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PRESENTA:

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Dedicated to
my lovely family

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"El agradecimiento es la memoria del corazón".
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Contents

Chapter 1

Introduction

In the study of complex holomorphic functions the Riemann surfaces, abstract objects, appear naturally, since holomorphic (analytic) functions may be defined between them. We recall that a complex function $f : A \subset \mathbb{C} \to \mathbb{C}$, where A is an open subset of $\mathbb{C}$, is:

(a) *Holomorphic* in A if f is differentiable (in the complex sense) for every $z \in A$, that is, the derivative

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

is defined and continuous.

(b) *Analytic* in A if for every $z_0 \in A$ there exists $r > 0$, with $D(z_0; r) \subseteq A$ and a power series $\displaystyle\sum_{n=0}^{\infty} a_n (z - z_0)^n$, converging in $D(z_0; r)$, such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad z \in D(z_0; r).$$

From Complex Analysis we know that holomorphic and analytic are equivalent; see [64], so we will use these concepts indistinctly through this thesis.

The thesis deals with the dynamical behavior of some classes of holomorphic functions defined on Riemann surfaces, so we shall give a definition of these objects taken from [57].

A *Riemann surface* M is a connected complex analytic manifold of complex dimension 1, which means that M is a connected Hausdorff space that is endowed with an atlas of charts. Furthermore, in some neighborhood U of an arbitrary point of M we can choose a coordinate chart, which maps U homeomorphically onto an open subset of the complex plane $\mathbb{C}$ with the following property: In the overlap $U \cap U'$ between two neighborhoods, each of these coordinate charts can be expressed as a holomorphic function of the other. Two Riemann surfaces M and M' are conformally isomorphic (or biholomorphic) if and only if there is a homeomorphism from M onto M', which is holomorphic in terms of the respective coordinate charts.

The complex plane $\mathbb{C}$ and the Riemann sphere $\mathbb{S} = \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ are the most basic Riemann surfaces. The uniformization Theorem [50] is one of the major results of 19th century in mathematics and states that *an arbitrary simply connected Riemann surface is conformally equivalent to one on the following three Riemann surface:*

(i) The open unit disc $\mathbb{D}(0,1)$.

(ii) The complex plane $\mathbb{C}$.

(iii) The Riemann sphere $\widehat{\mathbb{C}}$.

Based on the previous classification, we will focus in the behavior of the iterations of different classes of holomorphic function defined either in the complex plane $\mathbb{C}$ or in the Riemann sphere $\widehat{\mathbb{C}}$. The *iteration* of a holomorphic function f is the composition of the function f with itself n times, denoted by $f^n = \underbrace{f \circ f \ldots \circ f}_{n \quad times}$.

The class of transcendental meromorphic functions, denoted by $\mathscr{M}$, are functions $f : \mathbb{C} \to \widehat{\mathbb{C}}$ which are meromorphic in $\mathbb{C}$, that is, holomorphic except in poles, and f has one pole that is not an omitted value. In this class of functions ∞ is an essential singularity which is not in the set of exceptional points; see Page 20 for details.

The main objects in the study of the dynamics of functions in class $\mathscr{M}$ are:

(a) The *Fatou set* (or *Stable set*), which consists of all points $z \in \mathbb{C}$, such that the sequence of iterates of f is well defined and forms a normal family in some neighborhood of z in the sense of Montel.

(b) The *Julia set* (or *chaotic set*) is the complement of the Fatou set.

For functions in class $\mathscr{M}$ we recall some well known properties for the Fatou and Julia sets; see Page 23 for details:

(i) The Fatou set is open, so the Julia set is closed.

(ii) The Julia set is perfect and non-empty.

(iii) The Fatou and Julia sets are completely invariant under f.

(iv) If $f \in \mathscr{M}$ and z_0 is not an exceptional point, then the Julia set coincides with the closure of the backward orbit of z_0.

(v) The Julia set is the closure of the set of all repelling periodic points.

The Fatou set have components which can be either periodic, pre-periodic or wandering. Furthermore, there is a classification of the periodic components of the Fatou set; see Page 24 for details.

With the concepts stated before the dynamics of functions in the class $\mathscr{M}$ was first studied by Baker, Kotus and Lü in [11], [12], [13] and [14], where Montel's theory of normal families with its consequences and Picard's theorems were the main tools used in the study.

In this thesis we investigate a family of transcendental meromorphic functions with one not omitted pole and two parameters. We study the behavior of the family under certain assumptions on the parameters and the pole. A definition of a slice of the parameter space is given and some examples of approximations of the Fatou and Julia sets are shown. Moreover, some results related with the residual Julia set for transcendental meromorphic functions are given when the Fatou set has a $p-$cycle of Herman rings.

1.1 A brief review of Complex Dynamics

The origin of the theory of complex dynamics was founded in the last decades of the XIX century and the first years of the XX century. The ealier works in the area were two detailed versions of Newton's method, developed by J. Schröder and A. Cayley; see [24], [70] and [71]. The subsequent works elaborated in the XIX century were related to the behavior of the iteration of complex functions in the neighborhood of periodic points, with a distinguished mention to the work developed by G. Köenigs in [51], where he composed the first rigorous study of the iteration of complex functions; see [4] for more information.

In the early years of the XX century, the theory of complex dynamics presented a critical change with the contributions of P. Fatou [41] and G. Julia [48], who investigated the iteration of rational functions in the Riemann sphere using Montel's theory of normal families [58]. They partitioned the Riemann sphere in two sets (normality and non-normality) and proved basic properties for rational functions with degree at least 2. Nowadays, the set of normality is called the *Fatou set* and the set of non-normality is called the *Julia set* in honor of their contributions.

An open question posted by Fatou for rational functions was: *Are there components in set of normality such that they are neither periodic nor pre-periodic components?*, today these components are called *wandering domains*. D. Sullivan proved in [77] that this type of component cannot exists for rational functions. Moreover, he stated a "kind" of first classification of periodic components in the Fatou set: Attracting, parabolic, Siegel disc and Herman ring; see Section 3.3 for definitions.

Returning to the last years of the 1920s, P. Fatou [42] extended certain results obtained for rational functions to transcendental entire functions (complex functions which are analytic in the complex plane and infinite is an essential singularity). Fatou observed that the proofs of the properties for rational functions did not work for transcendental entire functions, since rational functions always have fixed points, while transcendental entire functions may not. He gave new proofs using different mathematical tools, nevertheless, he died in the year of 1929 leaving important open questions and properties related to the set of normality.

The following decades after the death of P. Fatou, the theory of complex dynamics had an intermittent development. Among the most important contributions were the results obtained by H. Cremer [27] and C. L. Siegel [76] related with certain domains in the Fatou set for rational functions, for more information in this topic see [5]. It is until the work developed by I. N. Baker in [6] that the study of the dynamics of the class of transcendental entire functions was continued. He investigated properties of the Fatou and Julia sets for functions in this class. Moreover, Baker in [9] constructed the first example of a transcendental entire function whose Fatou set contains a wandering component. Thus, the dynamics of rational functions and transcendental entire functions are different.

The theory of complex dynamics for transcendental meromorphic functions (complex functions which are analytic in the complex plane except in poles with an essential singularity at infinite) was investigated by I. N. Baker, J. Kotus and Lü between 1990 and 1992 in [11], [12], [13] and [14]. Many of the basic properties of the Fatou and Julia sets for transcendental meromorphic functions are similar to the properties for rational functions, but different proofs were required. After Baker, Kotus and Lü, many mathematicians have continued investigating the dynamics of transcendental meromorphic functions as well as the dynamics of other classes of functions with more than one essential singularity, for instance, A. Epstein [37], M. E. Herring [45] and A. Bolsch [20].

As we can observe, the theory of complex dynamics have had an irregular development, nevertheless, in recent years this theory has substantially gained interest due to use of computers and the implementation of programs such as C++, Mathematica, since the parameter plane of a family of functions dependent of one complex parameter could be graphed, where the parameter plane refers to the phase space in which the bifurcations of the dynamics are represented as one varies the parameter. The relevance of the parameter planes lies in the fact that we can handily obtain an imperishable source of paths to investigate and develop new results related to dynamics of families of functions.

1.2 Structure of the thesis

The dynamics of the complex sine family $g_\lambda(z) = \lambda \sin(z), \lambda \in \mathbb{C}$ was studied by P. Bhattacharya in 1969 [19] and by P. Dominguez and G. Sienra in 2002 [34]. The dynamics of the sine family was not investigated deeper before because it was assumed that it had a similar dynamical behavior to the exponential family, $E_\lambda = \lambda e^z, \lambda \in \mathbb{C}\backslash\{0\}$ investigated by R. Devaney in [30]. Both families of functions are transcendental entire, since they are analytic in $\mathbb{C}$ without poles. McMullen in [56] proved that for the exponential function the Lebesgue measure of the Julia set is zero, whereas for the sine function is not. The result determines that the dynamics of the families cannot be similar.

In this thesis we are interested in the dynamics of the sine family when it is perturbed by a simple pole z_0 and with a new real parameter $\mu > 0$ sufficiently small, that is,

$$f_{\lambda,\mu,z_0}(z) = \lambda \sin(z) + \frac{\mu}{z - z_0}, \text{where } \lambda \in \mathbb{C} \setminus \{0\}, \mu \in \mathbb{R}^+ \setminus \{0\} \text{ and } z_0 \in \mathbb{R}. \qquad (1)$$

Observe that the family f_{λ,μ,z_0} belongs to class of transcendental meromorphic functions, since f_{λ,μ,z_0} is a family of analytic functions except at the pole z_0 which is not an omitted value, that is $f_{\lambda,\mu,z_0}(z_0) = \infty$ and $f^{-1}_{\lambda,\mu,z_0}(z_0) \neq \emptyset$. We will be focused in the geometrical and topological properties of the Fatou and Julia sets of the family f_{λ,μ,z_0}. Observe that the family $g_\lambda(z) = \lambda \sin(z)$ is transcendental entire, so the families f_{λ,μ,z_0} and g_λ can not have the same dynamics.

The first result of the thesis is a generalization of the Theorem 1 presented in [36], which is stated as follows.

