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Evolución de espacios máximamente simétricos bajo el flujo de Ricci

Resumen

En este trabajo presentamos soluciones a las ecuaciones del flujo de Ricci en $3d$ y $4d$. Como primer caso obtenemos una solución a partir de un ansatz para la métrica y el campo vectorial de DeTurck en $3d$. Esta solución pertenece a la clase de espacios máximamente simétricos, y puede ser extendida a $4d$ siguiendo un tratamiento análogo al caso $3d$. Nuestras soluciones pueden ser divididas en dos escenarios: espacios máximamente simétricos con curvatura positiva, es decir espacios de Sitter, y espacios máximamente simétricos con curvatura negativa, es decir espacios Anti-de Sitter. El flujo incrementa la curvatura en los espacios de Sitter y la disminuye en los espacios Anti-De Sitter. También mostramos que entre ambos escenarios existe una "transición de fase" donde la curvatura es infinita. Una característica interesante de la solución para $n \geq 4$, es que las ecuaciones de flujo se satisfacen con signatura Euclídea o de Lorentz. También mostramos un efecto interesante del flujo en el cual la signatura de la métrica cambia cuando pasa de un espacio de Sitter a un espacio Anti-de Sitter a través del flujo.

Evolution of maximally symmetric spaces under the Ricci Flow

Abstract

In this work we present solutions to the Ricci flow equations in $3d$ and $4d$. First we solve the flow equations starting with an ansatz for the metric and the DeTurck vectorial field in $3d$. We note that our solution belongs to the family of maximally symmetric spaces and can be extended to $4d$ following an analogue treatment to the $3d$ case. Our solutions can be divided into two scenarios: maximally symmetric spaces with positive curvature i.e. de Sitter spaces, and maximally symmetric spaces with negative curvature i.e. Anti-de Sitter spaces. The flow increases the curvature of de Sitter spaces, and decreases the curvature of Anti-de Sitter spaces. We show that between both scenarios there is a "phase transition" where the curvature blows up. An interesting feature of the solution with $n \geq 4$ is that the flow equations are satisfied with an Euclidean or Lorentzian signature. Also we show an interesting effect of the flow consisting in a change of the signature of the metric when passing from a de Sitter space to an Anti-de Sitter space throughout the flow.

Content

Chapter 1

Introduction

Classifying manifolds has been an open problem from the beginning of topology. The works of Poincaré, Thurston, Hamilton, Perelman and many others to classify topological spaces have open the door to new fields in mathematics and also in physics. In this work we show an explicit solution to the Ricci flow that could have potential applications in physics.

The first motivation for us to study the Ricci flow in a physical context was the paper [1] where the Ricci flow was applied to a family of wormhole geometries. However in this paper the treatment was completely numeric. The results shown there are fairly compelling, and agree with more formal treatments of the flow [2], [3], nonetheless it will be very nice to get more information from an analytical solution. Thus as a starting point we tried to solve the Ricci flow equations with the purpose of finding a wormhole solution. Nevertheless we found a no less interesting solution to the Ricci flow equations, and in a similar way as in [1] we found a kind of critical phenomenon given by a "phase transition" between spaces with positive curvature and spaces with negative curvature at a critical value of the flow parameter.

In chapter 2 we present the basics of the Hamilton's Ricci flow. Then we show some special solutions known as Einstein manifolds and Ricci solitons. In particular we have found solutions that belong to the family of Einstein manifolds. Then we show some examples of applications of the Ricci flow in the context of general relativity. Also we make a brief resume of the paper [1] where the Ricci flow is applied to deform a family of wormhole geometries. In that work it is found a critical phenomenon given by a special value of a parameter that identifies different wormhole geometries. Also it is shown that the throat of the wormhole can expand or shrink. If the parameter is under the critical value the throat shrinks in finite "time", otherwise if the parameter is above the critical value the throat expands indefinitely. It is expected that the topology of the manifold will converge to that of a cylinder expanding monotonically in the case when the throat is expanding, if the throat pinches off it is expected that two asymptotically regions will evolve separately to flat spaces.

Inasmuch as our solutions belong to maximally symmetric spaces we make a brief review of this topic in chapter 3 . Then we start our treatment of the Ricci flow in $3d$ deducing our solution from a generic ansatz for the metric in $3d$ that possess just one arbitrary function that must be determined. For simplicity this function depends only on the flow parameter and the radial coordinate. Then from the flow equations we obtain a dynamical and a constriction equation. Once we have found a solution, we show that there is a fixed point of the flow which is a flat space. Then we verify that the solution represents a maximally symmetric space.

Once that we have found a solution in $3d$ we extend our solution to $4d$ noting that the $3d$ metric found previously has the form of the spatial part of a maximally symmetric space in $4d$ written in global coordinates. Then we see that a maximally symmetric space in $4d$ is also a solution to the Ricci flow. The deduction is analogue to the case in $3d$ but this time we have two functions that must be determined, then it is easy to see that the solution can be extended for any dimension. A very interesting feature of our solution is that the flow can be carried on either with an Euclidean or Lorentzian signature for $n \geq 4$. Although we can extend the solution to $4d$ and higher dimensions we show that there are no more fixed points for $n \geq 4$.

In the Apendix A we show some results that are needed to understand the key points of the evolution of the metric and other geometrical quantites under an uniparametric flow. Our treatment of the basics of Ricci flow is not exahustive and for the reader who is interested in more formal details of the work of Perelman and the Ricci flow we recommend to have a look on [4–11]. There are also many possibilities for physical applications, to see other interesting applications in physics check [1, 12–15].

Chapter 2

Hamilton's Ricci flow

2.1 History of Ricci flow

Geometric flows are important geometric partial differential equations that allow geometrical quantities to evolve under a certain class of heat equations. We can say that they appear with the so called curve shortening flow, proposed in 1956 by W. Mullins [16], to model the motion of idealized grain boundaries. Then after some extensions and due to its geometrical properties mathematicians as R. Hamilton and others realized that geometric flows could be used as a tool to prove the geometrization conjecture, which is a generalization of the work of H. Poincaré on the classification of 3-manifolds.

About 1900 as an attempt to classify $3-$manifolds, H. Poincaré stated that

- *Every simply connected and closed 3-manifold is homomorphic to the 3-sphere.*

This conjecture surprisingly remained without a proof for almost 100 years. In 2002 the mathematician G. Perelman published a series of 3 papers in which he proved the Thurston's geometrization conjecture which includes as a special case the Poincaré's conjecture. The aim of Thurston's geometrization conjecture is to classify 3-dimensional topological spaces and can be seen as the analogue of the uniformization theorem for two-dimensional surfaces. This theorem states that every simply connected Riemann surface has a conformally equivalent Riemannian metric with constant curvature (Euclidean, spherical, or hyperbolic). In three dimensions the difficulty to classify manifolds increases because not always it is possible to assign a single geometry to a whole topological space. The geometrization conjecture states that every closed 3-manifold can be decomposed into pieces that possess one of the eight types of fundamental geometric structures, i.e $\mathcal{R}^3, \mathcal{H}^3, \mathcal{S}^3$,

$\mathcal{S}^2 \times \mathcal{R}$, $\mathcal{H}^2 \times \mathcal{R}$, the geometry of the universal cover of $SL(2, \mathcal{R})$, the Nil geometry and the Solv geometry [7, 11, 17].

In 1982 R. Hamilton [4] introduced the so-called Ricci flow equation, as part of a program to solve the Thurston's conjecture, that year W. Thurston was awarded with the Fields Medal for his contributions to the study of 3-manifolds. The Ricci flow equation has been called the heat equation for metrics, due to its property of making metrics "homogeneous". Hamilton's work and subsequent papers have had a profound influence on modern geometric analysis, and recently Ricci flow has started to gain interest in physics due to its fascinating properties.

2.2 Hamilton's Ricci flow

In Hamilton's work [4] it is proved that *"given a smooth metric g_0 on a closed manifold $\mathcal{M}$, there exist $\epsilon > 0$ and a smooth family of metrics $g(\lambda)$ such that*

$$\frac{\partial g(\lambda)}{\partial \lambda} = -2Rc(g), \qquad \lambda \in [0, \epsilon] \tag{2.1}$$

$$g(0) = g_0, \tag{2.2}$$

here $Rc(g)$ denotes the Ricci tensor depending on the metric $g(\lambda)$".

Equation (2.1) is the so called Ricci flow. The main idea behind the Ricci flow is to find a mechanism that allows the metric tensor g evolve under a partial differential equation which resembles the heat equation. So as the λ parameter goes on, the metric tensor will become homogeneous throughout the flow.

In some cases the flow defforms a metric g into a metric that is distinguished by its curvature, as in the case of a two-dimensional manifolds where the flow deforms a metric to one of constant curvature, and thus gives a proof of the two-dimensional uniformisation theorem. In general, the topology of $\mathcal{M}$ may lead to the formation of singularities in finite time. Although singularities can appear, the behaviour of the flow is still giving us many information about the topology of the underlying manifold. The strategy to deal with singularities is to stop the flow once a singularity has formed, and then perform "surgery", as Perelman called it [10], on the evolved manifold, removing the singular regions and then making a connected sum of the remaining parts to continue the flow.

An inconvinient with the Ricci flow equation, is that it is only weakly parabolic (see [5], [6], [7] for a detailed explanation), so in order to prove short time existence D. DeTurck has used the known "DeTurck Trick" [18] which makes use of a family of diffeomorphisms ψ_λ to modify the Ricci flow. The main idea is to prove that this modified flow is strongly parabolic and then show that it is equivalent to the original Ricci flow. The Hamilton-DeTurck Ricci flow is defined by the equation

$$\partial_\lambda g_{ij} = -2R_{ij} + \nabla_i V_j + \nabla_j V_i, \tag{2.3}$$

where $g_{ij} = g_{ij}(\lambda, \vec{x})$ identifies a 3-metric with Euclidean signature, λ is the flow parameter, V_i is the DeTurck vector field which generates diffeomorphisms along the flow, and R_{ij} is the Ricci tensor.

2.3 Special solutions to the Ricci flow

2.3.1 Einstein manifolds

There is a class of special solutions to the Ricci flow called Einstein manifolds, which are solutions to the Einstein equations with cosmological constant σ and are given by the relation

$$R_{ij}(g(0)) = \sigma g_0. \tag{2.4}$$

As an exmple in a n-dimensional sphere of radius r (with $n > 1$), the metric can be written as $g = r^2 g_0$ where g_0 is the metric on the unit sphere. Then it can be shown that [7]

$$R_c = (n-1)g_0; \tag{2.5}$$

then substituting g and R_c in the flow equations (2.1) we obtain

$$\begin{aligned} \frac{\partial g}{\partial \lambda} &= -2R_c(g) \\ \frac{\partial (r^2 g_0)}{\partial \lambda} &= -2(n-1)g_0 \\ \frac{\partial r^2}{\partial \lambda} &= -2(n-1) \end{aligned} \tag{2.6}$$

then

$$r = \sqrt{\lambda_0 - 2(n-1)\lambda} \tag{2.7}$$

with λ_0 the initial radio of the sphere. It can be seen that as λ increases the manifold shrinks to a point until

$$\lambda = \frac{\lambda_0}{2(n-1)}. \tag{2.8}$$

An alternative example of this type would be if g_0 were a hyperbolic metric that is of constant sectional curvature -1. In that case $Rc(g_0) = -(n-1)g_0$, then the evolution will be

$$g(\lambda) = (\lambda_0 + 2(n-1)\lambda)g_0, \tag{2.9}$$

and the manifold will expand as λ grows up. We will treat in detail this kind of metrics in the next chapter.

