence are

$$\left[\varphi(x,t), \varphi^\dagger(x',t)\right] = \begin{cases} F(t,\Delta x) & 0 < \frac{\Delta x}{2} < t, \\ \frac{2t}{m} e^{-m\Delta x} & t \leq \frac{\Delta x}{2} < \infty, \end{cases} \tag{5.5a}$$

$$\left[\varphi(x,t), \pi_\varphi(x',t)\right] = \begin{cases} \frac{1}{2}\frac{\partial}{\partial t} F(t,\Delta x) & 0 < \frac{\Delta x}{2} < t, \\ \frac{1}{m} e^{-m\Delta x} & t \leq \frac{\Delta x}{2} < \infty, \end{cases} \tag{5.5b}$$

$$\left[\pi_\varphi(x,t), \pi_{\varphi^\dagger}(x',t)\right] = \begin{cases} J_0\left[2m\left(t^2 - (\frac{\Delta x}{2})^2\right)^{1/2}\right] & 0 \leq \frac{\Delta x}{2} < t, \\ 0 & t < \dfrac{\Delta x}{2} < \infty \\ & \text{or} \quad t = \dfrac{\Delta x}{2} = 0, \\ \frac{1}{2} & t = \frac{\Delta x}{2} \neq 0, \end{cases} \tag{5.5c}$$

With the aid of a software these fields commutators are shown in Figure 5.1. As it is explored in the next subsection, the condition of evaluating the field commutators at the same time t, and on the light cone implies that either $\Delta x = 0$ as long as $x = x' = t$, or $\frac{\Delta x}{2} = t$, rather than $\Delta x = t$; this is seen in Figure 5.3b.

Even though there is a leak exponentially decaying for space-like distances, for small values of m the commutator penetrates deeper outside the light cone, going to infinity as $m \to 0$; this is represented in Figure 5.2a.

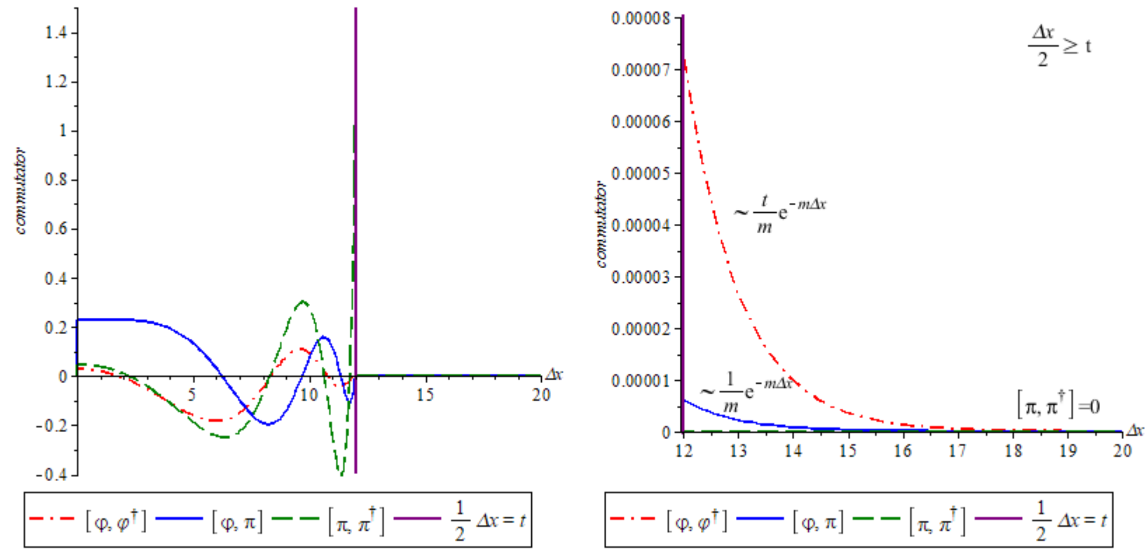
Unlike the usual formulation where $[\varphi(x,t), \pi_\varphi(x',t)] = i\delta(x - x')$, in the formulation at hand, the commutators have the following finite expression for $\Delta x \to 0$,

$$\left[\varphi(x,t), \varphi^\dagger(x',t)\right]\Big|_{x=x'} = \frac{8}{\pi}\int_0^\infty \frac{dk}{(k^2+M^2)^{\frac{3}{2}}} \sin\left(t\sqrt{k^2+M^2}\right), \tag{5.6a}$$

$$\left[\varphi(x,t), \pi_\varphi(x',t)\right]\Big|_{x=x'} = \frac{4}{\pi}\int_0^\infty \frac{dk}{k^2+M^2} \cos\left(t\sqrt{k^2+M^2}\right), \tag{5.6b}$$