Theorem A. *If λ, μ, z_0 are real parameters such that $0 < |\lambda| < 1$, $\mu > 0$ sufficiently small and $|z_0| \geq |\lambda| + \frac{\pi}{2}$, then the family $f_{\lambda,\mu,z_0}(z) = \lambda \sin(z) + \frac{\mu}{z-z_0}$, has an attracting completely invariant component in the Fatou set, which is multiply connected.*

In Theorem A the pole is in the real line with a restriction to the parameter λ. The proof and details of Theorem A are in Section 4.1.1. For the case when the parameter λ is a complex number with modulus less than 1, we state the following theorem, whose proof and details are in Section 4.1.2.

Theorem B. *If $\lambda \in \mathbb{C}$ is a complex parameter with $0 < |\lambda| < \frac{1}{1+e}$, $\mu > 0$ and z_0 are real parameters such that $\mu > 0$ sufficiently small and $|z_0| \geq |\Re(\lambda)| + \frac{\pi}{2}$, then the family $f_{\lambda,\mu,z_0}(z) = \lambda \sin(z) + \frac{\mu}{z-z_0}$, has an attracting completely invariant component in the Fatou set, which is multiply connected.*

The family f_{λ,μ,z_0} in (1) involves three parameters, so it is not possible to draw the parameter plane, but we can fix two of the three parameters, said z_0 and μ; and give values to the third parameter λ, for which the sequence of iterates of a critical point of the family f_{λ,μ,z_0} is bounded. The resulting set is called an *approximation of a slice of the parameter space.*

Using the *FractalStream* software, we plot some approximations of slices of the parameter space of the family f_{λ,μ,z_0} in Section 4.1.3. Moreover, we give different values to the parameters such that the conditions of Theorem A and Theorem B are satisfied, thus we plot the Fatou and Julia sets of the family f_{λ,μ,z_0} for both theorems.

Observe that in Theorems A and B the pole z_0 is never zero. For the case when the pole z_0 is zero the family can be written as follows:

$$f_{\lambda,\mu}(z) = \lambda \sin(z) + \frac{\mu}{z}, \lambda \in \mathbb{C} \setminus \{0\}, \mu \in \mathbb{R}^+ \setminus \{0\}. \tag{2}$$

For this case we prove some geometric properties of $f_{\lambda,\mu}$ using trigonometric complex identities, which are stated in Lemmas 4.5, 4.6, 4.7 and 4.8 in Chapter 4.

The main results of the family in (2) are in Section 4.2.2 which are as follows.

Proposition 4.9. *For $z = x \in \mathbb{R}$ and $\lambda, \mu \in \mathbb{R} \setminus \{0\}$, with $\mu > 0$ sufficiently small, the family $f_{\lambda,\mu}(z) = \lambda \sin(z) + \frac{\mu}{z}$ has two real fixed points, which are symmetric with respect to the imaginary axis.*

Proposition 4.10. *The family $f_{\lambda,\mu}(z) = \lambda \sin(z) + \frac{\mu}{z}$, with $\lambda, \mu, \in \mathbb{R} \setminus \{0\}$, belongs to class $\mathcal{B}$.*

From Section 4.2.3 to Section 4.2.6 we give conditions to the parameters λ and μ to obtain examples of different types of components in the Fatou set of $f_{\lambda,\mu}$, such as: attracting components (Section 4.2.3), parabolic components (Section 4.2.4), Siegel discs (Section 4.2.5) and combined components such as attracting with parabolic (Section 4.2.6). Moreover, we state some conjectures derived of the examples given.

Using the *FractalStream* software, we give some approximations of dynamical planes of the family $f_{\lambda,\mu}$ related with the conditions of the parameters λ and μ mentioned above. Also, we show examples of the Fatou and Julia sets of the family $f_{\lambda,\mu}$, for some parameters λ and μ given.

In Chapter 5, we investigate the residual Julia set of functions in class $\mathscr{M}$. First we define and give some properties of residual Julia set and by using the concept of island defined by Ahlfors and Theorem B in [31], we prove the following result.

Theorem C. *There exist $g \in \mathscr{M}$ which satisfies that the Fatou set of g has a p-cycle of Herman rings $H_1, H_2, \dots, H_p$, $p \geq 3$, removing the cases when*

(a) the p-cycle of Herman rings is nested and

(b) the p-cycle of Herman rings is almost nested,

such that the Julia set of g contains singleton components which are dense and buried.

With the technique developed by Shishikura in [74] and [75], along with the result obtained by Peter and Fagella in [40], we give examples which satisfy hypothesis of Theorem C.

For functions in class $\mathscr{M}$ there are examples of wandering components of any prescribe connectivity, either bounded or unbounded, which were constructed by using complex approximation in [12]. Concerning wandering components and the residual Julia set we state the following theorem, whose proof is in Section 5.3.2.

Theorem D. *There exist $h \in \mathscr{M}$ which satisfies that the Fatou set has a bounded wandering component which is neither simply connected nor nested multiply connected, such that the Julia set contains singleton components which are dense and buried.*

Observe that in Theorems C and D the Julia set is not connected. There is an interesting question whether the residual Julia set is non-empty when Julia set is connected? The following theorem is a partial answer to the question.

Theorem E. *Suppose that f is a transcendental meromorphic function, the Fatou set has only two attracting simply connected invariant bounded components U_1 and U_2, such that:*

(a) $\partial U_1 \cap \partial U_2 = \{0\}$, where zero is a non-omitted pole of f,

(b) U_1 and U_2 are symmetric with respect to the imaginary axis, and

(c) the imaginary axis is in the Julia set, where $f^n(iy) \to \infty$, with $y \in \mathbb{R}$.

Then the imaginary axis is a buried component, except for zero.

The thesis is organized as follows:

Chapter 2 contains some preliminaries such as analytic functions, the types of singularities of a complex function, the concept of topological conjugacy and some of its properties. Also, we state fundamental results related to normal families, such as Montel's theorem, Great Picard's theorem and Little Picard's theorem.

In Chapter 3 we define the class of transcendental meromorphic functions $\mathscr{M}$ and some results for this class of functions. We define the Fatou and Julia sets for a function $f \in \mathscr{M}$ and give some of its properties. We state the classification of the periodic components in the Fatou set. We also define the classes of finite type and bounded type for $f \in \mathscr{M}$, denoted by $\mathcal{S}$ and $\mathcal{B}$, respectively and state some of its properties.

In Chapter 4 we investigate the dynamics of the family $f_{\lambda,\mu,z_0}(z) = \lambda\sin(z) + \frac{\mu}{z-z_0}$, where $\lambda \in \mathbb{C}\setminus\{0\}$, $\mu \in \mathbb{R}^+\setminus\{0\}$ and $z_0 \in \mathbb{R}$. The investigation is divided in two cases, since the location of the pole changes the dynamics of the family f_{λ,μ,z_0}.

(i) First case: When the pole $z_0 \neq 0$. We prove Theorems A and B, mentioned before, and study some approximations of slices of the parameter space of the family $f_{\lambda,\mu,z_0}(z) = \lambda\sin(z) + \frac{\mu}{z-z_0}$. Also we give some examples of approximations of the Fatou and Julia sets of the family f_{λ,μ,z_0}, for some parameters λ, μ and z_0 such that they satisfy the conditions of Theorems A and B.

(ii) Second case: When the pole $z_0 = 0$. We analyze geometric properties of functions of the family $f_{\lambda,\mu}(z) = \lambda\sin(z) + \frac{\mu}{z}$ and give conditions on the parameters λ and μ for the existence of attracting component, parabolic component and Siegel discs contained in the Fatou set of $f_{\lambda,\mu}$. Finally, we show some examples of Fatou set and Julia set of the family $f_{\lambda,\mu}$.

Chapter 5 includes the definition and some properties of the residual Julia set for functions $f \in \mathscr{M}$, the technique developed by Shishikura in [74] and [75], the result given by Fagella and Peter in [40] and the proofs of Theorems C, D and E.

Finally, Appendix A contains more examples of functions in class $\mathscr{M}$ with a p-cycle of Herman rings which are constructed by using the techniques in [74] and [75] along with a result in [40]. We finalize the thesis with some conclusions and several ideas for a future research.

Chapter 2

Preliminaries

In this chapter we briefly describe some definitions and results related to complex analysis that are of particular importance for the following chapters in this thesis. The concepts and main results can be consulted in [3], [54], [64] and [65], among others, in case the reader wishes to know details. We shall assume that the reader has some familiarity with the elementary results in Topology used in this section, such as a domain, a compact set, Hausdorff space, etc., these concepts can be consulted in [28], [61] and [63].

2.1 Analytic functions

Let $A \subset \mathbb{C}$ and f the map given by $f : A \to \mathbb{C}$, that is, an assignment of a specific point $f(z) \in \mathbb{C}$ for each $z \in A$, where A is the domain of f and the range is the set of values that f assumes in $\mathbb{C}$. The map f is called a complex function of a complex variable which can be either:

(i) *Single-valued*, if f has an unique value $f(z)$; for each point $z \in A$; or

(ii) *Multi-valued* if f has two o more different values $f(z)$, for $z \in A$.

A multi-valued function f can be considered as a collection of single-valued functions using the concept of branches, where a *branch* (also called a *sheet*) is a portion of the range of a multi-valued function over which the function is single-valued. Combining all the branches gives the full structure of the function. It is often convenient to choose a particular branch of a function to work with, and this choice is often called the *principal branch*. Thus, throughout the thesis we will allow the use of single-valued and multi-valued functions with their respective considerations.

From now on we will write complex function instead of complex function of a complex variable.

Let f be a complex function, $f : A \subset \mathbb{C} \to \mathbb{C}$, where A is an open subset of $\mathbb{C}$. The function f is:

(a) *Holomorphic* in A if f is differentiable (in the complex sense), for every $z \in A$, that is, the derivative

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

is defined and continuous.

Observe that the definition requires all holomorphic functions to be single-valued, but it is possible to consider multiple-valued functions, provided they are restrict to a defined region U for which it is possible to select a single-valued and holomorphic branch of the function.

(b) *Analytic* in A if for every $z_0 \in A$ there exists $r > 0$, with $D(z_0; r) \subseteq A$ and a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, converging in $D(z_0; r)$, such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad z \in D(z_0; r).$$

From now and on, in this thesis we will use either the term complex function or function in instead of complex function of a complex variable.

The following theorem gives the equivalence between the previous concepts; see [64] for a proof.

Theorem 2.1. *A complex function f is analytic over a complex domain A if and only if f is holomorphic in A.*

Due to the previous result, we will use the concepts either analytic or holomorphic indistinctly throughout the thesis.