2.3.2 Ricci solitons

Ricci solitons are a class of special solutions to the Ricci flow and also are extensions of Einstein Manifolds. It is possible to obtain the Ricci Soliton equation considering the Ricci flow modified by a family of diffeomorphisms ψ_λ and defining

$$g(\lambda) = \sigma(\lambda)\psi_\lambda^*(g(0)), \tag{2.10}$$

with $\sigma(\lambda)$ a smooth function of λ. Taking the derivative of (2.10) and evaluating at $\lambda = 0$ yields

$$\frac{\partial g(\lambda)}{\partial \lambda} = \frac{d\sigma(\lambda)}{d\lambda}\psi_\lambda^* g(0) + \sigma(\lambda)\frac{\partial(\psi_\lambda^* g(0))}{\partial \lambda} \tag{2.11}$$

$$R_{ij}(g(0)) = \sigma'(0)g(0) + \mathcal{L}_V g(0) \tag{2.12}$$

where $V = \frac{d\psi_\lambda}{d\lambda}$. Setting $\sigma'(0) = 2\lambda$ this can be written as

$$-2R_{ij} + \alpha g_{ij} + \nabla_i V_j + \nabla_j V_i = 0, \tag{2.13}$$

Equation (2.13) is the equation of the Ricci soliton. A special case arrives when V is the gradient of a scalar function i.e. $V_i = \nabla_i f$. Then the equation becomes

$$R_{ij} + \alpha g_{ij} + \nabla_i \nabla_j f = 0. \tag{2.14}$$

Such flow is called a steady, expanding or shrinking gradient Ricci Soliton depending on whether $\alpha = 0$, $\alpha < 0$ or $\alpha > 0$ respectively. There are some explicit examples of Ricci solitons which are of great interest in physics and mathematics. One of them is the "Witten black hole" (cigar soliton in math literature) which is an steady gradient Ricci soliton and opens at infinity like a cylinder. Also there is a similar rotationally symmetric steady gradient soliton called the "Bryant soliton" which instead of opening like a cylinder at infinity opens asymptotically like a paraboloid [7].

2.4 Some applications of the Ricci flow in physics

The use of geometrical tools in Physics has played a key role to have a better understanding of the intrinsic properties of space-time. In recent years Ricci flow has started to gain attemption in Physics, in part thanks to the work of G. Perelman for solving the Thurston's geometrization conjecture. Ricci flow appears naturally in some contexts in physics, for example in string theory these flows are the world-sheet renormalization group trajectories. In the case of general relativity Ricci flow has been used to model the evolution of wormholes and black holes. The advantage of Ricci flow is that the flow parameter is not a coordinate and so could be identified with other physical quantities as energy or entropy. For example, in the paper [12] the Ricci flow was applied to general relativity. At high temperature they have shown that there are three saddle points derived from an action that are: a hot flat space and a large and small black hole. The flow was simulated numerically and shows that the small black hole has a GrossPerryYaffe-type negative mode and is unstable under the Ricci flow. The two flows seeded by this mode, lead to a large black hole and to hot flat space respectively, in the latter case via a topology-changing singularity. Ricci flow thus defines a unique path in the space of metrics connecting the small black hole to each of the other two saddle points.

Another interesting case of the use of the Ricci flow in physics is given in [13]. In this paper it is shown that the existence of solutions to the Ricci soliton equation (2.14) with a negative α on an m-dimensional Riemannian manifold with a boundary governed by a simple maximum principle. They show that in particular, if the vector field (De Turck vector) vanishes on all boundaries, this rules out the existence of non-trivial Ricci solitons. Then static solutions to the Einstein-DeTurck equation are considered by Euclidean continuation to a Riemannian problem which is elliptic, as well as how to implement boundary conditions for the Einstein-DeTurck equation in a variety of cases: boundaries with modified Dirichlet data, mixed Neumann-Dirichlet data (extrinsic curvature proportional to the induced metric), and ends where the analytically continued Lorentzian metric is asymptotically flat, Kaluza-Klein, locally AdS or has an extremal horizon. The main result is that, using the maximum principle for solutions with these boundary types, the existence of non-trivial Ricci solitons is actually ruled out. In other words, if one can solve the Einstein-DeTurck equation, for example, using the Ricci-DeTurck flow, one is guaranteed that the solution will solve the vacuum Einstein equations (with a non-positive cosmological constant).

2.4.1 Evolution of Morris-Thorne geometries under the Ricci flow

There are many other interesting examples of how Ricci flow can be applied to physics. Here we present a brief resume of the paper [1], where the Ricci flow was simulated numerically using the modified Frankel Duffort method to analize the behaviour of the throat of a family of wormhole geometries given by (2.23). The throat can expand or shrink depending on the value of the parameter α, under the critical value the throat shrinks in finite "time". As a result the two asymptotically flat spaces that were connected by the throat will evolve to two separated flat spaces. Above the critical value of the parameter the throat expands indefinitely, thus the topology of the manifold seems to be that of a cylinder with a monotonically increasing radius. The beahviour of the geometry under the flow is schematically presented in Figure (2.1).

Let the modified Ricci flow

$$\partial_\lambda g_{ab} = -2R_{ab} + 2\nabla_a V_b, \tag{2.15}$$

where $g_{ab} = g_{ab}(\lambda, x^c)$ identifies the metric of a 3-manifold with Euclidean signature, λ is the flow parameter and V_b is the DeTurck vectorial field which generates diffeomorphisms along the flow. For a spherically symmetric 3-manifold it is assumed the ansatz

$$ds^2 = e^{2X(\lambda,\rho)}\left[d\rho^2 + R^2(\lambda,\rho)d\Omega\right] \tag{2.16}$$

for the metric and $V_b = V(\lambda,\rho)\partial_\rho$ for the De Turck vectorial field. Calculating R_{ab} and substituting it into (2.15) one can obtain the flow equations

$$\dot{X}e^{2X} = \left[2\frac{(R'X' - R'' + RX'')}{R}\right] + V' - X'V, \tag{2.17}$$

$$\begin{aligned}\dot{X}e^{2X}R^2 + e^{2X}R\dot{R} = &-\left[1 - R'^2 - 3RR'X' - RR'' - R^2X'^2 - R^2X''\right]\\ &+ RR'V + R^2X'V.\end{aligned} \tag{2.18}$$

The equation for the $\phi\phi$ component of the flow is the same as (2.18) multiplied by $sen^2\theta$, and there is no dynamical equation for $V(\lambda,\rho)$. A wormhole geometry is defined by a metric (2.16) with two asymptotically flat regions as $\rho \pm \infty$. The combination $e^{X(\lambda,\rho)}R(\lambda,\rho)$ should be non- zero for all ρ, and the wormhole throat is located at the local minima $\rho = \rho_{th}$. For simplicity one can choose $X = 0$ and impose

$$\partial_\rho R(\lambda,\rho)|_{\rho=\rho_{th}} = 0 \quad \text{and} \quad \partial_\rho^2 R(\lambda,\rho)|_{\rho=\rho_{th}} > 0, \tag{2.19}$$

thus the line element is

$$ds^2 = d\rho^2 + R^2 d\Omega^2, \tag{2.20}$$

and the flow equations become

$$\begin{aligned}\dot{R} &= R'' + \frac{R'^2}{R} - \frac{1}{R} + VR',\\ V' &= -2\frac{R''}{R}.\end{aligned} \tag{2.21}$$

	Modified Dufort Frankel (MDF)
$\partial_\rho A$	$(A^j_{i+1} - A^j_{i-1})/2(\delta\rho)$
$\partial^2_\rho R$	$(R^j_{i+1} - R^{j+1}_i - R^{j-1}_i + R^j_{i-1})/\delta\rho^2$
$\partial_\lambda R$	$(R^{j+1}_i - R^{j-1}_i)/2\delta\lambda$
Accuracy	$\mathcal{O}(\delta\lambda^2, \delta\rho^2)$

Table 2.1: Summary of the modified DufortFrankel (MDF) finite-difference derivative approximations. The notationA^j_i refers to the value of the quantity $A(t, \rho)$ at the i_{th} spatial and j_{th} temporal nodes.

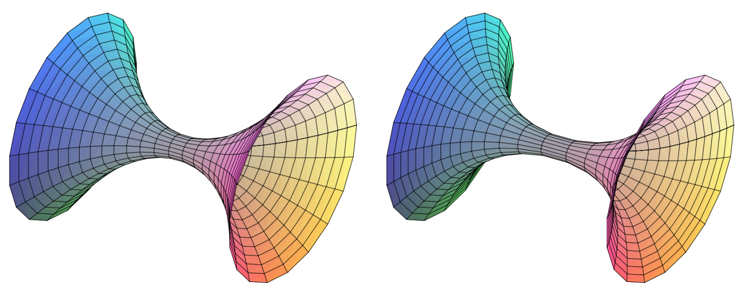

Figure 2.1: In this figure are shown the embedding diagrams of the Einstein-Rosen and Ellis wormholes in flat space. This embeddings can be obtained by equating the metrics: $\left[1 + \left(\frac{dZ}{dr}\right)^2\right] dr^2 + r^2 d\Omega = \frac{1}{1-\left(\frac{r_0}{r}\right)^\alpha} dr^2 + r^2 d\Omega^2$; therefore by integrating the equation for $Z(r)$ one can obtain the embeded surface [1]. (To see a detailed explanation of this embbeding have a look of [19]. This figure has been taken from [1].)

These equations are solved using the modified Frankel-Duffort method which is presented schematically in Table (2.4.1). It can be shown that the MDF approximation actually converges to the solution

$$\dot{R} = R'' + \frac{R'^2}{R} - \frac{1}{R} + VR' - \left(\frac{\delta\lambda}{\delta\rho}\right)^2 \ddot{R}; \tag{2.22}$$

then the MDF method solves a "slightly" different version of the original system of PDEs. Therefore in order to minimize the difference, the ratio $\epsilon = \delta\lambda/\delta\rho$ must be as small as possible.