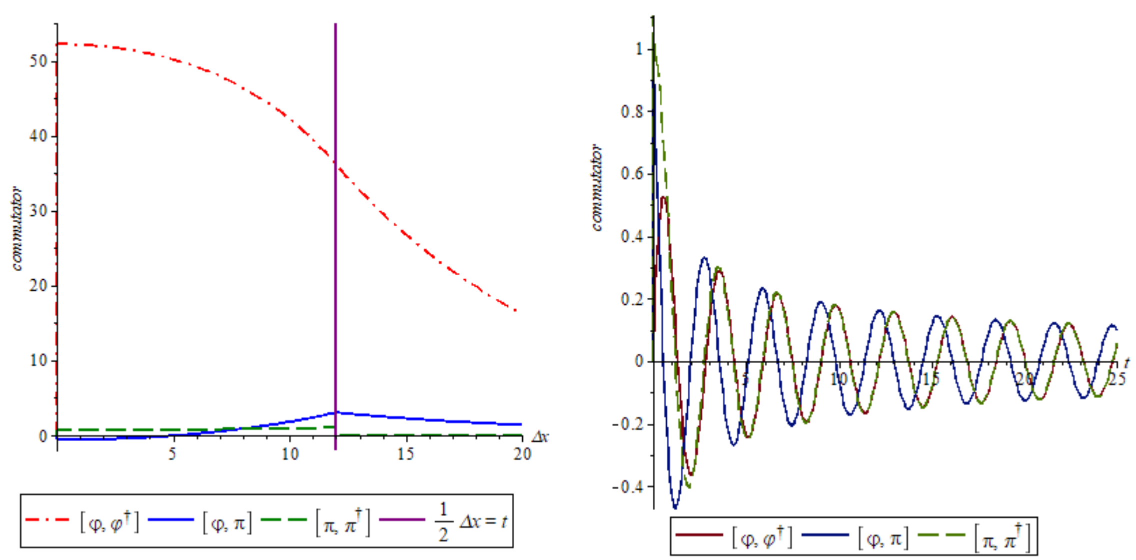
$$\left[\pi_\varphi(x,t), \pi_{\varphi^\dagger}(x',t)\right]\Big|_{x=x'} = \frac{2}{\pi}\int_0^\infty \frac{dk}{(k^2+M^2)^{\frac{1}{2}}} \sin\left(t\sqrt{k^2+M^2}\right); \tag{5.6c}$$

with $M \equiv 2m$, they are represented in Figure 5.2b; the unique analytical expression accessible at the moment is for the last commutator, $\left[\pi_\varphi(x,t), \pi_{\varphi^\dagger}(x',t)\right]\big|_{x=x'} = J_0[2mt]$.



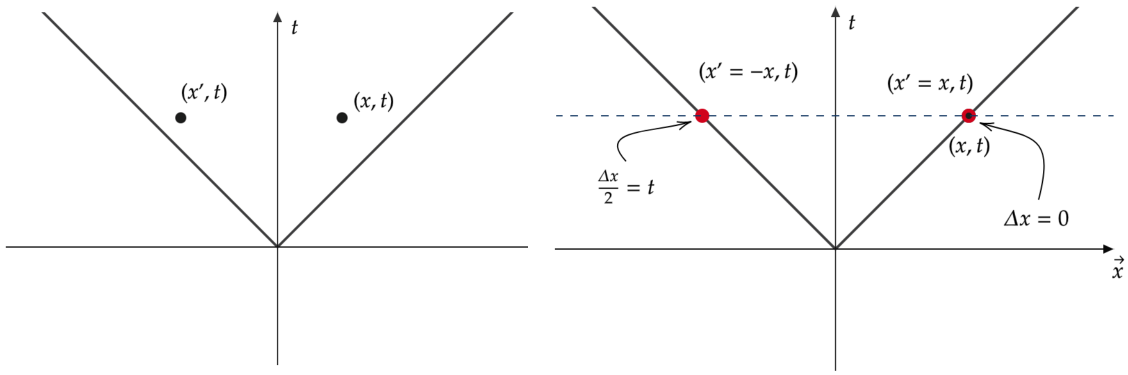
(a) (b)