We recall that the Riemann sphere is denoted by $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The distance between two points in the Riemann sphere is the *chordal metric*; see [52] for definition and details.

The derivative in the chordal metric, called the *spherical derivative*, is defined for $z \in \widehat{\mathbb{C}}$ such that $f(z) \neq \infty$ as follows:

$$f^{\sharp}(z) = \frac{|f'(z)|}{1 + |f(z)|^2}.$$

When $f(z) = \infty$, the spherical derivative is defined by the following equation:

$$f^{\sharp}(z) = \lim_{w \to z} f^{\sharp}(w).$$

Observe that $f^\sharp$ is continuous and $f^\sharp(z) = f(\frac{1}{f(z)})^\sharp$; see [72] for details.

The singularities of a complex function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ are special points in the domain of definition of the function, so we need to define them.

A point $z_0 \in \mathbb{C}$ is a *singularity of the complex function* f if the function f is not analytic in z_0. There are two types of singularities: isolated and not isolated. The formal definition is given as follows.

(a) A singularity z_0 an *isolated* of f if f is analytic in the punctured disc given by $\{z \in \mathbb{C} : 0 < |z - z_o| < R\}$ for some $R > 0$, but not at $z = z_0$.

(b) A singularity z_0 is a *non-isolated* of f if for every neighborhood U of z_0 it contains a singularity of f other than z_0.

By the Laurent Expansion Theorem, the complex function f can be written around the isolated singularity z_0 as follows:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

where both series on the right side of the above equation converges absolutely on a neighborhood of z_0. The terms of the series involving negative powers of $(z - z_0)$ is called the *principal part* of f at z_0 and the series concerning to positive powers is called the *analytic part* of f at z_0.

Using the Laurent's expansion, the classification of isolated singularities of a complex function f is given as follows; see [25] for details.

(i) If all the terms b_n in the principal part of f at an isolated singularity z_0 are zero, then the point z_0 is called a *removable singularity of f.*

(ii) If the principal part of f at z_0 contains at least one nonzero term but the number of such terms is finite, then z_0 is called a *pole of order k.* A pole of order $m = 1$ is called a *simple pole.*

(iii) If the principal part of f at z_0 has an infinite number of nonzero terms, then z_0 is called an *essential singularity.*

Some examples of isolated singularity are the following.

(a) The complex function $f(z) = \sin(z)/z$ has a removable singularity at $z = 0$, since we can define the sine function as a power series, that is,

$$\sin z = \sum_{k \geq 0} \frac{(-1)^k}{(2k+1)!}\, z^{2k+1},$$

so, for $z \neq 0$, we have that

$$f(z) = \frac{\sin z}{z} = \sum_{k \geq 0} \frac{(-1)^k}{(2k+1)!} z^{2k},$$

which is a convergent power series everywhere.

(b) The complex function $f(z) = 1/z^2$ has a pole of order 2 at $z = 0$.

(c) The complex function $f(z) = e^{\frac{1}{z}}$ has an essential singularity at $z = 0$.

A characterization of an isolated essential singularity is given in the Casorati-Weierstrass Theorem; see [65] for a proof.

Theorem 2.2 (Casorati-Weierstrass Theorem)**.** *If a complex function f is analytic in a punctured disc $D^* = D^*(z_0, r)$ and has an essential singularity at its center, then $f(D^*)$ is dense in the complex plane, that is, for U a neighborhood of z_0 and $w \in \mathbb{C}$, there exits a sequence $\{z_n\} \subset U \setminus \{z_0\}$ that converges to z_0 such that the limit of $f(z_n)$ converges to w.*

Some complex functions are characterized by their domain of analyticity, such is the case of the meromorphic functions whose definition taken from [3] is as follows.

Definition 2.1. A complex function $f : A \subset \mathbb{C} \to \mathbb{C}$ is *meromorphic in A* if f is analytic in A except in poles. More precisely, for every $a \in A$ there exist a neighborhood $|z-a| < \delta$, contained in A, such that either $f(a)$ is analytic in the whole neighborhood, or $f(z)$ is analytic, for $0 < |z-a| < \delta$, and the isolated singularity is a pole.

The quotient $h = f/g$ of two analytic complex functions in a domain A is a meromorphic function in A, if g is not identically zero. For instance, the rational functions are meromorphic functions, since they are the quotient of two polynomials. In general, the sum, the product and the quotient of meromorphic functions are meromorphic functions.

The following classes of complex functions will be used later in this thesis.

Definition 2.2. Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a complex function. The function f is called:

(i) A *transcendental entire function* if f is an analytic function in the whole complex plane and $f(\infty)$ is not defined.

(ii) A *transcendental meromorphic function* if f is a meromorphic function, which is not a rational function.

From the previous definitions, we can observe the following:

(1) The transcendental functions have an essential singularity at ∞, since it is neither pole nor removable; see [43] for details.

(2) The essential singularity ∞ can be either isolated or not isolated, for example:

   (i) For $f(z) = \lambda \sin(z) + \frac{1}{z}$, the function has just one pole, so ∞ is isolated.

   (ii) For $T(z) = \tan(z)$, the poles of the function are $z_n = \frac{(2k-1)\pi}{2}$, $k \in \mathbb{Z}$, so here ∞ is not isolated, since it is an accumulation point of the poles z_n.

(3) The meromorphic functions are divided into two disjoint sets: the *transcendental meromorphic functions* and the *rational functions*.

There exist distinguished values in the range of a holomorphic function which we give as follows.

Definition 2.3. Let $f : A \subset \mathbb{C} \to \mathbb{C}$ (or $\widehat{\mathbb{C}}$) be an holomorphic function.

(i) w is an *omitted value of f* if for every $z \in \mathbb{C}$ (or $\widehat{\mathbb{C}}$) the equation $f(z) - w = 0$ has no solutions.

(ii) w is a *Picard exceptional value of f* if $f(z) - w = 0$ has only finitely many solutions. The set of solutions is denoted by $\mathcal{P}(f)$.

Remark 2.1. If w is an omitted value of f, then w is a Picard exceptional value of f.

Examples of functions with omitted values or Picard exceptional values are as follows.

(i) The function $f(z) = e^z$ has two values in $\widehat{\mathbb{C}}$ which are $\{0, \infty\}$ and in $\mathbb{C}$ is $\{0\}$. The set of Picard exceptional values of f is given by $\mathcal{P}(f) = \{0, \infty\}$.

(ii) The function $h(z) = -e^z + \frac{1}{z}$ has no omitted values and $\mathcal{P}(h) = \{\infty\}$.

The analytic functions that are neither polynomials nor rational functions have an essential singularity in ∞. As an application of this fact we state the Picard's Great theorem and Picard's Little theorem for meromorphic functions. The reader can revise the theorems in [26].

Theorem 2.3 (Picard's Great Theorem). *Let $f : A \subset \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function and z_0 an essential singularity of f. If U is a punctured neighborhood of z_0, then for every $w \in \mathbb{C}$ except at most two values, the equation $f(z) = w$ has infinite solutions.*

The Picard's Great Theorem is a strengthened version of the Casorati-Weierstrass Theorem and it implies that if z_0 is an essential singularity of f, then one of the equations $f(z) = a$, $f(z) = b$, $f(z) = c$ has infinitely many solutions in each neighborhood of z_0. In the special case when $A = \hat{\mathbb{C}}$ and $c = z_0 = \infty$ we have the following result.

Theorem 2.4 (Picard's Little Theorem). *A non-constant entire function (analytic at the whole complex plane) assumes every finite complex value, with at most one exception.*

From the Picard's Little Theorem, a similar result is obtained for meromorphic functions.

Theorem 2.5. *A non-constant meromorphic function assumes every complex value with at most two exceptions.*

2.2 Singular values of a complex function

In this section, we give definitions of a particular set of points obtained from the derivative of an holomorphic function. These points play an important role in the study of the dynamics, as we will see in the following chapters.

Definition 2.4. Let $f : A \subset \mathbb{C} \to \mathbb{C}$ be an analytic function defined on open set A. We say that w is a *critical point of* f if w is a solution of $f'(z) = 0$, the value $f(w)$ is called a *critical value of* f

Definition 2.5. A point $w \in \mathbb{C}$ is called a *asymptotic value* of f if there exists a curve $\gamma : [0, \infty] \to \mathbb{C}$ such that $\gamma(t) \to \infty$ and $f(\gamma(t)) \to w$, when $t \to \infty$.

Examples of critical points, critical values and asymptotic values for different functions are the following:

(i) If $f(z) = \cos z$, then $f'(z) = -\sin z$. The critical points of f are of the form $z = k\pi, k \in \mathbb{Z}$. The critical values of f are the points $f(k\pi)$, that is, the set $\{1, -1\}$ are the critical values of f. The function has no asymptotic values.

(ii) Take $f(z) = e^z$ and the curve $\gamma : [0, \infty] \to \mathbb{C}$ given for $\gamma(t) = -t$. When $t \to \infty$, we can observe that $f(\gamma(t)) = e^{-t} \to 0$. Thus 0 is a asymptotic value of the exponential function. The function has neither critical points nor critical values.

With the definitions given above, we shall define the set of the singular values of a holomorphic complex function f as follows.

Definition 2.6. The *set of the singular values* of a holomorphic function f is the closure of the set of asymptotic values and critical values. We denote this set as follows:

$$SV(f) = \overline{\{\text{critical values and asymptotic values}\}}.$$

Following the Iversen's Theorem in [47] we also state a more detailed definition of the singular values for functions in Definition 2.2; see [18].

Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be an analytic function and $a \in \widehat{\mathbb{C}}$. Consider the open discs $B(a,r)$ of radius r with center a. For every $r > 0$ we can choose a component U_r of the preimage $f^{-1}(B(r,a))$ such that $r_1 < r_2$ implies that $U_{r_1} \subset U_{r_2}$. Two possibilities can occur: (1) The intersection $\bigcap_{r>o} U_r = \{z\}$ for an unique z in $\mathbb{C}$ and (2) $\bigcap_{r>0} U_r = \emptyset$.