The 3-metric given as initial data for the Ricci flow has spatial section

$$ds^2 = \frac{1}{1 - \frac{1}{b(r)}} dr^2 + r^2 d\Omega^2, \tag{2.23}$$

at $\lambda = 0$, this 3-metric can be transformed into the areal radius gauge (2.20) by the coordinate transformation $r = R(0,\rho)$ with $R'(0,\rho) = f(R(0,\rho))$, $f = 1 - \frac{1}{b(r)}$, where $R(0,\rho)$ is the initial profile for the areal radius metric function. The $f(r)$ functions considered are

$$f(r) = 1 - \left(\frac{r_0}{r}\right)^\alpha, \tag{2.24}$$

where r_0 is a length scale and α is a dimensionless parameter. The spatial section of the metric known as the Einstein-Rosen bridge [20] corresponds to $\alpha = 1$. Another special case is the Ellis wormhole [21] with $\alpha = 2$. In figure 2.2 are shown the results of the simulations for the evolution of the areal radius function $R(t,\rho)$. For the Ellis case ($\alpha = 2$) the areal radius diverges for all ρ as $t \to \infty$, while for the Einstein-Rosen bridge $\alpha = 1$, R decreases and the wormhole throat pinches off at finite "time". In general it appears that for values of α above the critical value the topology of the manifold tends to a cylinder with a monotonically increasing radius. On the other hand for α under the critical value the throat pinches off at finite "time" and the two asymtotically regions evolve separately to flat spaces. The critical value of alpha is determined numerically and is approximately $\alpha \approx 1.259$ [1].

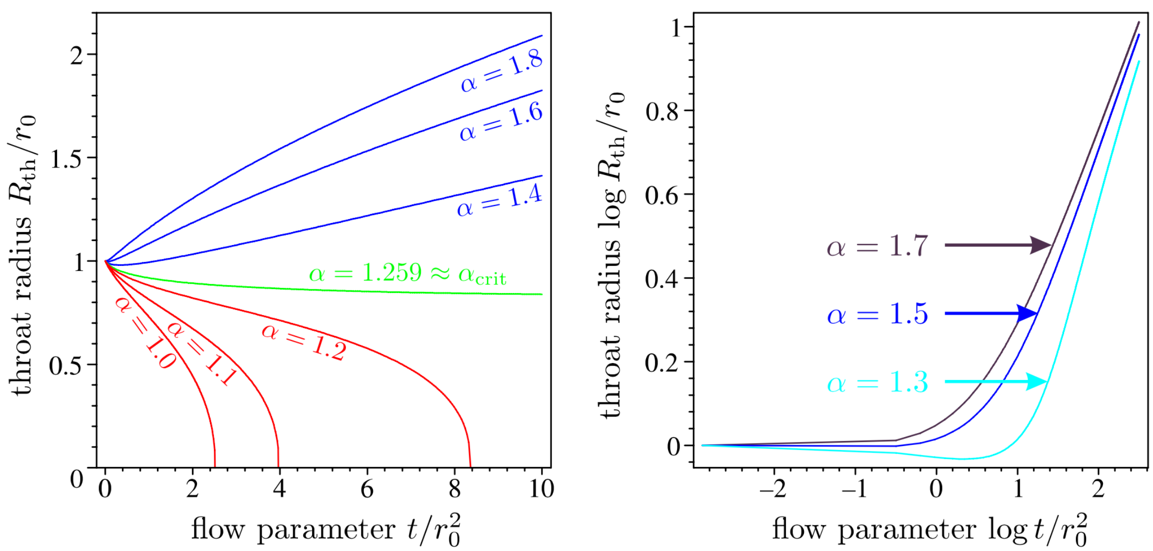

Figure 2.2: In the left panel it can be seen a critical behaviour for the troath of the wormhole. There is a critical $\alpha \approx 1.259$ which separates scenarios with shrinking or expanding troath. In the right image there is a plot of the troath radius for various α above the critical value, all of them expand as the parameter "t" (the flow parameter) grows up (This figure has been taken from [1]).

Chapter 3

Maximally symmetric spaces and exact solutions to the Ricci flow

3.1 Maximally symmetric spaces

In this section we present the notion of a maximally symmetric space. This topic can be found in many text books, here we present a brief resume of the treatment given in [22] which is very intuitive.

The best example to start a treatment of a maximally symmetric spaces is of course $\mathbb{R}^n$. This space posses the highest number of symmetries that a space can have and that is the reason why they are called maximally symmetric spaces. Lets consider the isometries of this space, which are translations and rotations in n-dimensions. A translation is a function that moves an object a certain distance, the object should not be altered in any other way i.e. it is not rotated, reflected or re-sized. In a translation, every point of the object must be moved in the same direction and for the same distance. As there are n independent axes along which it can be moved, then there are n total translations.

On the other hand any rotation is a motion of a certain space that preserves at least one point. It can describe, for example, the motion of a rigid body around a fixed point. A rotation is different from other types of motions for example translations or reflections. A clockwise rotation is a negative magnitude and a counter-clockwise is a positive magnitude.

A rotation, centred at p, is a transformation that leave p invariant, in other words any rotation is a motion of a certain space that preserves at least one point. They can be thought of as moving one of the axes through p into one of the others. In an space of dimension n there are n axes, and for each axis there are $n-1$ other axes into which it can be rotated, a rotation of y into x should not be counted as separate from a rotation of x into y, so the total number of independent rotations is $\frac{1}{2}n(n-1)$. Thus we have

$$n + \frac{1}{2}n(n-1) = \frac{n}{2}(n+1), \tag{3.1}$$

independent symmetries of $\mathbb{R}^n$. This argument only refers to the behavior of the symmetry in a neighborhood of p, not in the whole manifold; so even in the presence of curvature the counting should be the same. If the metric signature is not Euclidean, some of the rotations will actually be boosts, but again the counting will be the same. The number of isometries is the number of linearly independent Killing vector fields, therefore we refer to an n-dimensional manifold with $\frac{1}{2}n(n+1)$ Killing vectors as a maximally symmetric space. Another example of a maximally symmetric space is a n-dimensional sphere. If a manifold is maximally symmetric, the curvature is the same everywhere in every direction. Hence, if we know the curvature of a maximally symmetric space at one point, we know it everywhere. There is a small number of possible maximally symmetric spaces; they are classified by the curvature scalar R, the dimensionality n, and the metric signature, and possible issues related to the global topology.

Since the geometry looks the same in all directions, the curvature tensor should look the same in all directions as well. Since the geometry is maximally symmetric, the same is required to be true for the Riemann tensor; that is, the components of $R_{\rho\sigma\mu\nu}$ should not change under a Lorentz transformation. But there are unique tensors that do not change their components under a Lorentz transformation, those are the metric, the Kronecker delta, and the Levi-Civita tensor. If we choose locally inertial coordinates at some point p, so that $g_{\mu\nu} = \eta_{\mu\nu}$, then in these coordinates and at this point, the components of $R_{\rho\sigma\mu\nu}$ will be proportional to a tensor constructed from these invariant tensors [22]. Thus there is a unique possibility:

$$R_{\rho\sigma\mu\nu} \propto g_{\rho\sigma}g_{\mu\nu} - g_{\rho\nu}g_{\sigma\mu}. \tag{3.2}$$

This is a tensorial relation, so it must be true in any coordinate system. As in a maximally symmetric space all points posses the same geometrical properties, it must be true at any other point as well. The proportionality constant can be fixed by contracting both sides twice, the left-hand side becomes R, and the right-hand side is $n(n-1)$. Then the equation for any maximally symmetric space, at any point, in any coordinate system is

$$R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}\left(g_{\rho\sigma}g_{\mu\nu} - g_{\rho\nu}g_{\sigma\mu}\right). \tag{3.3}$$

The basic classification of such spaces is simply whether R is positive, zero, or negative, since the magnitude of R represents an overall scaling of the size of the space. For Euclidean signatures, the flat maximally symmetric spaces are planes or appropriate higher-dimensional generalizations, while the positively curved ones are spheres. Maximally symmetric Euclidean spaces of negative curvature are hyperboloids, denoted by $\mathbb{H}^n$. These are less familiar because even a two-dimensional hyperboloid cannot be isometrically embedded in $\mathbb{R}^3$. Let's examine this two-dimensional hyperboloid briefly.

One simple way of representing $\mathbb{H}^2$ is the Poincaré half-plane, which is the $y > 0$ section of a two-dimensional region with coordinates (x, y) and metric $ds^2 = \frac{c^2}{y^2}\left(dx^2 + dy^2\right)$. The geometry of the Poincaré half-plane is of course different from that of the upper half of $\mathbb{R}^2$, despite the use of similar coordinates.

One can obtain representations of de Sitter and anti de Sitter space-times by embedding d dimensional hypersurfaces in a $d+1$ dimensional flat space-time. Then the de Sitter space-time of curvature scale a can be defined as the hyperboloid

$$-X_0^2 + \sum_{i=1}^{d} X_i^2 = a^2, \tag{3.4}$$

embedded in a $d+1$ dimensional Minkowski space-time. An AdS space-time is in turn obtained by considering the hyperboloid

$$X_0^2 + X_d^2 - \sum_{i=1}^{d-1} X_i^2 = l^2, \tag{3.5}$$

of radius of curvature $l > 0$ embedded in the $d+1$ dimensional space-time

$$ds^2 = -\left(dX_0^2 + dX_d^2\right) + \sum_{i=1}^{d-1} dX_i^2. \tag{3.6}$$

Let us now construct line elements for the AdS space in d dimensions. A global parametrisation is constructed as follows. Given the form of (3.5) we consider two spheres

$$X_0^2 + X_d^2 = r_1^2, \qquad \sum_{i=1}^{d-1} X_i^2 = r_2^2, \tag{3.7}$$

such that

$$r_1^2 - r_2^2 = l^2. \tag{3.8}$$

This equation is solved setting $r_1 = l\cosh\left(\frac{u}{l}\right), r_2 = l\sinh\left(\frac{u}{l}\right)$ where $u \in [0, \infty)$. Using this and the parametrisation in polar spherical coordinates, the embedding in flat space is thus defined in coordinates, $(t, u, \theta_1 ... \theta_{d-2})$ as,

$$\begin{aligned} X_0 &= l\text{cosh}\left(\frac{u}{l}\right)\text{sin}(t) \\ X_1 &= l\text{sinh}\left(\frac{u}{l}\right)\text{cos}\theta_1 \\ &\vdots \\ X_{d-1} &= l\text{sinh}\left(\frac{u}{l}\right)\text{sin}\theta_1 ... \text{cos}\theta_{d-2} \\ X_d &= l\text{cosh}\left(\frac{u}{l}\right)\text{cos}(t) \end{aligned} \tag{3.9}$$

then by replacing these expressions into the line element (3.6) we obtain

$$ds^2 = -l^2\text{cosh}^2\left(\frac{u}{l}\right)dt^2 + du^2 + l^2\text{sinh}^2\left(\frac{u}{l}\right)\Omega^2_{d-2}. \tag{3.10}$$

The boundary lies at $u \to \infty$. This is the global parametrisation of AdS since all points of the hyperboloid are taken into account exactly once. This metric is a solution to the Einstein equations with cosmological constant,

$$2\Lambda = -\frac{(d-1)(d-2)}{l^2}. \tag{3.11}$$

Note that the timelike coordinate is an angular coordinate $t \in [-\pi, \pi]$. This means that AdS is a spacetime with closed timelike curves. We can get around this since the space is not simply connected (i.e. the time circle cannot be topologically reduced to a point), we can unwrap the circle of the time coordinate and take a new coordinate $t \in (-\infty, \infty)$ with t in each 2π interval. This means that we are effectively taking infinite copies of the hyperboloid. Equation (3.10) is the universal covering of AdS space [23].