Figure 5.1: (a) Commutators (5.5) as function of Δx for fixed m and t. The vertical purple line represents the light cone; its left hand side corresponds to the inside region of the cone. (b) Zoom of the outside region of the light cone of Figure 5.1a.



(a) (b)

Figure 5.2: (a) Commutators with the same parameters m and t as in Figure 5.1 reduced by a factor $\frac{1}{10}$. (b) Commutators as function of t for $\Delta x = 0$ and fixed m



(a) (b)

Figure 5.3: (a) Two arbitrary points in space-time for time-like distances and equal times. (b) Points along the light cone with the equal time condition.

5.3 Quantum fields inside and outside the light cone

As it is seen in equations (5.4), the contribution modulated by ρ shows a change of behavior from the integral function $F(t, \Delta x)$ to an exponential decay when we "cross" the boundary $\frac{\Delta x}{2} = t$. In this section, the behavior of the commutators in the regions in which they are split is discussed.

Firstly, let be an observer S and two arbitrary points in space-time (x, t) and (x', t); since the commutators are evaluated at the same time by S, the two points are also restricted to this condition as shown in Figure 5.3a. If the observer S wants to evaluate the fields along the light cone then it holds that $|x| = t$, which implies that $x' = \pm x$ because they have the same time coordinate. In terms of Δx this means that in the light cone the spatial distance is either $\Delta x = 0$ or $\frac{\Delta x}{2} = t$ as Figure 5.3b illustrates; therefore, the maximum time-like separation is such that $\Delta x = 2t$. Then, the decaying exponential in (5.5) for $t \leq \frac{\Delta x}{2}$ is a sort of leak outside the light cone.

Despite both of them decay as $\Delta x \to \infty$, there is a slight difference in $F(t, \Delta x)$ evaluated along the light cone if $\Delta x = 0$ or $\frac{\Delta x}{2} = t$, this is

$$\Delta x = 0 \longrightarrow F(t, \Delta x) = \frac{8}{\pi} \int_0^\infty \frac{dk}{(k^2 + 4m^2)^{\frac{3}{2}}} \sin\left(t\sqrt{k^2 + 4m^2}\right), \tag{5.7a}$$

$$t = \frac{\Delta x}{2} \longrightarrow F(t, \Delta x) = \frac{\Delta x}{m} e^{-m\Delta x}. \tag{5.7b}$$

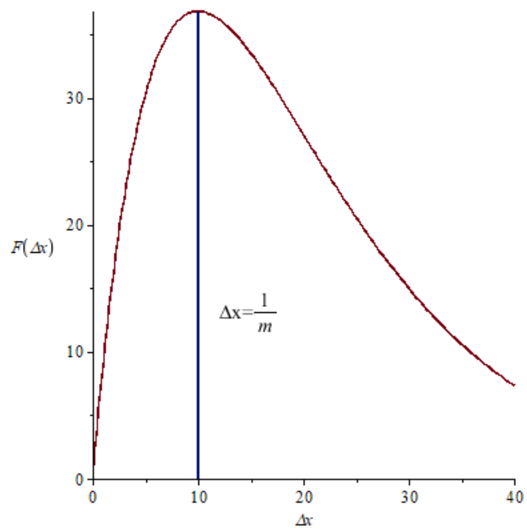


Figure 5.4: $F(t, \Delta x)$ evaluated in the light cone for $\frac{\Delta x}{2} = t$. The maximum of the function occurs at $\Delta x = \frac{1}{m}$.

Whereas the expression (5.7a) is shown in Figure 5.2b, the expression (5.7b) is drawn in Figure 5.4, which shows that the time dependent contribution of the field commutators penetrates into a region of the order $\sim \frac{1}{m}$ outside the light cone.