(1) If $\bigcap_{r>o} U_r = \{z\}$, $z \in \mathbb{C}$, then $a = f(z)$. In this case we have two possibilities:

  (a) If $a \in \mathbb{C}$ and $f'(z) \neq 0$ or if $a = \infty$ and z is a simple pole of f, then z is called an *ordinary point* or *regular point.*

  (b) If $a \in \mathbb{C}$ and $f'(z) = 0$ or if $a = \infty$ and z is a multiply pole of f, then z is called a *critical point* and a is called a *critical value.* Also, we say that f^{-1} has a *algebraic singularity over a.*

(2) For the case $\bigcap_{r>0} U_r = \emptyset$, we say that the election of $r \longmapsto U_r$ define a *transcendental singularity of f^{-1} over a* and a is called the *projection of the transcendental singularity of f^{-1}*, or the transcendental singularity belongs over a. The sets U_r are called neighborhood of the singularity U. The transcendental singularities are classified as follows:

  (a) A transcendental singularity over a is called *direct* if for some $r > 0$ we have that $f(z) \neq a$, for $z \in U_r$. In other case, the transcendental singularity over a is called *indirect.*

  (b) A direct singularity a is called a *logarithmic singularity* if the restriction $f : U_r \to B(a,r) \setminus \{a\}$ is a universal covering for some $r > 0$.

The collection of $\bigcap_{r>o} U_r$ is called the *tracts of f* and the points described in both cases *singularities of the inverse function* which we denote as $sing(f^{-1})$.

The above definitions can be extended for the case when $a = \infty$ by using $B(\infty, r) = \{z \in \widehat{\mathbb{C}} : |z| > \frac{1}{r}\}$. Also, the projections of transcendent singularities coincide with the asymptotic values of f; see [18] for details.

Examples of the above types singularities are the following:

(i) The function $f(z) = e^z$ has one logarithmic singularity over 0, since $f(z) \neq 0$ and the universal covering space of the punctured complex plane is the complex plane itself, and the cover is given by the exponential map.

(ii) A example of an indirect singularity is given by the function $f(z) = \frac{\sin(z)}{z}$. Observe that $\frac{\sin(z)}{z} \to 0$ as $z \to +\infty$ along the positive real axis and the U_r containing tail of positive real axis contains infinitely many zeros of $f(z)$, which are of the form $z = k\pi$, $k \in \mathbb{Z}$. Also, with some calculations, we obtain that 0 is a limit point of critical values of f. Thus $f(z) = \frac{sin(z)}{z}$ has an indirect transcendental over 0.

2.3 Topological conjugacy

A fundamental mathematical tool in the study of the complex dynamics is the concept of *analytic conjugation*, which was introduced by Schröder's in his work mentioned in the introduction; see Page 3.

Let $U, V \in \mathbb{C}$, we say that the function $f : U \to U$ is *topologically conjugate* to a function $g : V \to V$ if and only if there exists homeomorphisms $\phi : U \to V$ and $\psi : U \to V$ such that $g(\phi(z)) = \psi(f(z))$, that is, the following diagram commutes in Figure 2.1.

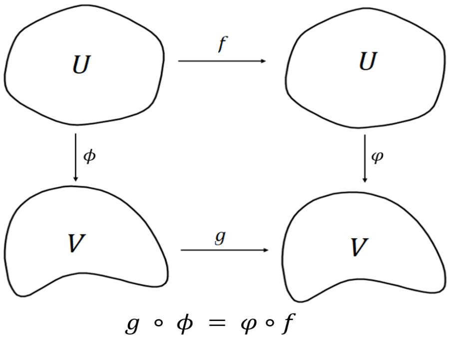

Figure 2.1: Diagram of a topological conjugacy

Remark 2.2. With the definition given above, the topological conjugacy receives different names, depending on the characteristics of the functions ϕ and ψ.

(i) For the case when ϕ and ψ are conformal functions and $\phi \equiv \psi$, we say that f is *conformally conjugate* to g. Moreover, the conjugation can be represented by the equation

$$g \circ \phi = \phi \circ f.$$

(ii) For the case when ϕ and ψ are analytic functions and $\phi \equiv \psi$, we say that f is *analytically conjugate* to g. Moreover, the conjugation can be represented by the equation

$$g \circ \phi = \phi \circ f.$$

In gratitude to the contributions of Schröder in complex dynamics, the equation, mentioned above, is called *Schröder's equation*.

In some cases, it is more convenient to study the conjugated function than the original function, for an example we refer to the reader to Lemma 1 in [34].

2.4 Normal families

In 1912 P. Montel in [58] introduced the concept of normal families. This section is focused on normal families and their consequences, since this will be used in Chapter 3. We begin the section with the concepts of *locally uniform convergence*, *equicontinuity* and *locally bounded*, which can be revised in [23], [26] and [65].

Definition 2.7. Let f_n be a sequence of holomorphic functions defined on a open set $A \subset \mathbb{C}$. The sequence f_n converges *uniformly on* A to a function f if for every $\epsilon > 0$, there exists an index $N \in \mathbb{N}$ such that for all $n > N$ and all $z \in A$

$$|f_n(z) - f(z)| < \epsilon.$$

Definition 2.8. Let f_n be a sequence of functions defined on a open set $A \subset \mathbb{C}$. The sequence f_n converges *locally uniformly on* A to f if for every point $z_0 \in A$, there exists a neighborhood $V(z_0) \subset A$ on which f_n converges uniformly to f.

Observe that locally uniform convergence is equivalent to compact convergence, meaning uniform convergence on every compact $K \subset A$; see [79] for details.

Definition 2.9. Let $\mathcal{F}$ be a family of holomorphic functions defined on a open set $A \subset \mathbb{C}$. $\mathcal{F}$ is *equicontinuous* (on A) if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < \epsilon$, for all $f \in \mathcal{F}$, and all x_1, $x_2 \in A$ such that $|x_1 - x_2| < \delta$.

Definition 2.10. Let $\mathcal{F}$ be a family of holomorphic functions defined on a open set $A \subset \mathbb{C}$. The family of functions $\mathcal{F}$ is *pointwise bounded* in $D \subset \mathbb{C}$ if for each fixed z in U there exists a constant $M \in \mathbb{R}$ such that the set of values $\{f(z) : f \in \mathcal{F}\}$ is a bounded set in $\mathbb{C}$, that is $|f(z)| \leq M$ for $z \in A$.

Definition 2.11. Let $\mathcal{F}$ be a family of holomorphic functions defined on a open set $A \subset \mathbb{C}$. The family of functions $\mathcal{F}$ is *locally uniformly bounded* in $D \subset \mathbb{C}$ if for every $f \in \mathcal{F}$ is uniformly bounded in every compact set in D, that is, for every compact subset K contained in D there exits a constant $m = m(K)$, such that $|f(z)| \leq m$, for every point $z \in K$ and for every $f \in \mathcal{F}$.

The above definitions can also be considered for meromorphic functions. Now, we state the following two theorems; see [72] for the proofs.

Theorem 2.6. *If $f_n \to f$ converges locally uniformly, where f_n and f are holomorphic functions defined on some open set $A \subset \mathbb{C}$, then $f_n' \to f'$ locally uniformly.*

Theorem 2.7 (Hurwitz's Theorem)**.** *If $f_n \to f$ locally uniformly as in Theorem 2.6 and $f_n(z) \neq 0$ for all $z \in A$, then either $f \neq 0$, for all $z \in A$, or f is constant.*

We state the definition of normal family for holomorphic functions as follows.

Definition 2.12. A family $\mathcal{F}$ of holomorphic functions f defined on a domain $D \subset \mathbb{C}$ is called *normal*, if every sequence f_n of function in $\mathcal{F}$ contains either a subsequence f_{nk} such that converges locally uniformly on D to a function f, or a subsequence f_{nk} which tends to ∞ (i.e. given $\delta > 0$, $|f_{nk}(z)| \geq \delta$ for $k \geq k_0$ and all $z \in A$) on every compact subset of D. The family $\mathcal{F}$ is called *normal at a point* $z_0 \in D$ if z_0 has a neighborhood U, such that $\mathcal{F}$ is normal in U.

From Definition 2.12 we state the following remarks.

(i) It is not difficult to show that the family $\mathcal{F}$ is normal in $D \subset \mathbb{C}$ if and only if the family $\mathcal{F}$ is normal at each point of D.

(ii) Definition 2.12 does not require that the limit functions of the convergent subsequences belong to the family $\mathcal{F}$. In particular, ∞ could be a limit function. Observe that it is the only difference between *normal* and *sequentially compact*; see [61] for a definition of sequentially compact.

(iii) If $\mathcal{F}$ is a normal family of analytic functions, then the limit function, say F, of f_{nk} is either an analytic function or identically ∞.

(iv) If $\mathcal{F}$ is a normal family of analytic functions and $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$, then $\mathcal{F}'$ need not to be normal. For instance, consider the family of functions $\mathcal{F}$ given by $f_n(z) = nz^2 - n^2$ defined on $\mathbb{C}$. The family $\mathcal{F}$ is normal, since $f_n \to \infty$ uniformly on every compact subset of $\mathbb{C}$. But $f_n'(z) = 2nz$ and $\mathcal{F}'$ is not normal, since $f_n'(z) \to \infty$ for $z \neq 0$ but $f_n \to 0$ for $z = 0$.

Using Definition 2.9 and the spherical derivative, denoted by $f^\sharp$; see Pages 10 and 17, respectively; we shall state Marty's theorem, which provides an equivalent condition of normality for a family of meromorphic functions. The proof can be consulted in [65].

Theorem 2.8 (Marty's Theorem). *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D \subset \hat{\mathbb{C}}$. Then $\mathcal{F}$ is called normal in D if and only if $\{f^\sharp\}_{f\in\mathcal{F}}$ is locally uniformly bounded in D.*

An important theorem for meromorphic functions which relates normality and equicontinuity is Arzelà-Ascoli's Theorem; see [26] for a proof.

Theorem 2.9 (Arzelà-Ascoli's Theorem). *A family of meromorphic functions $\mathcal{F}$ defined in a domain $A \subset \widehat{\mathbb{C}}$ is a normal family in A if and only if the family $\mathcal{F}$ is both equicontinuous and pointwise bounded in this set.*

The main results concerning to normal families that we will use in the thesis are the following two theorems due to Montel. The proofs can be consulted in [26] and [72].