3.2 Exact solutions to the Ricci flow

In this section we are going to construct explicit solutions to the Ricci flow in $3d$ and $4d$. We start with a general ansatz for $3d$ metric with Euclidean signature. Once a solution is found it is verified that it correspnds to a maximally symmetric space with positive or negative curvature depending on our election for the λ parameter. The fixed points of the flow are flat spaces and there is a singular point that divides spaces with positive scalar curvature from those with negative scalar curvature.

In the next step we make an extension to $4d$, first noting that our solution in $3d$ has the same form of the spatial part of an AdS space in $4d$ written in global coordinates. Then we perform a more detailed deduction of the solution starting from a general anstaz for a metric in $4d$. We found that the flow can be carried on even for metrics with Lorentzian signature. In both $3d$ and $4d$ cases, there is a “phase transition” between spaces with positive scalar curvature and those with negative scalar curvature.

3.2.1 Solution in 3d

Now we are going to solve the Ricci-DeTurck flow in $3d$ starting with the ansatz

$$ds^2 = d\rho^2 + P(\rho, \lambda) d\Omega^2, \tag{3.12}$$

note that this could be also an ansatz for a wormhole geometry if we require that

$$P(\lambda, \rho) > 0, \tag{3.13}$$

and

$$\partial_\rho P(\lambda, \rho)|_{\rho=\rho_{th}} = 0 \quad \text{and} \quad \partial_\rho^2 P(\lambda, \rho)|_{\rho=\rho_{th}} > 0. \tag{3.14}$$

The non-trivial Christoffel symbols and components of the Ricci tensor are

$$\begin{aligned} \Gamma^\theta_{\rho\theta} &= \tfrac{P'}{2P} \\ \Gamma^\rho_{\theta\theta} &= -\tfrac{P'}{2} \\ \Gamma^\phi_{\rho\phi} &= \tfrac{P'}{2P} \\ \Gamma^\phi_{\theta\phi} &= \cot(\theta) \\ \Gamma^r_{\phi\phi} &= -\tfrac{1}{2}\sin^2(\theta) P' \\ \Gamma^\theta_{\phi\phi}\theta &= -\cos(\theta)\sin(\theta) \end{aligned} \tag{3.15}$$

$$\begin{aligned} R_{\rho\rho} &= \tfrac{P'^2 - 2PP''}{2P^2} \\ R_{\theta\theta} &= 1 - \tfrac{P''}{2} \\ R_{\phi\phi} &= -\tfrac{1}{2}\sin^2(\theta)\left(P'' - 2\right) \end{aligned} \tag{3.16}$$

using this, the flow equations can be written as

$$\dot{P} = P'' + VP' - 2, \tag{3.17}$$

$$V' = -\frac{P''}{P} + \frac{P'^2}{2P^2}. \tag{3.18}$$

So we have a dynamical equation and a constriction equation for P. To solve this system we are going to use the ansatz

$$V = w_1^2(\lambda)\rho. \tag{3.19}$$

By substituting it into equation (3.18) and solving for P we obtain

$$P = w_3(\lambda)\cos^2\left[\frac{w_1(\lambda)}{\sqrt{2}}\left(\rho - 2w_2(\lambda)\right)\right]. \tag{3.20}$$

Here $w_1(\lambda), w_2(\lambda), w_3(\lambda)$ are functions of λ that we are going to determine using equation (3.17). By substituting this solution into this equation and making use of some trigonometric identities we obtain

$$\begin{aligned} -\left(\dot{w_3}+4\right) = &\left(\dot{w_3}+2w_1^2w_3\right)\cos\left[\sqrt{2}w_1(\rho-2w_2)\right] - \\ &\sqrt{2}w_3\sin\left[\sqrt{2}w_1(\rho-2w_2)\right]\left[(\dot{w_1}-w_1^3)\rho - 2(w_1\dot{w_2})\right]; \end{aligned} \tag{3.21}$$

to satisfy this equation we require

$$\dot{w_3}+4 = 0, \tag{3.22}$$

$$\dot{w_3}+2w_1^2w_3 = 0, \tag{3.23}$$

$$\dot{w_1}-w_1^3 = 0, \tag{3.24}$$

$$w_1\dot{w_2} = 0, \tag{3.25}$$

by solving this system we obtain

$$w_1(\lambda) = \frac{1}{\sqrt{2(\lambda_0-\lambda)}}, \qquad w_2 = k\sqrt{2(\lambda_0-\lambda)}, \qquad w_3 = 4(\lambda_0-\lambda), \tag{3.26}$$

with k and τ_0 arbitrary constants. Thus the solution can be written as

$$P(\lambda,\rho) = 4(\lambda_0-\lambda)\cos^2\left[\frac{\rho}{2\sqrt{\lambda_0-\lambda}} + k_0\right], \tag{3.27}$$

with $k_0 = \sqrt{2}k$ an arbitrary constant. It is interesting to note that this is an Einstein metric, which is a solution to the Einstein equations with a cosmological constant and are given by the condition

$$R_{ij}(g_0) = \sigma g_0. \tag{3.28}$$

It is known that if a metric g is such that $g(0) = g_0$ and satisfies (3.28), then $g_{ij}(\lambda) = (1-2\sigma\lambda)g_0$ is a solution of (2.1). We can see that our metric is an Einstein metric noting that

$$R_{ij} = \begin{pmatrix} \frac{1}{2(\lambda_0-\lambda)} & 0 & 0 \\ 0 & 2\cos^2\left[\frac{\rho}{2\sqrt{\lambda_0-\lambda}}+k_0\right] & 0 \\ 0 & 0 & 2\cos^2\left[\frac{\rho}{2\sqrt{\lambda_0-\lambda}}+k_0\right]\sin^2[\theta] \end{pmatrix}, \quad \sigma = \frac{1}{2\lambda_0} \tag{3.29}$$

and

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4(\lambda_0 - \lambda)\cos^2\left[\frac{\rho}{2\sqrt{\lambda_0-\lambda}} + k_0\right] & 0 \\ 0 & 0 & 4(\lambda_0 - \lambda)\cos^2\left[\frac{\rho}{2\sqrt{\lambda_0-\lambda}} + k_0\right]\sin^2[\theta] \end{pmatrix}. \quad (3.30)$$

If we want to know more about the geometrical properties of this metric, it will be useful to have a look on the curvature. The Ricci curvature is given by

$$R(\lambda) = \frac{3}{2(\lambda_0 - \lambda)}, \quad (3.31)$$

Thus we can see that the Ricci curvature tends to cero for big values of the λ parameter. Also the curvature blows up when $\lambda = \lambda_0$. Now lets investigate if there are fixed points for this flow. The fixed points are given by the condition

$$\frac{\partial g(\lambda)}{\partial \lambda} = 0, \quad (3.32)$$

this is a system of equations that must be satisfied simultaneously for all ρ. In this case, equations (3.32) are reduced to

$$4\cos^2\left(\frac{\rho}{2\sqrt{\lambda_0 - \lambda}} + k_0\right) + \frac{\rho}{\sqrt{\lambda_0 - \lambda}}\sin\left(\frac{\rho}{\sqrt{\lambda_0 - \lambda}} + 2k_0\right) = 0. \quad (3.33)$$

To satisfy this equation for every ρ we require $\lambda \to \pm\infty$, the second term vanish due to the coefficient and we obtain $4\cos^2(k_0)$ for the first term. Therefore if we take $k_0 = \frac{n\pi}{2}$, the first term vanish and the equation is satisfied for all ρ. Making this election for k_0 turns our solution into

$$P = 4(\lambda_0 - \lambda)\sin^2\left[\frac{\rho}{2\sqrt{\lambda_0 - \lambda}}\right], \quad (3.34)$$

then the line element can be written as

$$ds^2 = d\rho^2 + 4(\lambda_0 - \lambda)\sin^2\left[\frac{\rho}{2\sqrt{\lambda_0 - \lambda}}\right]d\Omega^2. \quad (3.35)$$

Note that if we multiply the second term of (3.35) by ρ^2/ρ^2 we can make use of

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1, \quad (3.36)$$

when $\lambda \to \pm\infty$ the argument of $\sin(\cdot)$ tends to zero and the metric is

$$ds^2 = d\rho^2 + \rho^2 d\Omega^2; \quad (3.37)$$

thus the fixed points are flat metrics.

It is also interesting to note that our solution is a maximally symmetric space i.e. it satisfies

$$R_{abcd} = \frac{R}{n(n-1)} \left(g_{ac} g_{bd} - g_{ad} g_{bc} \right) \tag{3.38}$$

here $n = 3$ is the dimension of the space. Using (3.31), (3.35) and the fact that the only non zero components of the Riemann tensor are

$$\begin{aligned} R_{r\theta r\theta} &= \sin^2 \left[\frac{\rho}{2\sqrt{\lambda_0 - \lambda}} \right] \\ R_{r\phi r\phi} &= \sin^2 \left[\frac{\rho}{2\sqrt{\lambda_0 - \lambda}} \right] \sin^2 \theta \\ R_{\theta\phi\theta\phi} &= 4(\lambda_0 - \lambda) \sin^4 \left[\frac{\rho}{2\sqrt{\lambda_0 - \lambda}} \right] \sin^2 \theta \end{aligned} \tag{3.39}$$

one can verify by inspection that equation (3.38) is satisfied. Then equation (3.38) tell us that we have a de Sitter or anti de Sitter space for each λ depending whether $R > 0$ or $R < 0$. So far we have considered that λ is in the interval $(-\infty, \lambda_0)$, but bhat happens if we consider $\lambda \in (\lambda_0, \infty)$? In this case the argument of the function $\sin(\cdot)$ in equation (3.34) becomes imaginary, then we can use the identity

$$sin(\mathtt{i}z) = -\frac{\sinh(z)}{\mathtt{i}} \tag{3.40}$$

$$\Rightarrow sin^2(\mathtt{i}z) = -\sinh^2(z), \tag{3.41}$$

thus the line element can be written as

$$ds^2 = d\rho^2 + 4(\lambda - \lambda_0) \sinh^2 \left[\frac{\rho}{2\sqrt{\lambda - \lambda_0}} \right] d\Omega^2. \tag{3.42}$$

Then our solution to the Ricci flow equation take us from a maximally symmetric space with positive "constant" curvature (here we use the word constant in the sense that for each λ R is constant) to another maximally symmetric space with negative constant curvature. i.e. our solution take us from a de Sitter space to an Anti-de Sitter space. The flow will face a singularity when passing from positive to negative curvature, this occurs when $\lambda = \lambda_0$.