Chapter 6

Conclusions

Although the canonical formalism is a well established method of quantization for the free scalar field theory, it suffers from some pathologies like ultraviolet divergences, and non-preservation of the canonical structure in the case of dissipative field theories. Therefore, this work proposes a non-canonical formalism, by taking advantage of the freedom for the power s of the measure, and the algebra between creation and annihilation operators.

By virtue of taking the general integration measure determined by the ultraviolet regulator s, and the deformed and extended algebra (3.13), it is possible to obtain an infinite family of non-divergent solutions for the field commutators evaluated in the same space-time point, and to remove the ultraviolet divergence in the zero-point energy; furthermore, the dependence on time, background dimension, and mass scale are recovered in the theory. This new scheme of quantization also removes the necessity of the normal ordering operation, since every object containing the ultraviolet regulator introduced in this work can be smeared to smoother distributions by choosing a suitable value of s. In the case of the vacuum energy, this leads to a finite value that is consistent with general relativity, since the vacuum energy is determined precisely by the mass of the field.

Up to the moment, we have a complete description for the time dependent contribution just for the (1+1)-theory, but it leads to interesting conceptual consequences, just as the *leak* into the outside region of the light cone, which is present in the usual theory at the level of the free propagator.

Appendix A

Proof of the Lorentz invariance of the measure

For the real scalar field, the field operator

$$\varphi(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^D}} \int d^{D+1}k \; \delta\left(k^2 - m^2\right) \Theta\left(k^0\right) A(k) e^{-i\mathbf{k}\cdot\mathbf{x}}, \tag{A.1}$$

is invariant under orthochronous Lorentz transformations since it depends only on scalar quantities. Nevertheless, it is useful to show explicitly that the measure is in fact, Lorentz invariant.

First of all, it is evident that $d^{D+1}k$ is a Lorentz invariant quantity because of the unitary nature of the proper Lorentz transformations, *i.e.*, $\det \lambda = 1$. On the other hand, $\delta(k^2 - m^2)$ depends only in the scalar $k^\mu k_\mu = m^2$. Therefore, the measure

$$\int d^{D+1}k \; \delta\left(k^2 - m^2\right)\Big|_{k^0>0}, \tag{A.2}$$

is Lorentz invariant.

By using the property of the Dirac delta,

$$\delta(k^2 - m^2) = \frac{1}{2\omega_k}(\delta(k^0 - \omega_k) + \delta(k^0 + \omega_k)),$$

and integrating over the time component k^0, it is obtained

$$\int d^{D+1}k \; \delta\left(k^2 - m^2\right)\Big|_{k^0>0} = \int \frac{d^D k}{2k^0}\Big|_{k^0=\omega_k}. \tag{A.3}$$

Then, the measure $\frac{d^D k}{2\omega_k}$, and as a consequence, the field operator definition

$$\varphi(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^D}} \int \frac{d^D k}{2\omega_k} \left(A(k) e^{-i\mathbf{k}\cdot\mathbf{x}} + B^\dagger(k) e^{i\mathbf{k}\cdot\mathbf{x}} \right), \tag{A.4}$$

are explicitly Lorentz invariant.

Appendix B

Units of some important objects

At the beginning of this work it was established the use of the so called *natural units*, where $\hbar = c = 1$; nevertheless, this obscures the different units obtained for each value of s. Therefore, it is important to recover the physical constants in order to know what to expect from the constants α, β, σ, and ρ.
First of all, the units of the fields are determined by the Lagrangian density,

$$\mathcal{L} = \hbar^2 \ \partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 c^2 \varphi^\dagger \varphi. \tag{B.1}$$

Therefore the fields are

$$\varphi = \frac{c}{\hbar} \int \frac{d^D k}{\sqrt{(2\pi)^D} \omega_k^{\frac{s}{2}}} \left[a(\vec{k}) e^{-i(\mathbf{k}\cdot\mathbf{x})} + b^\dagger(\vec{k}) e^{i(\mathbf{k}\cdot\mathbf{x})} \right], \tag{B.2a}$$

$$\pi_\varphi = \frac{\hbar^2}{c^2} \frac{\partial}{\partial t} \varphi; \tag{B.2b}$$