Theorem 2.10 (Montel's Theorem). *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D \subset \widehat{\mathbb{C}}$. Let $a, b, c \in \widehat{\mathbb{C}}$ be pairwise distinct and suppose that $f(z) \neq a, b, c$ for all $z \in D$ and all $f \in \mathcal{F}$. Then $\mathcal{F}$ is normal in D.*

Theorem 2.11 (Montel's Theorem)**.** *Let $\mathcal{F}$ be a family of holomorphic functions in a domain $D \subset \widehat{\mathbb{C}}$. Suppose that there exists $M > 0$, such that $|f(z)| \leq M$, for all $z \in D$, and all $f \in \mathcal{F}$. Then $\mathcal{F}$ is normal in D.*

Observe that Theorem 2.11 is a special case of Theorem 2.10. In the thesis we will refer to the Theorem 2.11 as the Montel's theorem.

Chapter 3

Class of Transcendental Meromorphic Functions

The iteration of transcendental meromorphic functions with at least one not omitted pole was initially investigated by Baker, Kotus and Lü in [11], [12], [13] and [14] in the last decades of the last century, since then many mathematicians have been investigated the area. In this chapter, we define the iteration of transcendental meromorphic functions and some of their properties. Also we define the Fatou and Julia sets for transcendental meromorphic functions, the principal properties of these sets and the classification of the periodic components in the Fatou set. In the last section of this chapter we mention two classes of functions introduced by Eremenko and Lyubich in [38] and some results concerning these classes.

Following the notation in [16], the class of transcendental meromorphic functions, denoted by $\mathscr{M}$, is defined as follows:

$$\mathscr{M} = \{f : \mathbb{C} \to \widehat{\mathbb{C}} \mid f \text{ is transcendental meromorphic with at least one not omitted pole}\}.$$

Some examples of functions in class $\mathscr{M}$ are the following.

(1) The family $f_{\lambda,\mu}(z) = \lambda \sin(z) + \frac{\mu}{z},\ \lambda, \mu \in \mathbb{C} \setminus \{0\}$, has a single pole in $z = 0$, which is not an omitted value. For every $z \neq 0$, the family $f_{\lambda,\mu}$ is well defined and holomorphic.

(2) The family $f_\lambda(z) = \lambda \tan(z),\ \lambda \in \mathbb{C} \setminus \{0\}$ has an infinite countable set of not omitted poles of the form $w = \frac{(2k-1)\pi}{2}, k \in \mathbb{Z}$. Outside these set the family $f_\lambda(z)$ is well defined and holomorphic.

Observe that ∞ is the only essential singularity for functions in the class $\mathscr{M}$.

3.1 Iteration of functions in class $\mathscr{M}$

Let f be a function in class $\mathscr{M}$. The n-th iterate of f is defined as the composition of f with itself n times, that is, $\underbrace{f \circ f \ldots \circ f}_{n \quad times} = f^n$, for $n \in \mathbb{N}$ and $f^0 = Id$.

Observe that class $\mathscr{M}$ is not closed under composition, that is, if $f \in \mathscr{M}$ does not imply that $f^n \in \mathscr{M}$. For instance, take the family:

$$f_{\lambda,\mu}(z) = \lambda \sin(z) + \frac{\mu}{z}, \text{where } \lambda, \mu \in \mathbb{C} \setminus \{0\}.$$

The second iterate of $f_{\lambda,\mu}$ is given by

$$f^2_{\lambda,\mu}(z) = f_{\lambda,\mu} \circ f_{\lambda,\mu}(z) = \lambda \sin(\lambda \sin(z) + \frac{\mu}{z}) + \frac{\mu}{\lambda \sin(z) + \frac{\mu}{z}}.$$

Observe that $f^2_{\lambda,\mu}$ is analytic in $\mathbb{C}$ except in zero and in the set of solutions of the equation $\lambda \sin(z) + \frac{\mu}{z} = 0$, which is a countable set of essential singularities. Thus, ∞ is not the only essential singularity of $f^2_{\lambda,\mu}$. Therefore, $f^2_{\lambda,\mu}$ is not in class $\mathscr{M}$.

Definition 3.1. Let D be a domain in $\widehat{\mathbb{C}}$. The *rank* n of a point $z \in D$ is the greatest integer n, such that $f^n(z)$ is defined as $f(f^{n-1}(z))$ and $f^j(z) \in D$, for $j = 1, 2, ..., n-1$, while $f^n(z) \notin D$. The case when $n = \infty$ is allowed.

Remark 3.1. The points of rank at least n form an open set D_n, that is,

$$D = D_1 \supset D_2 \supset D_3 \supset ... \supset D_\infty = \bigcap_{n=1}^{\infty} D_n.$$

For a finite D_n is the domain of definition of f^n

We state the following result due to Radström, which will be used in the development of the following chapters; see [19] for a proof.

Lemma 3.1. *The set* $\widehat{\mathbb{C}} \setminus D_\infty$ *consists of* $\widehat{\mathbb{C}} \setminus D$ *together with all predecessor of points* $\widehat{\mathbb{C}} \setminus D$

To guarantee that the n-th iterate of f belongs to class $\mathscr{M}$, we have to consider the domain of the definition of this iterate as $D_{f^n} = \mathbb{C} \setminus \{z \in \mathbb{C} : f^i(z) = \infty, \ i \leq n\}$.

Definition 3.2. Let $f : D \subset \mathbb{C} \to \widehat{\mathbb{C}}$ be a function in class $\mathscr{M}$. For $z_0 \in D$, we define the following sets.

(a) The *forward orbit of* z_0 *under* f as the set $O^+(z_0) = \{f^n(z_0) : n \in \mathbb{N}\}$.

(b) The *backward orbit of* z_0 *under* f as $O^-(z_0) = \bigcup_{n=1}^{\infty}\{z \in \mathbb{C} : f^n(z) = z_0\}$.

(c) The *orbit of* z_0 *under* f as $O(z_0) = O^+(z_0) \cup \{z_0\} \cup O^-(z_0)$.

Observe that $O^{-}(\infty)$ is a infinite set. Indeed, $f^{-3}(\infty)$ is infinite as a consequence of the Picard's Little Theorem. Thus, the largest open set where all iterates of a function $f \in \mathscr{M}$ are well defined is $\widehat{\mathbb{C}} \backslash \overline{O^{-}(\infty)}$. We remark that by $f^{-1}(z)$ we mean all the branches of the inverse function.

Definition 3.3. A set D is *invariant* under f if $f(D) \subset D$. The set D is *completely invariant* if $f^{-1}(D) = D = f(D)$, where $f^{-1}(D)$ denotes the inverse function of $f(D)$.

Definition 3.4. Let $f \in \mathscr{M}$ and $S_p(f) = \bigcup_{k=0}^{p-1} f^k((SV) \setminus A_k(f))$, where SV is the set of the singular values of f; see Page 14; and $A_k(f) = \{z \in \mathbb{C} : f^k$ is not analytic at $z\}$. The set $P(f) = \bigcup_{p=1}^{\infty} S_p(f)$ is defined as the *post-singular set of* f.

Remark 3.2. The post-singular set of a function $f \in \mathscr{M}$ is formed by the singular values of f and their images under iteration, except at points where f is not analytic. Moreover, $sing(f^{-p}) \subset S_p(f)$ and $S_p(f) \subset S_{p+1}(f)$, where $sing(f^{-p})$ are the singularities of the inverse of the $p-$th iterate of f; see Page 14 for a definition.

Definition 3.5. Let $f \in \mathscr{M}$. A point $z_0 \in D \subset \mathbb{C}$ is called *Fatou exceptional point* if $O^{-}(z_0)$ is finite. The set of exceptional points is denoted by $E(f)$.

Observe that $E(f)$ can be defined in general for any holomorphic function. For instance, for the function $f(z) = ze^z$, the set $E(f) = \{0\}$.

The set $E(f)$ is an example of a dynamical property that is invariant under conjugation, that is, if f and g are conjugate by a homeomorphism ϕ, said $g \circ \phi = \phi \circ f$, then $E(g) = \phi(E(f))$.

The following theorem gives a characterization of the Fatou exceptional points depending of the domain D, where the analytic function is defined. The proof can be consulted in [17].

Theorem 3.1. *Let f be an analytic function defined on D. One of the following are satisfied.*

(i) If $D = \widehat{\mathbb{C}}$, then $|E| \leq 2$. If $|E| = 1$, then f is conjugate to a polynomial, with the exceptional point being mapped to ∞ by the conjugacy. If $|E| = 2$, then f is conjugate to $g(z) = z^d$ for some $d \in \mathbb{Z} \backslash \{0, \pm 1\}$, with the exceptional set being mapped to $\{0, \infty\}$ by the conjugacy.

(ii) If $D = \mathbb{C}$, then $|E| \leq 1$. If $|E| = 1$, then there exists $m \geq 0$ and an entire function h such that f is conjugate to $g(z) = z^m e^{h(z)}$, with the exceptional point being mapped to 0 by the conjugacy.

(iii) If $D = \mathbb{C}^* = \widehat{\mathbb{C}} \setminus \{0, \infty\}$, then $E = \emptyset$.

By using Picard Great's Theorem, it follows that $|\widehat{\mathbb{C}} \setminus (D \cup E(f)| \leq 2$ in all three cases, that is, there exists at most two Fatou exceptional points.

3.2 Periodic points and their classification

The fixed points of an analytic function have an important role in the study of the complex dynamics, in this thesis we will give a definition of fixed points.

Definition 3.6. Let $f \in \mathscr{M}$ and $D \subset \mathbb{C}$. We say that $z_0 \in D$ is a *periodic point of period* n *of* f when $f^n(z_0) = z_0$, $n \in \mathbb{N}$, and $f^i(z_0) \neq z_0$ for $i = 1, ..., n-1$. If $n = 1$, z_0 is called a *fixed point*. If z_0 is a periodic point of period n of f, that is, $\{z_0, f(z_0), \ldots, f^{n-1}(z_0)\}$ is called a *cycle* of periodic points.

To obtain fixed points of a function $f \in \mathscr{M}$, it is necessary to solve the equation $f(z) = z$. In general, it is not easy to find the solutions of the equation by simple calculations, thus the problem could be solved through numerical methods.

The periodic points have associated a complex number which is called the multiplier and its definition is as follows.

Definition 3.7. Let $f \in \mathscr{M}$ and $z_0 \neq \infty$ a periodic point of f of period n. The *multiplier of* z_0 is $\lambda = (f^n)'(z_0) = \prod_{j=0}^{n-1} f'(f^j(z_0))$. In the case when $z_0 = \infty$ the multiplier is defined by $\lambda = \frac{d}{dz}\frac{1}{f^n(\frac{1}{z})}\Big|_{z=0}$.