The curvature can be positive or negative depending on our election of λ and λ_0. The curvature is positive when λ is in the range $(-\infty, \lambda_0)$ and negative in the range (λ_0, ∞). At the critical point $\lambda = \lambda_0$, the curvature blows up and the Riemann components are not well defined. It seems that the space faces a "phase transition" when the geometry changes from positive constant curvature to negative constant curvature.

In order to have a feeling of how these spaces are deformed we are going to make an embedding in an Euclidean and Minkowski space for the de Sitter and Anti-de Sitter spaces respectively. Let's start with a coordinate transformation for the de Sitter space

$$r = 2\sqrt{\lambda_0 - \lambda}\sin\left(\frac{\rho}{2\sqrt{\lambda_0 - \lambda}}\right), \tag{3.43}$$

by differentiating this relation we can arrive to

$$\begin{aligned} d\rho^2 &= \frac{dr^2}{\cos^2\left[\arcsin\left(\frac{r}{2\sqrt{\lambda_0-\lambda}}\right)\right]} \\ &= \frac{dr^2}{1 - \frac{r^2}{4(\lambda_0-\lambda)}}, \end{aligned} \tag{3.44}$$

substituting it into the line element (3.35) we obtain

$$ds^2 = \frac{dr^2}{1 - \frac{r^2}{4(\lambda_0-\lambda)}} + r^2 d\Omega^2. \tag{3.45}$$

Now we are going to embed this space in an slice of a 4 dimensional Euclidean space, let

$$ds^2 = dZ^2 + dr^2 + r^2 d\Omega^2, \tag{3.46}$$

and take $\theta = \pi/2$, thus

$$ds^2 = dZ^2 + dr^2 + r^2 d\phi^2, \tag{3.47}$$

we can rewrite this as

$$ds^2 = \left[1 + \left(\frac{dZ}{dr}\right)^2\right] dr^2 + r^2 d\phi^2; \tag{3.48}$$

now we are going to identify this metric with the slice of (3.45) at $\theta = \pi/2$, equating the coefficients of dr^2 we have

$$\left[1 + \left(\frac{dZ}{dr}\right)^2\right] = \frac{1}{1 - \frac{r^2}{4(\lambda_0-\lambda)}}, \tag{3.49}$$

solving for Z we obtain

$$Z = \pm\sqrt{4(\lambda_0 - \lambda) - r^2}, \tag{3.50}$$

this is the equation of an sphere, remember that

$$Z^2 + r^2 = l^2, \tag{3.51}$$

where $l^2 = 4(\lambda_0 - \lambda)$ is the radius of the sphere. Note that as λ approaches λ_0 this radius becomes smaller until $\lambda = \lambda_0$ where the sphere collapses. In figure 3.2 it is shown the profile of the sphere and its evolution under the flow.

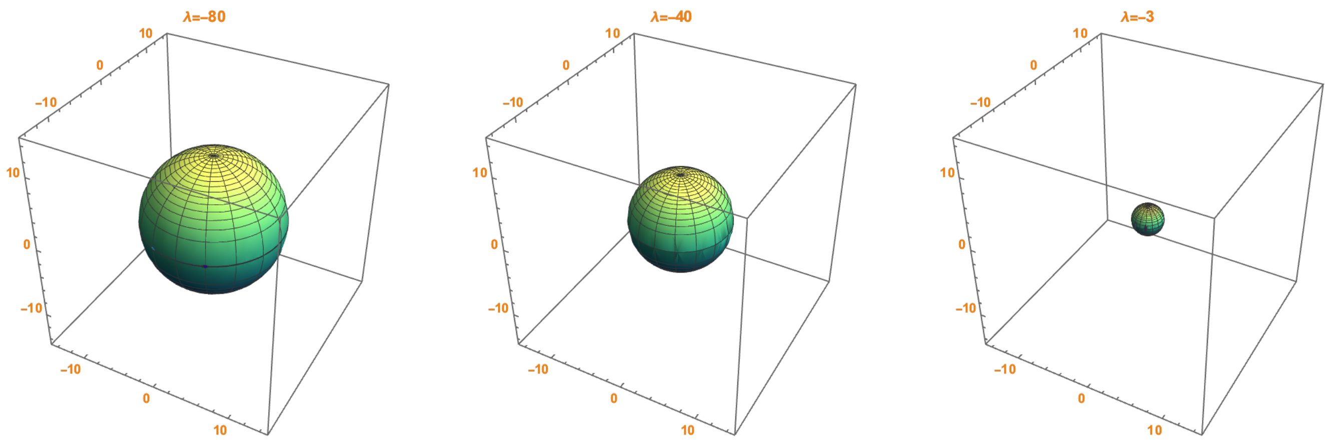

Figure 3.1: In this figure it is shown the De Sitter space embedded in a $4d$ Euclidean space for $\lambda = -80, \lambda = -40, \lambda = -3$.

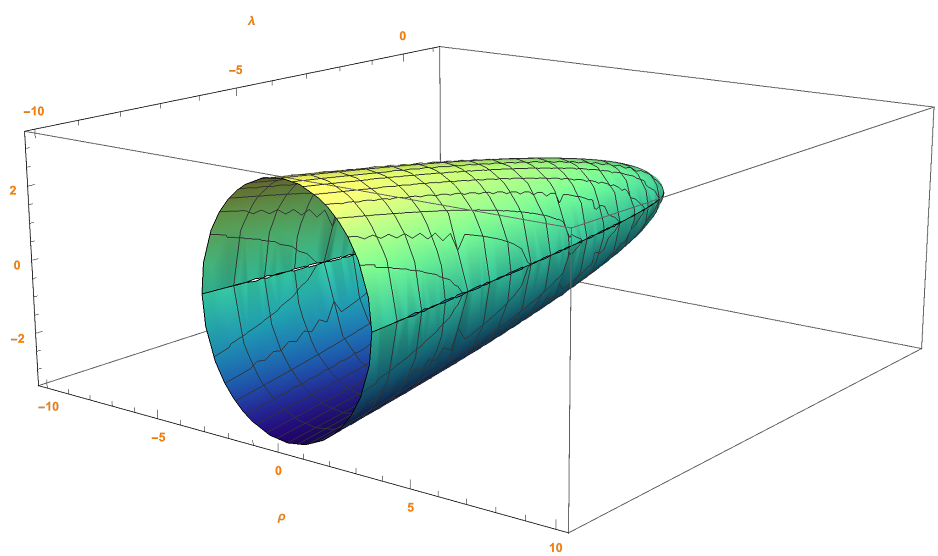

Figure 3.2: In this figure it is shown the profile of the sphere evolving under the flow. At infinity this will look like a flat space, then the curvature increases as λ approaches λ_0, at the same time the sphere is contracting until it collapses and the curvature blows up.

To visualize the evolution of the AdS under the flow we proceed in a similar way but this time we have $\sinh(\cdot)$ instead of $\sin(\cdot)$, and we are going to embed it in a $4d$ Minkowski space. Let's start with the coordinate transformation

$$ds^2 = -dW^2 + dr^2 + r^2 d\Omega^2, \tag{3.52}$$

$$r = 2\sqrt{\lambda_0 - \lambda}\sinh\left(\frac{\rho}{2\sqrt{\lambda - \lambda_0}}\right); \tag{3.53}$$

by differentiating this relation we can we arrive to

$$\begin{aligned} d\rho^2 &= \frac{dr^2}{\cosh^2\left[\text{arcsinh}\left(\frac{r}{2\sqrt{\lambda-\lambda_0}}\right)\right]} \\ &= \frac{dr^2}{1 + \frac{r^2}{4(\lambda-\lambda_0)}} \end{aligned} \tag{3.54}$$

by substituting this expression into the line element (3.35) we obtain

$$ds^2 = \frac{dr^2}{1 + \frac{r^2}{4(\lambda-\lambda_0)}} + r^2 d\Omega^2 \tag{3.55}$$

now we are going to rewrite (3.52) as

$$ds^2 = \left[1 - \left(\frac{dW}{dr}\right)^2\right] dr^2 + d\Omega^2 \tag{3.56}$$

equating the coefficients of dr^2 we have

$$\left[1 - \left(\frac{dW}{dr}\right)^2\right] = \frac{1}{1 - \frac{r^2}{4(\lambda-\lambda_0)}} \tag{3.57}$$

and solving for W we obtain

$$W = \pm\sqrt{r^2 - 4(\lambda - \lambda_0)} \tag{3.58}$$

and this is the equation for an hyperboloid of one sheet

$$r^2 - W^2 = l^2 \tag{3.59}$$

where $l^2 = 4(\lambda - \lambda_0)$ is the radius of curvature. Note that the minimum value of λ is λ_0, therefore this hyperboloid expands as λ increases. In figure 3.3 it is shown the hyperboloid for some fixed values of λ. In figure 3.4 it is shown the profile of the hyperboloid and its evolution under the flow.

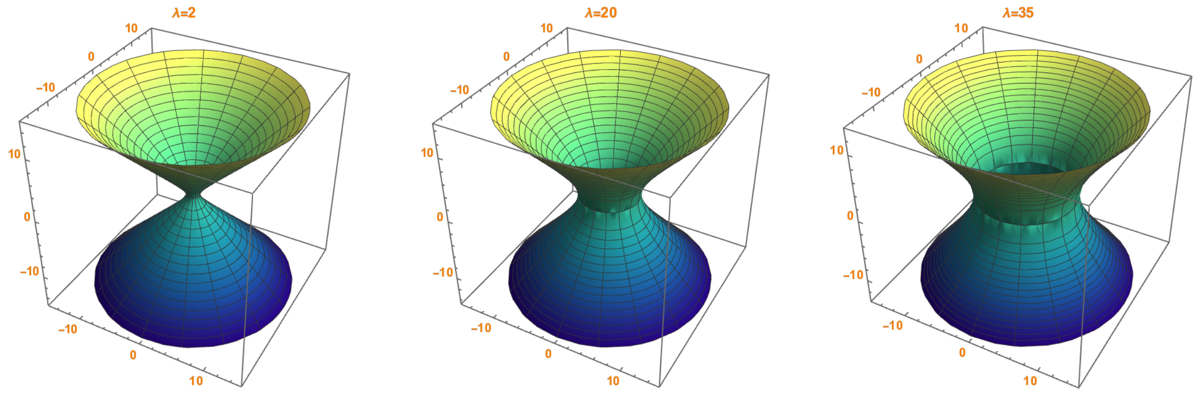

Figure 3.3: In this figure it is shown the AdS space embedded in a $4d$ Euclidean space for $\lambda_0 = 1, \lambda = 20$.