with $\omega_k = c\sqrt{k^2 + \frac{m^2 c^2}{\hbar^2}}$.
Since the Lagrangian density has units of $[\mathcal{L}] = E/V$, where E denotes energy and V volume, *i.e.*, $V \equiv L^D$, the field must have the units $[\varphi] = V^{-1/2} M^{-1/2}$, with M the mass units.
With this, the field commutator $[\varphi, \pi_\varphi]$ is restricted to take the units, $[[\varphi, \pi_\varphi]] = ML^2 V^{-1} T^{-1}$, with L and T, length and time units respectively. In other words, $[[\varphi, \pi_\varphi]] = \hbar/V$. Finally, as a consequence of these requirements, $[a(k)] = [b(k)] = M^{1/2} V^{1/2} T^{-s/2} L$ and $[\alpha] = \hbar/T^{s-1}$, with the same for β, σ and ρ.

Appendix C

Definite integrals used for the (3+1)-simplest deformation

In the case of the *simple deformation* problem for three spatial dimensions, we deal with the integral

$$\left[\varphi(\vec{x},t),\pi_{\varphi}\left(\vec{x'},t\right)\right]\sim\int_{0}^{\infty}dk\frac{k\sin(k|\Delta\vec{x}|)}{\left(\sqrt{k^{2}+m^{2}}\right)^{s-1}}, \tag{C.1}$$

whose solution is found in the reference [8],

- For s odd,

$$\int_{0}^{\infty}dk\frac{k\sin(k|\Delta\vec{x}|)}{\left(\sqrt{k^{2}+m^{2}}\right)^{s-1}}=\frac{\pi}{2}\frac{(-1)^{\frac{s-3}{2}}}{\left(\frac{s-3}{2}\right)!}\frac{d^{\left(\frac{s-3}{2}\right)}}{dz^{\left(\frac{s-3}{2}\right)}}e^{-|\Delta\vec{x}|\sqrt{z}};\quad z\equiv m^{2}. \tag{C.2}$$

- For s even,

$$\int_{0}^{\infty}dk\frac{k\sin(k\xi)}{\left(\sqrt{k^{2}+m^{2}}\right)^{s-1}}=\frac{-\sqrt{\pi}}{2^{\frac{s-2}{2}}m^{\frac{s-2}{2}}\Gamma\left(\frac{s-1}{2}\right)}\frac{d}{d\xi}\left[\xi^{\frac{s-2}{2}}K_{\frac{s-2}{2}}(m\xi)\right],\quad s\geq 2; \tag{C.3}$$

with $\xi\equiv|\Delta\vec{x}|$.

Bibliography

[1] R. J. Crewther, *Introduction to quantum field theory* , arXiv:hep-th/9505152.

[2] I. J. R. Aitchison, *Nothing's plenty the vacuum in modern quantum field theory*, Contemporary Physics, 26:4, 333-391, (1985).

[3] S. E. Rough, and H. Zinkernagel, *The quantum vacuum and the cosmological constant problem*, Stud. in Hist. and Phil. of Mod. Phys. 33, 663, (2002).

[4] A. Zee, *Quantum Field Theory in a Nutshell*, 2nd ed., Princenton University Press, (2010).

[5] D. Tong, *Lectures on Quantum Field Theory*, www.damtp.cam.ac.uk

[6] W. Greiner, and J. Reinhardt, *Field Quantization*, Springer, (1996).

[7] A. Lahiri, and P.B. Pal, *A First Book of Quantum Field Theory*, 2nd ed., Alpha Science International, (2004).

[8] I.S. Gradshteyn, and I.M Ryzhik, *Tables of Integrals, Series, and Products*, 7th ed. Academic Press, (2007).

[9] M. Asorey, *A Concise Introduction to Quantum Field Theory*, International Journal of Geometric Methods in Modern Physics 16 (2019).

[10] E. Celeghini, M. Rasetti, G. Vitiello, *Quantum dissipation*. Ann. Phys. 215, 156 (1992).

[11] R. Cartas-Fuentevilla, J. Berra-Montiel, and O. Meza-Aldama, *Hyperbolic ring based formulation for thermo field dynamics, quantum dissipation, entanglement, and holography*. Eur. Phys. J. C 80, 603 (2020).

[12] E. Abdalla, M.C.B. Abdalla, and K.D. Rothe, *Non-Perturbative Methods in 2 Dimensional Quantum Field Theory*, World Scientific, (2001).