The classification of a periodic point $z_0 \in \mathbb{C}$ of period n of a function $f \in \mathscr{M}$ is given in terms of its multiplier.

(a) z_0 is *super-attracting point* if $\lambda = 0$.

(b) z_0 is *attracting* if $0 < |\lambda| < 1$.

(c) z_0 is *repelling* if $|\lambda| > 1$.

(d) z_0 is *rationally indifferent* if $|\lambda| = 1$ and λ is a root of unit.

(e) z_0 is *irrationally indifferent* if $|\lambda| = 1$ and λ is not a root of unit.

From the previous classification we make the following remarks.

(i) It follows from the chain rule that all point of a cycle of periodic points have the same multiplier.

(ii) If f and g are conjugate by a homeomorphism ϕ and if $z_0 \in \mathbb{C}$ is a periodic point of $f \in \mathscr{M}$, then $\phi(z_0)$ is a periodic point of g with the same period and multiplier of f.

3.3 Fatou and Julia sets for functions in class $\mathscr{M}$

As explained in the introduction, the main goal of complex dynamics is to understand the possible behaviors generated by the iterates of a holomorphic function. As it turns out, a family of functions $\mathcal{F}$, in this case a family of transcendental meromorphic functions, has associated two different dinamically sets: The set of initial conditions whose iterates is controlled (the Fatou set), and its complement, formed by chaotic orbits (the Julia set). We will start the section with the following formal definitions.

Definition 3.8. The *Fatou set of* $f \in \mathscr{M}$, denoted by $F = F(f)$, is defined by

$$F(f) = \{z \in \mathbb{C} : \{f^n\}_{n \geq 1} \text{ is well defined and normal in some neighborhood of } z\}.$$

The Julia set of $f \in \mathscr{M}$, denoted by $J = J(f)$, is the complement of the Fatou set, that is, $J = \mathbb{C} \setminus F(f)$.

We will state some properties of the Fatou and Julia sets, which were proved by Baker, Kotus and Lü between 1990 and 1991 for functions in the class $\mathscr{M}$; see [11], [12], [13] and [14] for results and more details concerning the Fatou and Julia sets.

Theorem 3.2. *Let $f \in \mathscr{M}$. The following statements hold.*

(i) The Fatou set is open, so the Julia set is closed.

(ii) The Julia set is perfect and non-empty set.

(iii) F and J are completely invariant, that is, $z \in F$ if and only if $f(z) \in F$ and $z \in J$ if and only if $f(z) \in J$.

(iv) $F(f^n) = F(f)$ and $J(f^n) = J(f)$, for all $n \in \mathbb{N}$.

(v) The Julia set is the closure of the set of all repelling periodic points.

The following theorem gives a characterization of the Julia set. The proof can be consulted in [17].

Theorem 3.3. *Let $f \in \mathscr{M}$. If $z_0 \in J(f)$ and $z_0 \notin E$, then $J(f) = \overline{O^-(z_0)}$.*

From Property (i) of Theorem 3.2, we assure that the Fatou set has connected components. The following classification of a connected component U in the Fatou set for $f \in \mathscr{M}$ is given as follows; see [12].

- If there exists some $n \in \mathbb{N}$ such that $f^n(U) \subset U$, then U is called a *periodic component* of $F(f)$. The minimum n is the period of the component U. In particular, if $n = 1$, then U is called an invariant component.
- If $f^m(U)$ is periodic, for some integer $m \geq 0$, then U is called a *pre-periodic component* of $F(f)$.

- If U is neither periodic nor pre-periodic, U is a *wandering domain.*

As we mention in Chapter 1, an open question proposed by P. Fatou concerned the existence of components in the Fatou set which are neither periodic nor pre-periodic was solved by D. Sullivan in [77]. He proved that this type of component, called wandering domains, cannot be exist for rational functions; see [77]. Nevertheless, for functions in class $\mathscr{M}$ there are examples of wandering components of any prescribe connectivity, either bounded or unbounded, which were constructed by using complex approximation in [12].

The classification of periodic components in the Fatou set for a function $f \in \mathscr{M}$ and it is given as follows; see [16] for details:

(a) U contains an attracting periodic point z_0 of period n. Then $f^{nk}(z) \to z_0$ for every z in U as k tends to infinity. U is called the *attracting component.*

(b) ∂U has a periodic point z_0 of period n and $f^{nk}(z) \to z_0$ for every z in U as k tends to infinity. Then $(f^n)'(z_0) = 1$. (For $z_0 = \infty$ we have $(g^n)'(0) = 1$ where $g(z) = \frac{1}{f(\frac{1}{z})}$.) In this case, U is called either a *Leau domain* or *parabolic component.*

(c) There exits an analytic homeomorphism $\varphi : U \to D$, where D is the unit disc, such that φ conjugates f^n with $e^{2\pi\alpha}z$ for some $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. In this case, U is called a *Siegel disc.*

(d) There exits an analytic homeomorphism $\varphi : U \to A$, where A is an annulus, $A = \{z : 1 < |z| < r\}$, $r > 1$, such that φ conjugates f^n with $e^{2\pi\alpha}z$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. In this case, U is called a *Herman ring.*

(e) There exists $z_0 \in \partial U$ such that $f^{nk}(z) \to z_0$ for every z in U as k tends to infinity, but $f^n(z_0)$ is not defined. In this case, U is called a *Baker domain.*

The following important results related to the components in the Fatou set for functions f in class $\mathscr{M}$ were proved by Baker, Kotus and Lü in [12] and [14].

Theorem 3.4. *If $f \in \mathscr{M}$ and U an invariant component in the Fatou set of f, then the component U has connectivity $1, 2$ or ∞. Moreover, if U is doubly connected, then U is a Herman ring.*

Theorem 3.5. *Let $f \in \mathscr{M}$ and U a component in the Fatou set of f, that is, $U \subset F(f)$.*

(i) If U is an attracting component or a parabolic component, then U contains a singular value of f.

(ii) If U is a Siegel disc or a Herman ring, then $\partial U \subset \overline{\mathcal{O}^+(SV(f))}$.

3.4 The classes $\mathcal{S}$ and $\mathcal{B}$

Following the ideas of Eremenko and Lyubich in [38] and [39] where they defined for transcendental entire functions, the classes of functions $\mathcal{S}$ and $\mathcal{B}$, called of finite type and bounded type, respectively, we state the following classes.

Definition 3.9. The class $\mathcal{S}$ is the set of functions $f \in \mathscr{M}$ of finite type, that is, the class $\mathcal{S}$ consists of functions $f \in \mathscr{M}$ for which the set of the singular values $SV(f)$ is finite.

Definition 3.10. The class $\mathcal{B}$ is the set of functions $f \in \mathscr{M}$ of bounded type, that is, the class $\mathcal{B}$ consists of functions $f \in \mathscr{M}$ for which the set of singular values $SV(f)$ is contained in a bounded set in $\mathbb{C}$.

Some examples of families of functions in class $\mathscr{M}$ that belong to the classes $\mathcal{S}$ and $\mathcal{B}$ are:

(i) The family $T_\lambda(z) = \lambda \tan(z), \lambda \in \mathbb{C} \setminus \{0\}$ has a finite set of singular values which are the asymptotic values $\{i\lambda, -i\lambda\}$. Thus family T_λ belongs to the class $\mathcal{S}$.

(ii) The family $h_\lambda(z) = \frac{\lambda \sin(z)}{z}, \lambda \in \mathbb{C} \setminus \{0\}$ has an asymptotic value in $z = 0$ and a infinite set of critical values w bounded, that is, $\{w : |w| \leq 1\}$. Therefore, the family $h_\lambda(z) = \frac{\lambda \sin(z)}{z}$ belongs to the class $\mathcal{B}$.

Baker, Kotus and Lü proved in [11] and [12] the following two theorems.

Theorem 3.6. *If $f \in \mathscr{M} \cap \mathcal{S}$, then f hast at most two completely invariant components in the Fatou set of f.*

Theorem 3.7. *If $f \in \mathscr{M} \cap \mathcal{S}$, then the Fatou set of f does not contain neither wandering domains nor Baker domains.*

The conclusion in the last theorem does not hold in general for $f \in \mathscr{M} \cap \mathcal{B}$. An example is the function $f(z) = \frac{1}{z} - e^{-z}$, since in [11] it was proved that f has a Baker domain of period 2, and $f \in \mathscr{M} \cap \mathcal{B}$. In this example, the critical values of f accumulate at 0, which is also one of the limits corresponding to the cycle of Baker domains.

Rippon and Stallard in [68] studied the properties of functions in the following class.

$$\mathcal{B}_n = \{f : f \text{ is a transcendental meromorphic function with } S_n(f) \text{ bounded}\},$$

where $S_n(f) = \cup_{j=0}^{n-1} f^j(SV(f) \setminus A_j(f))$, with $A_j(f) = \{z \in \mathbb{C} : f^j \text{ is not analytic at } z\}$.

Observe that $\mathcal{B}_1 = \mathcal{B}$. The main results obtained in [68] are the following:

Theorem 3.8. *If* $f \in \mathcal{B}_n$, *then there is no component of* $F(f)$ *in which* $f^{mn}(z) \to \infty$ *as* $m \to \infty$.

Corollary 3.1. *If* $f \in \mathcal{B}_n$, *then* f *has no Baker domains of period* n.

Rippon and Stallard in [69] defined a Baker wandering domain as a wandering component U of $F(f)$ such that, for n large enough, U_n is a bounded multiply connected component of $F(f)$ which surrounds 0, and $U_n \to \infty$ as $n \to \infty$. For this type of component in [62] the following theorem was proved.

Theorem 3.9. *If* $f \in \mathscr{M}$ *and* f *has a finite asymptotic value, then* f *not have Baker wandering domains.*

Chapter 4

A study of the dynamics of the family $f_{\lambda,\mu,z_0}(z) = \lambda\sin(z) + \frac{\mu}{z-z_0}$, where $\lambda \in \mathbb{C}\setminus\{0\}$, $\mu \in \mathbb{R}^+\setminus\{0\}$ and $z_0 \in \mathbb{R}$

The family $g_\lambda(z) = \lambda\sin(z), \lambda \in \mathbb{C}\setminus\{0\}$, has interesting dynamical properties which are related to the Fatou and Julia sets. For instance, in [34] Domínguez and Sienra gave a description of the Fatou set for values of the parameter λ inside the unit disc, obtaining that the Fatou set consists of a simple connected completely invariant component. For values of λ on the unit circle of parabolic type ($\lambda = e^{i2\pi\theta}, \theta = p/q, (p,q) = 1$), they proved that: (i) if q is even, there is one q-cycle of Fatou components. (ii) If q is odd, there are two $q-$cycles of Fatou components. Moreover, the Fatou components of such cycles are bounded.