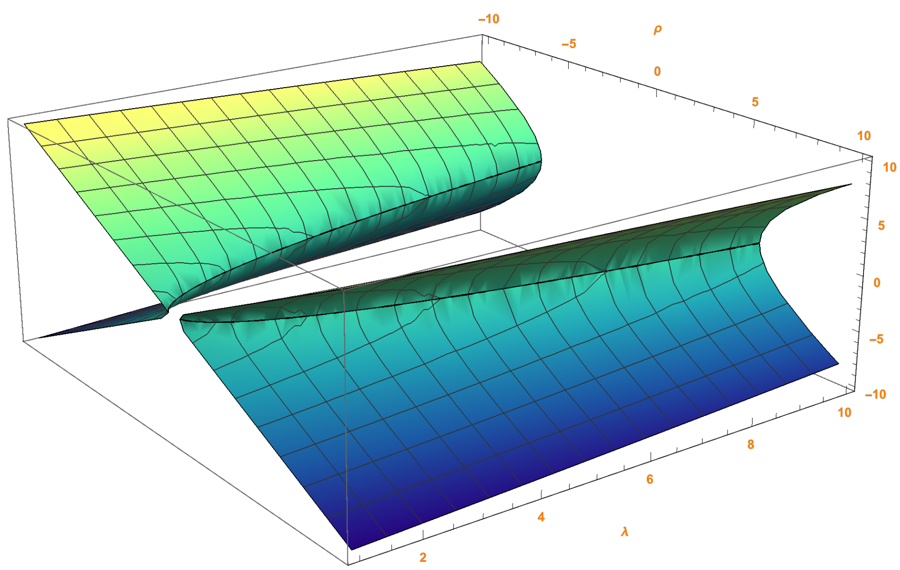

Figure 3.4: In this figure it is shown the profile of the hyperboloid evolving under the flow. At $\lambda = \lambda_0$ the curvature is infinite, then as λ increases the curvature decrease. The space expands indefinitely and locally will looks as a flat space

3.2.2 Solution in 4d

To this point we know the key features of the geometrical properties of our solution. It will be of interest to ask what happens if we want to extend the solution to$4d$ metric with a temporal component. Let us remember that a universal covering of AdS space is given by [23]

$$ds^2 = -l^2 \cosh^2\left(\frac{u}{l}\right) dt^2 + du^2 + l^2 \sinh\left(\frac{u}{l}\right) d\Omega_{d-2}, \tag{3.60}$$

where d is the space-time dimension and l is a parameter. Our solution in $3d$ has exactly the same form as the spacial part of (3.60) if we identify u with ρ and l with $4(\lambda_0 - \lambda)$. Then a natural question is to ask if equation (3.60) satisfies the Ricci flow equation. Surprisingly, the answer is yes. A heuristically way to see that this is true, is to substitute directly into the flow equations (2.15) and fix the coefficient that multiplies $(\lambda_0 - \lambda)$. Making this arrangements we arrive to

$$ds^2 = -6(\lambda_0 - \lambda)\cos^2\left[\frac{\rho}{\sqrt{6(\lambda_0 - \lambda)}}\right] dt^2 + d\rho^2 + 6(\lambda_0 - \lambda)\sin^2\left[\frac{\rho}{\sqrt{6(\lambda_0 - \lambda)}}\right] d\Omega^2, \tag{3.61}$$

for $\lambda \in (-\infty, \lambda_0)$ and

$$ds^2 = 6(\lambda - \lambda_0)\cosh^2\left[\frac{\rho}{\sqrt{6(\lambda - \lambda_0)}}\right] dt^2 + d\rho^2 + 6(\lambda - \lambda_0)\sin^2\left[\frac{\rho}{\sqrt{6(\lambda - \lambda_0)}}\right] d\Omega^2, \tag{3.62}$$

for $\lambda \in (\lambda_0, \infty)$, with the curvature and the DeTurck vector given by

$$R(\lambda) = \frac{2}{\lambda_0 - \lambda}, \qquad V(\rho, \lambda) = \frac{\rho}{2(\lambda_0 - \lambda)}. \tag{3.63}$$

Both spaces are maximally symmetric, again the curvature is positive for $\lambda \in (-\infty, \lambda_0)$ and negative for $\lambda \in (\lambda_0, \infty)$. It is interesting to note that (3.61) is a Lorentzian metric while (3.62) is Euclidean, this can be interpreted as a consequence of Ricci flow. Thus the behaviour of our space under the Ricci flow is to go from a de Sitter space with Lorentzian signature to an Anti-de Sitter space with Euclidean signature. As one can expect the transition is not smooth and the curvature faces a singularity at $\lambda = \lambda_0$.

One can perform a more carefully study of the extension to $4d$ of this solution by finding a solution from a generic metric in $4d$. Let us start with the ansatz

$$ds^2 = Q(\lambda, \rho)dt^2 + d\rho^2 + P(\lambda, \rho^2)d\Omega^2, \tag{3.64}$$

and

$$V(\lambda, \rho) = w_1^2(\lambda)\rho + u(\lambda), \tag{3.65}$$

Note that we have added a function $u(\lambda)$ in order to have the more general ansatz for the DeTurck field in the linear case. There is an infinity of possibilities for choosing this vector field, always that the flow equations are satisfied. For example we could choose a quadratic function for this vector field and tried to solve the flow equations.

The non-trivial Christoffel symbols and components of the Ricci tensor are

$$\begin{array}{rcl} \Gamma^{\rho}_{tt} &=& -\frac{Q'}{2} \\ \Gamma^{t}_{t\rho} &=& \frac{Q'}{2Q} \\ \Gamma^{\theta}_{\rho\theta} &=& \frac{P'}{2P} \\ \Gamma^{\rho}_{\theta\theta} &=& -\frac{P'}{2} \\ \Gamma^{\phi}_{\rho\phi} &=& \frac{P'}{2P} \\ \Gamma^{\phi}_{\theta\phi} &=& \cot(\theta) \\ \Gamma^{r}_{\phi\phi} &=& -\frac{1}{2}\sin^2(\theta)P' \\ \Gamma^{\theta}_{\phi\phi} &=& -\cos(\theta)\sin(\theta) \end{array} \tag{3.66}$$

and

$$\begin{array}{rcl} R_{tt} &=& \frac{Q'^2}{4Q} - \frac{P'Q'}{2P} - \frac{Q''}{2} \\ R_{\rho\rho} &=& \frac{P'^2}{2P^2} + \frac{Q'^2-2QQ''}{4Q^2} - \frac{P''}{P} \\ R_{\theta\theta} &=& -\frac{P'Q'}{4Q} - \frac{P''}{2} + 1 \\ R_{\phi\phi} &=& -\frac{\sin^2(\theta)(P'Q'+2Q(P''-2))}{4Q}, \end{array} \tag{3.67}$$

then the flow equations can be written as

$$\dot{Q} = \frac{P'Q'}{P} - \frac{Q'^2}{2Q} + Q'' + Q'\left(w_1^2\rho + u\right), \tag{3.68}$$

$$0 = -\frac{P'^2}{P^2} + \frac{2P''}{P} - \frac{Q'^2 - 2QQ''}{2Q^2} + 2w_1^2, \tag{3.69}$$

$$\dot{P} = -2 + \frac{P'Q'}{2Q} + P'' + P'\left(w_1^2\rho + u\right). \tag{3.70}$$

We are going to solve equation (3.69) by setting

$$\frac{P'^2}{P^2} - \frac{2P''}{P} = \frac{4}{a}w_1^2, \tag{3.71}$$

$$\frac{Q'^2}{Q^2} - \frac{2Q''}{Q} = \frac{4(a-2)}{a}w_1^2, \tag{3.72}$$

thus the solutions for P and Q are

$$P = \cos^2\left[\frac{w_1}{\sqrt{a}}\rho - w_2\right]w_3, \tag{3.73}$$

$$Q = \cos^2\left[\sqrt{\frac{(a-2)}{a}}w_1\rho - w_4\right]w_5; \tag{3.74}$$

by substituting (3.73) and (3.74) into (3.70)we obtain

$$\begin{aligned} &\dot{w}_3\cos^2(x) - 2w_3\cos(x)\sin(x)\left[\frac{\dot{w}_1}{\sqrt{a}}\rho - \dot{w}_2\right] = \\ &-2 - 2\frac{\sqrt{a-2}w_1^2w_3}{a}\tan(y)\cos(x)\sin(y) + 2\frac{w_1^2w_3}{a}\left[\sin^2(x) - \cos^2(x)\right], \\ &-2\frac{w_1w_3}{\sqrt{a}}\cos(x)\sin(x)\left[w_1^2\rho + u\right] \end{aligned} \tag{3.75}$$

where

$$x = \frac{w_1}{\sqrt{a}}\rho - w_2, \qquad y = \sqrt{\frac{a-2}{a}}w_1\rho - w_4. \tag{3.76}$$

In order to satisfy (3.75) we need to gather similar terms and then cancel the coefficients that multiply the trigonometric functions, nonetheless in the second line of (3.75) the argument of the tangent function is not the same of the other trigonometric functions. We can overcome this difficulty by setting

$$y = x - \frac{n\pi}{2}, \tag{3.77}$$

this condition determines the value of $a = 3$ and $w_4 = w_2 + \frac{n\pi}{2}$,this also implies that

$$\tan(y) = -\cot(x), \tag{3.78}$$

hence we can rewrite equation (3.75) as

$$\begin{aligned} &\cos^2(x)\left[\dot{w}_3 + 2w_1^2w_3\right] + 2\frac{w_3\rho}{\sqrt{3}}\cos(x)\sin(x)\left[-\dot{w}_1 + w_1^3\right] \\ &+ 2w_3\cos(x)\sin(x)\left[\dot{w}_2 + \frac{w_1u}{\sqrt{3}}\right] + \left[2 - \frac{2w_1^2w_3}{3}\right] = 0. \end{aligned} \tag{3.79}$$

This implies that

$$\dot{w}_3 + 2w_1^2w_3 = 0, \tag{3.80}$$

$$\dot{w}_1 - w_1^3 = 0, \tag{3.81}$$

$$\dot{w}_2 + \frac{w_1u}{\sqrt{3}} = 0, \tag{3.82}$$

$$2 - \frac{2w_1^2w_3}{3} = 0; \tag{3.83}$$

similarly, substituting (3.73) and (3.74) into (3.68) and using (3.78) we obtain

$$\frac{2w_5\rho}{\sqrt{3}}\cos(y)\sin(y)\left[w_1^3-\dot{w}_1\right]+2w_5\cos(y)\sin(y)\left[\dot{w}_4+\frac{w_1u}{\sqrt{2}}\right]+\cos^2(y)\left[\dot{w}_5+2w_1^2w_5\right]=0, \tag{3.84}$$

this also implies that

$$w_1^3-\dot{w}_1=0, \tag{3.85}$$

$$\dot{w}_4+\frac{w_1u}{\sqrt{2}}=0, \tag{3.86}$$

$$\dot{w}_5+2w_1^2w_5=0, \tag{3.87}$$

by solving the equations for the w_i functions we obtain

$$w_1=\frac{1}{\sqrt{2(\lambda_0-\lambda)}}, \tag{3.88}$$

$$w_3=6(\lambda_0-\lambda), \tag{3.89}$$

$$w_5=\kappa(\lambda_0-\lambda), \tag{3.90}$$

with κ an arbitrary constant. We have just one equation for w_2 and w_4 due to the fact that $w_4=w_2+\frac{n\pi}{2}$

$$\dot{w}_2+\frac{u}{\sqrt{6(\lambda_0-\lambda)}}=0, \tag{3.91}$$

where u is an arbitrary function of λ. Now we can write the line element in $4d$

$$ds^2=\kappa(\lambda_0-\lambda)\cos^2\left[\frac{\rho}{\sqrt{6(\lambda_0-\lambda)}}-\tilde{w}\right]dt^2+d\rho^2+6(\lambda_0-\lambda)\sin^2\left[\frac{\rho}{\sqrt{6(\lambda_0-\lambda)}}-\tilde{w}\right] \tag{3.92}$$

with $\tilde{w}(\lambda)$ an arbitrary function.