In this chapter we are interested in studying the dynamics of a family which is a perturbation of g_λ adding a not omitted pole z_0 and a new parameter μ to g_λ, we obtain:

$$f_{\lambda,\mu,z_0}(z) = \lambda\sin(z) + \frac{\mu}{z-z_0}, \text{ where } \lambda \in \mathbb{C}\setminus\{0\}, \mu \in \mathbb{R}^+\setminus\{0\} \text{ and } z_0 \in \mathbb{R}.$$

The family of functions f_{λ,μ,z_0} belongs to class $\mathscr{M}$ and f_{λ,μ,z_0} has an isolated essential singularity at ∞; see Page 12 for details.

The study of the dynamics of the family f_{λ,μ,z_0} will be divided in two cases.

- Case I. When the pole $z_0 \neq 0$, which is not close to the parameter $\lambda \in \mathbb{C}\setminus\{0\}$, and $\mu \in \mathbb{R}^+\setminus\{0\}$ is sufficiently small.
- Case II. When the pole $z_0 = 0$, $\lambda \in \mathbb{C}\setminus\{0\}$ and $\mu \in \mathbb{R}^+\setminus\{0\}$ is sufficiently small.

The division in two cases is due to the fact that the family f_{λ,μ,z_0} has different geometric properties and dynamic behaviors with respect to the location of the pole, as we will see throughout this chapter.

4.1 Case I: Dynamics of the family f_{λ,μ,z_0} when $z_0 \neq 0$

In this section we will prove Theorem A and Theorem B stated in Section 1.2.

4.1.1 Theorem A

In [31] it was proved that for the family $f(z) = \lambda\sin(z) + \epsilon/(z-\pi)$, $0 < \lambda < 1$ and $\epsilon > 0$, for sufficiently small $\epsilon > 0$, the Fatou set is a single completely invariant attracting domain of infinite connectivity.

Domínguez, Vázquez and Montes de Oca in [36] proved that if the pole is of the form $z_0 = k\pi$ with $k \in \mathbb{Z}\setminus\{0\}$, then the result mentioned above is also valid. The result is stated as follows.

Theorem 4.1. *If λ, μ are real parameters such that $0 < |\lambda| < 1$, $\mu > 0$ sufficiently small, then the family $f_{\lambda,\mu}(z) = \lambda\sin(z) + \frac{\mu}{z-k\pi}, k \in \mathbb{Z}\setminus\{0\}$, has an attracting completely invariant component in the Fatou set which is multiply connected.*

A generalization of Theorem 4.1 is one of the main results in this section.

Theorem A. *If λ, μ, z_0 are real parameters such that $0 < |\lambda| < 1$, $\mu > 0$ sufficiently small and $|z_0| \geq |\lambda| + \frac{\pi}{2}$, then the family $f_{\lambda,\mu,z_0}(z) = \lambda\sin(z) + \frac{\mu}{z-z_0}$, has an attracting completely invariant component in the Fatou set, which is multiply connected.*

To prove Theorem A, we will need to prove some lemmas that describe certain properties of the family of functions f_{λ,μ,z_0}.

Preliminary Lemmas

Consider the family of one real variable $f_{\lambda,\mu,z_0}(x) = \lambda\sin(x) + \frac{\mu}{x-z_0}$ with real parameters $0 < |\lambda| < 1$, $\mu > 0$ sufficiently small and $|z_0| \geq |\lambda| + \frac{\pi}{2}$.

For different fixed values of λ, μ and z_0, there are different functions which are shown in the graph of Figure 4.1 in different colors.

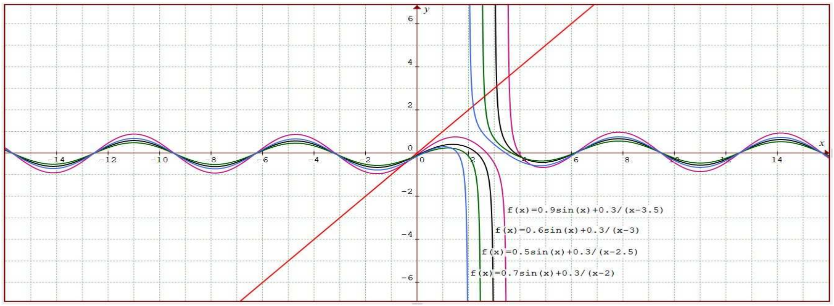

Figure 4.1: Real functions $f_{\lambda,\mu,z_0}(x)$ for different $(\lambda, \mu,\ z_0)$ with two fixed points

Observe that the graph in Figure 4.1 provides information about the existence of two real fixed points for λ, μ and z_0 given, so we will prove the following lemma.

Lemma 4.1. *If λ, μ, z_0 are real parameters such that $0 < |\lambda| < 1$, $\mu > 0$ sufficiently small and $|z_0| \geq |\lambda| + \frac{\pi}{2}$, then the family $f_{\lambda,\mu,z_0}(x) = \lambda \sin(x) + \frac{\mu}{x - z_0}$ has two real fixed points.*

Proof. Let λ, μ and z_0 be fixed and take $F(x) = f_{\lambda,\mu,z_0}(x) - x$. We shall consider only the case when $0 < \lambda < 1$ and $z_0 > \lambda + \frac{\pi}{2}$, since for the other case $-1 < \lambda < 0$ the proof is analogue.

The function $F(x)$ is well defined and continuous for each real value x different to z_0. Evaluating $F(x)$ at $x = 0$ we obtain $F(0) = \lambda \sin(0) + \frac{\mu}{0 - z_0} - 0 = -\frac{\mu}{z_0} < 0$, since $z_0, \mu > 0$.

Observe that $\lim\limits_{x \to -\infty} F(x) = \infty$, since the function $f(x) = \sin(x)$ is bounded, $\lim\limits_{x \to -\infty} \frac{\mu}{(x - z_0)} = 0$ and $\lim\limits_{x \to -\infty} -x = \infty$. Thus for all $M > 0$ exists $N < 0$ such that for every $x < N$, the function satisfies that $F(x) > M$. In particular, if we assume $M = 1$, then $F(x) > 1$. By the Intermediate Value Theorem, exists $t_1 \in \mathbb{R}$, with $t_1 < 0$, such that $F(t_1) = 0$. Therefore, $f_{\lambda,\mu,z_0}(t_1) = t_1$.

Now, using that $\lim\limits_{x \to z_0^+} \mu/(x - z_0) = \infty$ for μ and $z_0 \in \mathbb{R}^+$, we obtain that $\lim\limits_{x \to z_0^+} F(x) = \infty$. Thus for all $M > 0$ exists $\delta > 0$ such that for each $x \in (z_0, z_0 + \delta)$ satisfies that $F(x) > M$. In particular, if we suppose that $M = 1$, then exists $\delta > 0$ and $x_1 \in (z_0, z_0 + \delta)$ such that $F(x_1) > 1$.

By similar arguments, as above, $\lim\limits_{x \to \infty} F(x) = -\infty$. Therefore, for all $M < 0$ exists $N > 0$, such that for every $x > N$ the function F satisfies that $F(x) < M$. Now, assuming that $M = -1$, then exists $x_2 > N$ such that $F(x_2) < -1$. Thus, $F(x_1)F(x_2) < 0$. By the Intermediate Value Theorem, exists $t_2 \in (x_1, x_2)$ such that $F(t_2) = 0$. So, we conclude that $f_{\lambda,\mu,z_0}(t_2) = t_2$. Thus the lemma is proved.

■

The derivative of F is given by $F'(x) = \lambda \cos(x) - \frac{\mu}{(x - z_0)^2} - 1 < 0$ and considering the conditions of λ, μ and z_0 given in Lemma 4.1, there are only two real fixed points t_1 and t_2. By some easy calculations we obtain that: (i) one of the fixed points is attracting, so it belongs to the Fatou set, and it is the nearest to 0. (ii) The another fixed point is repelling, so it belongs to the Julia set; see Figure 4.1.

Following the idea in [31], we define a region **T** which contains the real axis, the parameters $\pm\lambda$ and the two fixed points t_1 and t_2 described above. Consider $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$ and define $\mathbf{S} = \{z \in \mathbb{C} : |\Im(z)| < \alpha\}$, where $\Im(z)$ denotes the imaginary part of the point z. Now, take $0 < \rho < \delta$ with $\delta = \min\{\alpha, \frac{|z_0 - \lambda|}{2}\}$ and

$0 < \mu \in \mathbb{R}$ sufficiently small, such that $\mu < \alpha\rho(1-|\lambda|)$, and consider the following set

$$\mathbf{T} = \mathbf{S} \cap \{z \in \mathbb{C} : |z - z_0| > \rho\},$$

which is shown in Figure 4.2.

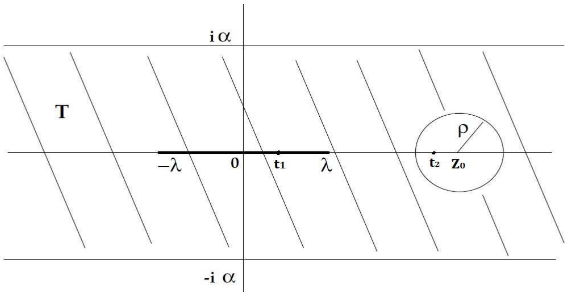

Figure 4.2: Region **T**

In what follows we will prove two lemmas related to the region **T**.

Lemma 4.2. *The family* $f_{\lambda,\mu,z_0}(z) = \lambda\sin(z) + \frac{\mu}{z-z_0}$, $\lambda, \mu, z_0 \in \mathbb{R}\setminus\{0\}$ *is uniformly bounded in the region* **T**.

Proof. Let $z = x + iy \in \mathbf{T}$. We consider two cases: (a) $y = 0$ and (b) $y \neq 0$.