As we already know this is a maximally symmetric space and one can see by inspection that equation (3.38) is satisfied using the line element and

$$\begin{aligned}
R_{t\rho t\rho} &= \tfrac{1}{6}\kappa\cos^2\left(\tfrac{\rho}{\sqrt{6(\lambda_0-\lambda)}}+\tilde{w}\right),\\
R_{t\theta t\theta} &= \tfrac{1}{4}\kappa\,(\lambda_0-\lambda)\sin^2\left(\tfrac{2\rho}{\sqrt{6(\lambda_0-\lambda)}}+2\tilde{w}\right),\\
R_{\rho\theta\rho\theta} &= \sin^2\left(\tfrac{\rho}{\sqrt{6(\lambda_0-\lambda)}}+\tilde{w}\right),\\
R_{t\phi t\phi} &= \tfrac{1}{4}\kappa\,(\lambda_0-\lambda)\sin^2(\theta)\sin^2\left(\tfrac{2\rho}{\sqrt{6(\lambda_0-\lambda)}}+2\tilde{w}\right),\\
R_{\rho\phi\rho\phi} &= \sin^2(\theta)\sin^2\left(\tfrac{\rho}{\sqrt{6(\lambda_0-\lambda)}}+\tilde{w}\right),\\
R_{\theta\phi\theta\phi} &= 6\,(\lambda_0-\lambda)\sin^2(\theta)\sin^4\left(\tfrac{\rho}{\sqrt{6(\lambda_0-\lambda)}}+\tilde{w}\right).
\end{aligned} \tag{3.93}$$

Note that the solution (3.92) is well defined for $\lambda\in(-\infty,\lambda_0)$, of course we can extend this solution to the interval $\lambda\in(\lambda_0,\infty)$ due to the arbitrariness of the function $\tilde{w}$. So far we have obtained a general solution to the Ricci flow in $4d$, which is a maximally symmetric space. This is a more general solution than the found previously in a speculative thinking about the behaviour of a maximally symmetric space. We can recover the nice form of equation (3.61) by setting $\kappa=6$ and choosing $u(\lambda)=0$. There is a more interesting feature about this solution and it is the fact that the κ constant is completely arbitrary, thus κ can be positive or negative, which implies that the flow will exist either with Euclidean or Lorentzian signature.

Even more, if we start the flow with a determined signature, we can see that due to the structure of the solution, after passing the singularity the metric will suffer a signature change. To be more specific when λ exceeds λ_0, $\lambda_0-\lambda$ becomes negative and $\sin^2(\mathtt{i}z)$ becomes $-\sinh^2(z)$, thus the change in the sign of $\lambda_0-\lambda$ is absorbed by the $\sinh^2(\cdot)$. Conversely in the case of the cosine function we have $\cos(\mathtt{i}z)=\cosh(z)$ and the change in sign is maintained. Having in mind this and the fact that the flow can be carried on with an Euclidean or Lorentzian signature we can say that the flow will always exhibit a "phase transition" from an Euclidean to Lorentzian metric or vice versa, no matter which one is deformed first.

Now let us investigate if this solution has fixed points. Remember that the fixed points are given by $\frac{\partial g}{\partial\lambda}=0$, thus

$$\frac{\partial g_{tt}}{\partial\lambda}=-\cos(x)-\frac{\rho}{\sqrt{6(\lambda_0-\lambda)}}\sin(x)+2(\lambda_0-\lambda)\dot{\tilde{w}}\sin(x)=0, \tag{3.94}$$

$$\frac{\partial g_{\theta\theta}}{\partial\lambda}=-\sin(x)+\frac{\rho}{\sqrt{6(\lambda_0-\lambda)}}\cos(x)-2(\lambda_0-\lambda)\dot{\tilde{w}}\cos(x)=0, \tag{3.95}$$

with

$$x = \frac{\rho}{\sqrt{6(\lambda_0 - \lambda))}} - \tilde{w}. \tag{3.96}$$

Both equations must be satisfied simultaneously, unfortunately this is not the case, note also that the κ constant does not appear in equation (3.94). To see that there is no choice of $\tilde{w}$ that can satisfy both equations simultaneously let us decompose $\sin x$ and $\cos x$ separating the parts that depends on ρ and $\tilde{w}$, then we can write equation (3.94) as

$$\cos\left(\frac{\rho}{\sqrt{6(\lambda_0-\lambda)}}\right)\left[-\cos(\tilde{w}) - 2(\lambda_0-\lambda)\dot{\tilde{w}}\sin(\tilde{w})\right] + \sin\left(\frac{\rho}{\sqrt{6(\lambda_0-\lambda)}}\right)\left[\sin(\tilde{w}) + 2(\lambda_0-\lambda)\dot{\tilde{w}}\cos(\tilde{w})\right]$$
$$-\frac{\rho}{\sqrt{6(\lambda_0-\lambda)}}\left[\sin\left(\frac{\rho}{\sqrt{6(\lambda_0-\lambda)}}\right)\cos(\tilde{w}) - \cos\left(\frac{\rho}{\sqrt{6(\lambda_0-\lambda)}}\right)\sin(\tilde{w})\right] = 0, \tag{3.97}$$

and in a similar way we have for (3.95)

$$\sin\left(\frac{\rho}{\sqrt{6(\lambda_0-\lambda)}}\right)\left[-\cos(\tilde{w}) - 2(\lambda_0-\lambda)\dot{\tilde{w}}\sin(\tilde{w})\right] + \cos\left(\frac{\rho}{\sqrt{6(\lambda_0-\lambda)}}\right)\left[\sin(\tilde{w}) - 2(\lambda_0-\lambda)\dot{\tilde{w}}\cos(\tilde{w})\right]$$
$$+\frac{\rho}{\sqrt{6(\lambda_0-\lambda)}}\left[\cos\left(\frac{\rho}{\sqrt{6(\lambda_0-\lambda)}}\right)\cos(\tilde{w}) + \sin\left(\frac{\rho}{\sqrt{6(\lambda_0-\lambda)}}\right)\sin(\tilde{w})\right] = 0. \tag{3.98}$$

Each term in both equations must vanish independently and simultaneously, but if we look in particular at the first term in (3.97) and the first term of (3.98) we will get

$$\dot{\tilde{w}} = -\frac{\cot(\tilde{w})}{2(\lambda_0-\lambda)}, \quad \text{and} \quad \dot{\tilde{w}} = \frac{\tan(\tilde{w})}{2(\lambda_0-\lambda)} \tag{3.99}$$

but this can not happen simultaneously, therefore we conclude that there are no fixed points in $4d$. Although we have shown that there are no fixed points in $4d$, it is interesting to note that this solution can be generalized to $n \geq 4$. If we add more dimensions the only change will be a modification in the dimension of $d\Omega_{d-2}$ and we will have similar solutions for P and Q. As we have shown, the line element of a maximally symmetric space can be written as (3.60), therefore that will be a solution for the Ricci flow in n dimensions with $\rho = u$ and $l^2 = 2(n-1)(\lambda_0 - \lambda)$.

Chapter 4

Discussion

In this work we have found an explicit solution to the Ricci flow in $3d$ and $4d$. This solutions are maximally symmetric spaces and therefore are solutions to the Einstein equations with cosmological constant. In our treatment we have found that the Ricci curvature is constant in the sense that if we take a value for the flow parameter we will get a defined constant value for the curvature. The flow develops a singularity at finite $\lambda = \lambda_0$ and this is the key point that divides spaces with positive constant curvature from those with negative constant curvature. Here we could say that we have two solutions to the Ricci flow in $3d$, one of them gives us a de Sitter space and the second one gives us an Anti-de Sitter space. This would be the primary logical conclusion from the mathematical point of view. But we are tempted to think that we have one solution that experiences a "phase transition" at a critical point. The behaviour of the flow could be resumed as follows: starting from a maximally symmetric space with positive curvature, the flow increases the curvature at the same time that the space is contracting until it shrinks to a point (the critical value) and then emerges as a maximally symmetric space, this time with negative constant curvature.

This space expands indefinitely to infinity where it reaches a fixed point of the flow. We have shown that the fixed points of the flow in $3d$ are spaces that locally can be represented with an Euclidean metric but it is not clear if the topology is either of a plane or a cylinder. We suspect that we have a plane for the fixed point of the flow that deforms the AdS space, on the contrary for the fixed point of the flow that deforms the dS space we could say that the topology is that of a cylinder, these two scenarios could be related by a change in topology at the critical value.

An even more interesting feature of our solution is the change in the signature of the metric. As we have shown, passing trough the critical point from positive to negative curvature yields a change in the signature of the metric. In other words, it was shown that we can start with an Euclidean metric and by means of the flow, after passing a singularity we will obtain a pseudo-Euclidean metric. On the other hand starting with a pseudo-Euclidean metric we will obtain a Euclidean metric after passing the singular point. This reasoning is supported by the idea that the solution to the flow is one connecting two spaces and not two different solutions. To support this claim it would be very nice to find a physical system which could be modelled by this “phase transition”; by the moment we could state this just as a conjecture.

In four dimensions we have no fixed points to the flow, thus the space does not asymptote to an Euclidean or Minkowsky space. The more convincing explanation is that these spaces expand indefinitely. There are more options that should be studied, for example we can add a term to normalize (the normalized Ricci flow includes a term that is proportional to the metric multiplied by the scalar curvature and preserves the volume of the manifold) the flow and look if these maximally symmetric solutions remain with this additional term. Furthermore this term will make the flow preserve the volume so it will be interesting to look how different will be the solution for the de Sitter space since as we said previously, our solution shrinks to a point.

Also would be very interesting to find a solution that could represent an AdS black hole, this solution could be of interest by itself from a mathematical viewpoint, but also will be of great interest in the AdS/CFT correspondence. We left this for possible future work.

Apendix A

Evolution of geometrical quantities

A.1 Scaling of geometrical quantities

It is of interest to know how the geometric quantities associated to the metric scale when the metric is scaled by a constant factor C.

If $\tilde{g} = Cg$ are two Riemannian metrics on an n-manifold $\mathcal{M}^n$, related by a scaling factor C, then geometrical quantities associated with the metric scale as follows:

1. $\tilde{g}^{ij} = C^1 g^{ij}$.

2. $\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k$.

3. $\tilde{R}_{ijk}^l = Rl_{ijk}$

4. $\tilde{R}_{ijkl} = CR_{ijkl}$.

5. $\tilde{R}_{ij} = R_{ij}$.

6. $\tilde{R} = C^{-1}R$.

7. The volume elements: $d\tilde{\mu} = C^{n/2}d\mu$.

A.2 Metrics evolving under a parameter

When the metric depends on a parameter, as in our case, it will be helpful to know how the geometric quantities associated withe the metric changes when the metric evolves. The following results are taken from [24], and are also derived more formally in [7].

Let $g_{ij}(\lambda)$ be a parameter-dependent Riemannian metric, and

$$\frac{\partial g_{ij}(\tau)}{\partial \tau} = h_{ij}(\tau). \tag{A.1}$$

Then the various geometric quantities evolve according to the following equations:

1. Inverse metric

$$\frac{\partial g^{ij}}{\partial \tau} = -h^{ij} = -g^{ik}g^{jl}h_{kl} \tag{A.2}$$

Proof: We know that $g^{ij}g_{jk} = d^i_k = constant$, then

$$\begin{aligned} \partial_\tau \left(g^{ij}g_{jk}\right) &= 0 \\ (\partial_\tau g^{ij})g_{jk} + g^{ij}\left(\partial_\tau g_{jk}\right) &= 0 \\ \partial_\tau g^{ij} &= -h^{ij}. \end{aligned} \tag{A.3}$$

2. Christoffel Symbols

$$\frac{\partial \Gamma^k_{ij}}{\partial \tau} = \frac{1}{2}g^{kl}\left(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}\right) \tag{A.4}$$

Proof:

$$\begin{aligned} \partial_\tau \Gamma^k_{ij} =& \frac{1}{2}\left(\partial_\tau g^{kl}\right)\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right) \\ & \frac{1}{2}g^{kl}\left(\partial_i \partial_\tau g_{jl} + \partial_j \partial_\tau g_{il} - \partial_l \partial_\tau g_{ij}\right) \end{aligned} \tag{A.5}$$

Now we can work in normal coordinates about a point p, in normal coordinates we have

(a) $g_{ij} = \delta_{ij}$ at p.

(b) If $v \in \mathbb{R}^n$ then the curve $\gamma_v(t) = tv$ is a geodesic for as long as it exists.

(c) $\frac{\partial g_{ij}(p)}{\partial x^k} = 0 \forall i, j, k$ and $\Gamma^k_{ij} = 0$ at p. Thus

$$\nabla_k F^{j_1, j_2, \ldots j_l}_{i_1, i_2, \ldots i_k} = \frac{\partial F^{j_1, j_2, \ldots j_l}_{i_1, i_2, \ldots i_k}}{\partial x^k} \tag{A.6}$$

then

$$\frac{\partial \Gamma^k_{ij}}{\partial \tau} = \frac{1}{2}g^{kl}\left(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}\right) \tag{A.7}$$

Although the Christoffel symbols are not the coordinates of a tensor quantity, their derivatives are. Hence both sides of this equation are the coordinates of tensorial quantities, so it does not matter what coordinates we evaluate them in. In particular, the equation is true for any coordinates, not just normal coordinates, and about any point p.

3. Riemann curvature tensor

$$\frac{\partial R^l_{ijk}}{\partial \tau} = \frac{1}{2} g^{lp} \begin{Bmatrix} \nabla_i \nabla_j h_{kp} + \nabla_i \nabla_k h_{jp} - \nabla_i \nabla_p h_{jk} \\ \nabla_j \nabla_i h_{kp} - \nabla_j \nabla_k h_{ip} + \nabla_j \nabla_p h_{ik} \end{Bmatrix} \tag{A.8}$$

Proof : Each term can be expressed using the result of formula (A.4) so:

$$\begin{aligned} R^l_{ijk} &= \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^p_{jk}\Gamma^l_{ip} - \Gamma^p_{ik}\Gamma^l_{jp} \\ \partial_\tau R^l_{ijk} &= \partial_i \left(\partial_\tau \Gamma^l_{jk}\right) - \partial_j \left(\partial_\tau \Gamma^l_{ik}\right) \\ &\quad + \left(\partial_\tau \Gamma^p_{jk}\right)\Gamma^l_{ip} + \Gamma^p_{jk}\left(\partial_\tau \Gamma^l_{ip}\right) - \left(\partial_\tau \Gamma^p_{ik}\right)\Gamma^l_{jp} - \Gamma^p_{ik}\left(\partial_\tau \Gamma^l_{jp}\right). \end{aligned} \tag{A.9}$$

Once again we work in normal coordinates so that $\Gamma^k_{ij}(p) = 0$. This gives us

$$\partial_\tau R^l_{ijk}(p) = \nabla_i \left(\partial_\tau \Gamma^l_{jk}\right)(p) - \nabla_j \left(\partial_\tau \Gamma^l_{ik}\right)(p). \tag{A.10}$$

both sides are tensors, so the equation holds in any coordinates. Plugging the result of formula (A.4) on the right hand side yields the result.

4. Ricci tensor

$$\frac{\partial R_{ij}}{\partial \tau} = \frac{1}{2} g^{pq} (\nabla_q \nabla_i h_{jp} + \nabla_q \nabla_j h_{ip} - \nabla_q \nabla_p h_{ij} - \nabla_i \nabla_j h_{qp}) \tag{A.11}$$

Proof: This can be easily done by using (A.8) and taking the trace over i, l, then the desired result is obtained.

5. Scalar curvature

$$\frac{\partial R}{\partial \tau} = \Delta H + \nabla^p \nabla^q h_{pq} - h^{pq} R_{pq} \tag{A.12}$$

where $H = g^{pq} h_{pq}$.

Proof: Using formulae (A.11) and (A.2) we can see:

$$\begin{aligned} \partial_\tau R &= \partial_\tau \left(g^{ij} R_{ij}\right) \\ &= \left(\partial_\tau g^{ij}\right) R_{ij} + g^{ij}\left(\partial_\tau R_{ij}\right) \\ &= -h^{ij} R_{ij} + g^{ij} \left(\frac{1}{2} g^{pq} \left(\nabla_q \nabla_i h_{jp} + \nabla_q \nabla_j h_{ip} - \nabla_q \nabla_p h_{ij} - \nabla_i \nabla_j h_{qp}\right)\right) \\ &= -\Delta H + \nabla^p \nabla^q h_{pq} - h^{pq} R_{pq} \end{aligned} \tag{A.13}$$

where we have used $\nabla g = 0$ and $\Delta = g^{ij} \nabla_i \nabla_j$.

6. Volume element

$$\frac{\partial d\mu}{\partial \lambda} = \frac{H}{2} d\mu \tag{A.14}$$

to show this the formula

$$d\mu = \sqrt{\det g_{ij}} dx^1 \wedge dx^2 \wedge ...dx^n \tag{A.15}$$

is required. First it is nedeed to calculate the variation in the determinant of a matrix det$A$, when the matrix A itself varies. Because A will end up being a metric, we may assume that it is symmetric and hence we can choose a basis in which A is diagonalized with eigenvalues $\sigma_i \neq 0$. Then $A_{ij} = \sigma_i \delta_{ij}$, and det$A = \Pi_i \sigma_i$. If we then vary the entry A_{ij}, the determinant will not change unless $i = j$. In that case, we have

$$\frac{\partial detA}{\partial A_{ii}} = \frac{\partial \left(\Pi_j \sigma_j\right)}{\partial \sigma_i} \tag{A.16}$$

Therefore by the chain rule,

$$\begin{aligned} \frac{d(detA)}{d\sigma} &= \sum_{i,j}^{n=0} \frac{\partial(detA)}{\partial A_{ij}} \frac{dA_{ij}}{d\sigma} \\ &= \sum_{i,j}^{n=0} \delta_{ij} \frac{1}{\sigma_i} detA \frac{dA_{ij}}{d\sigma} \\ &= \left(A^{-1}\right)^{ij} \frac{dA_{ij}}{d\sigma} detA \end{aligned} \tag{A.17}$$

This formula manifestly does not depend on the basis we choose as traces are basis-independent. It now follows by the chain rule that

$$\begin{aligned} \frac{\partial \left(d\mu\right)}{\partial \lambda} &= \frac{\partial}{\partial \lambda} \left(\sqrt{\det g_{ij}} dx^1 \wedge dx^2 \wedge ...dx^n\right) \\ &= \frac{1}{2\sqrt{detg_{ij}}} g^{ij} h_{ij} detg_{ij} dx^1 \wedge dx^2 \wedge ...dx^n \\ &= \frac{H}{2} d\mu \end{aligned} \tag{A.18}$$

with $H = g^{ij} h_{ij}$.

7. Volume of a manifold

$$\frac{d}{d\tau} \int_{\mathcal{M}} d\mu = \int_{\mathcal{M}} \frac{H}{2} d\mu. \tag{A.19}$$

This follows from formula (A.14) by taking the derivative under the integral sign.

8. Total scalar curvature on a closed manifold $\mathcal{M}$:

$$\frac{d}{d\tau}\int_{\mathcal{M}} Rd\mu = \int_{\mathcal{M}} \left(\frac{1}{2}RH - h^{ij}R_{ij}\right) d\mu. \tag{A.20}$$

This follows from formulae (A.12) and (A.14)

$$\begin{aligned}\frac{\partial}{\partial\lambda}\int_{\mathcal{M}} Rd\mu &= \int \mathcal{M}\left(\frac{\partial R}{\partial\lambda}d\mu\right) + R\left(\frac{\partial d\mu}{\partial\lambda}\right) \\ &= \int_{\mathcal{M}} \left(-\Delta H + \nabla^p\nabla^q h_{pq} - h^{pq}R_{pq} + \frac{1}{2}RH\right) d\mu \qquad \text{(A.21)} \\ &= \int_{\mathcal{M}} \left(\frac{1}{2}RH - h^{pq}R_{pq}\right) d\mu.\end{aligned}$$

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