(a) Observe that $1 - |\lambda| < 1$, since $0 < |\lambda| < 1$. Taking $\mu < \alpha\rho(1-|\lambda|)$ we have $\frac{\mu}{\rho} < \alpha(1-|\lambda|) < \alpha < 1$, so we obtain that $\frac{\mu}{\rho} < 1$. For $z = x \in \mathbf{T}$ we have

$$|f_{\lambda,\mu,z_0}(z) - \lambda\sin(x)| \leq \lambda|\sin(x) - \sin(x)| + \frac{\mu}{|z-z_0|} = \frac{\mu}{\rho} < \alpha < 1.$$

(b) For $z = x + iy \in \mathbf{T}$, $y \neq 0$, we have that $|y| < |\alpha| < 1$. Thus, we consider $0 < |\lambda| < \lambda' < 1$ such that $|\lambda||\cos(z)| < \lambda'$, for every $z \in \mathbf{S}$. Therefore,

$$\begin{aligned}|f_{\lambda,\mu,z_0}(z) - \lambda\sin(x)| &\leq |\lambda\sin(z) - \lambda\sin(x)| + \frac{\mu}{|z-z_0|}\\ &= |\lambda\sin(x+iy) - \lambda\sin(x)| + \frac{\mu}{|x+iy-z_0|}.\end{aligned}$$

Fixing $0 < y < \alpha$ and restricting to the function $\sin(x+iy)$ with respect to y in the interval $[0,\alpha]$, we know by the Mean-Value Theorem that there exists $t \in (0,\alpha)$. Thus for $z = x + iy \in \mathbf{T}$ we have

$$\begin{aligned}|\lambda\sin(x+iy) - \lambda\sin(x)| + \frac{\mu}{|x+iy-z_0|} &= |\lambda|\alpha|\cos(x+it)| + \frac{\mu}{|x+iy-z_0|}\\ &< \alpha\lambda' + \frac{\mu}{\rho}\\ &< \alpha\lambda' + \alpha(1-|\lambda|)\\ &< \alpha\lambda' + \alpha(1-\lambda') = \alpha < 1.\end{aligned}$$

In both cases (a) and (b), we have that $\lambda \sin(x) \in [-\lambda, \lambda]$, for $0 < \lambda < 1$ and $x \in \mathbb{R}$. Thus it implies that the image of the function f_{λ,μ,z_0} is bounded in a ball with center $\lambda \sin(x)$ and radius $r > 0$ with $r < \alpha < 1$, that is, $B(\lambda \sin(x), r) \subset \mathbf{T}$. We have proved that the family f_{λ,μ,z_0} is uniformly bounded in $\mathbf{T}$. ■

Lemma 4.3. *The singular values of the family* f_{λ,μ,z_0} *are contained in* $\mathbf{T}$.

Proof. We claim that the family f_{λ,μ,z_0} has no finite asymptotic values.

Indeed, let $\Gamma : [0, \infty) \to \widehat{\mathbb{C}}$ be any path which tends to ∞. Now, evaluating $\lim f(\Gamma(t))$, as $\Gamma(t) \to \infty$, we have $\lim\limits_{\Gamma(t)\to\infty} (\Gamma(t) - z_0)^{-1}\mu = 0$, thus $\lim\limits_{\Gamma(t)\to\infty} f_{\lambda,\mu,z_0}(\Gamma(t))$ should have a limit L if, and only if $\lambda \sin(\Gamma(t)) \xrightarrow{\Gamma\to\infty} L$, but it is satisfied only for $L = \infty$, since the family $\lambda \sin(z)$ has no finite asymptotic values. Thus, we have proved the family f_{λ,μ,z_0} has no asymptotic values, thus the singular values of f_{λ,μ,z_0} are critical values. In order to find them, we have to solve the following equation:

$$f'_{\lambda,\mu,z_0}(z) = \lambda \cos(z) - \frac{\mu}{(z - z_0)^2} = 0,$$

that is,

$$\lambda \cos(z) = \frac{\mu}{(z - z_0)^2}. \tag{3}$$

We consider the following two cases.

Case (a). When $z \in \mathbb{C}$ such that $|f'_{\lambda,\mu,z_0}(z)| = 0$ and $|z - z_0| \leq \rho$.

Case (b). When $z \in \mathbb{C}$ such that $|f'_{\lambda,\mu,z_0}(z)| = 0$ and $|z - z_0| \geq \rho$.

Case (a): Take $0 < \rho < \delta = \min\{\alpha, \frac{z_0-\lambda}{2}\}$ with $0 < \rho < \frac{1}{\cosh(\alpha)}$ and the points $z \in \mathbb{C}$ in the neighborhood $B(z_0, \rho)$.

Since $|f'_{\lambda,\mu,z_0}(z)| = 0$, it follows from Equation (3) that

$$|\lambda \cos(z)| = \left|\frac{\mu}{(z - z_0)^2}\right|. \tag{4}$$

Thus, multiplying Equation (4) by $|z - z_0|$ we obtain:

$$|z - z_0||\lambda \cos(z)| = \left|\frac{\mu}{z - z_0}\right| \leq \rho|\lambda \cos(z)|.$$

On the one hand, observe that

$$\begin{aligned} |\cos(z)|^2 &= [\cos(x)\cosh(y)]^2 + [\sin(x)\sinh(y)]^2 \\ &= \cos^2(x)\cosh^2(y) + \sin^2(x)[\cosh^2(y) - 1] \\ &= \cosh^2(y) - \sin^2(x) \leq \cosh^2(y). \end{aligned}$$

Therefore, $|\cos(z)|^2 \leq \cosh^2(y)$. Now, $\cosh(y) \geq 1 > 0$ and $|\cos(z)| > 0$, so we conclude that $|\cos(z)| \leq \cosh(y)$. For the complex function $\sin(z)$, the boundedness is similar. Thus, we obtain:

$$|\cos(z)| \leq \cosh(y) \quad \text{and} \quad |\sin(z)| \leq \cosh(y).$$

Thus,

$$\begin{aligned} |f_{\lambda,\mu,z_0}(z)| &\leq |\lambda\sin(z)| + \left|\frac{\mu}{z-z_0}\right| \\ &< |\lambda\sin(z)| + \rho|\lambda\cos(z)| \\ &\leq |\lambda|\cosh(y) + \rho\lambda' \\ &< |\lambda|/\rho + \alpha\lambda' < \alpha. \end{aligned}$$

The last inequality depends of an appropriate choice of ρ, that is, the size of the neighborhood of the pole z_0. Thus, we can conclude that $f_{\lambda,\mu,z_0} \in \mathbf{T}$.

Case (b): By hypothesis $|z_0| \geq |\lambda| + \frac{\pi}{2}$, thus $\frac{|z_0-\lambda|}{2} \geq \frac{\pi}{4}$, so we consider two cases: Case (i): $0 < \alpha < \frac{\pi}{4}$ and Case (ii) $\frac{\pi}{4} \leq \alpha < 1$. We recall that $0 < \alpha < 1$.

Case (i). Since $0 < \rho < \delta = \min\{\alpha, \frac{|z_o-\lambda|}{2}\}$, we obtain that $0 < \alpha^2\rho^2 < \rho^2$, thus

$$|\cos(z)| = \left|\frac{\mu}{\lambda(z-z_0)^2}\right| < \frac{|\mu|}{\lambda\rho^2} \leq \frac{|\mu|}{\lambda\alpha^2\rho^2}.$$

On the other hand

$$|\sin(z)| = |\sqrt{1-\cos^2(z)}| \leq \left|\sqrt{1-\frac{\mu^2}{\lambda^2\alpha^4\rho^4}}\right| \leq 1 + \frac{\mu^2}{\lambda^2\alpha^4\rho^4}.$$

Thus, $\sin(z) = \pm(1+\eta)$, where $|\eta| < \frac{\mu^2}{\lambda^2\alpha^4\rho^4}$ (if μ was chosen sufficiently small).

Now, since $|f_{\lambda,\mu,z_0}(z) - \lambda\sin(z)| = |\frac{\mu}{z-z_0}| < \frac{|\mu|}{\rho}$ and

$$|f_{\lambda,\mu,z_0}(z) - \lambda\sin(z)| = |f_{\lambda,\mu,z_0}(z) \pm \lambda(1+\eta)|,$$

we obtain

$$|f_{\lambda,\mu,z_0}(z) \pm \lambda| < \frac{\mu^2}{\lambda\alpha^4\rho^4} + \frac{|\mu|}{\rho}.$$

Since μ is sufficiently small, we have that the critical values (from critical points $z \in \mathbb{C}$ that are outside the neighborhood $B(z_0,\rho)$, $0 < \alpha < \frac{\pi}{4}$) are to a distance less than $\frac{\mu^2}{\lambda\alpha^4\rho^4} + \frac{|\mu|}{\rho}$ from λ and $-\lambda$, which are contained in the real axes in the region $\mathbf{T}$.

Therefore, we conclude that the critical values of f_{λ,μ,z_0} with the conditions given in Case (i) are contained in $\mathbf{T}$.

Case (ii): Consider $\rho \geq \frac{\pi}{4}$, where $\rho < \delta$ with $\delta = \min\{\alpha, \frac{|z_0-\lambda|}{2}\}$. Observe that

$$|\cos(z)| = \left|\frac{\mu}{\lambda(z-z_0)^2}\right| < \frac{|\mu|}{\lambda\rho^2} \leq \frac{16|\mu|}{\lambda\pi^2} < \frac{2|\mu|}{\lambda}.$$

Thus $\sin(z) = \pm(1+\eta)$, where $|\eta| < \frac{4\mu^2}{\lambda^2}$ (if μ was chosen sufficiently small).

Further, it implies that

$$|\sin(z)| = |\sqrt{1-\cos^2(z)}| \leq \left|\sqrt{1-\frac{4\mu^2}{\lambda^2}}\right| \leq 1 + \frac{4\mu^2}{\lambda^2}.$$

Now, since

$$|f_{\lambda,\mu,z_0}(z) - \lambda\sin(z)| = \left|\frac{\mu}{z-z_0}\right| < \frac{|\mu|}{\rho}$$

and

$$|f_{\lambda,\mu,z_0}(z) - \lambda\sin(z)| = |f_{\lambda,\mu,z_0}(z) \pm \lambda(1+\eta)|,$$

we obtain

$$|f_{\lambda,\mu,z_0}(z) \pm \lambda| < \frac{4\mu^2}{\lambda} + \frac{4|\mu|}{\pi}.$$

Since μ is sufficiently small, we have that the critical values, which are outside the neighborhood $B(z_0,\rho)$, are to a distance less than $\frac{4\mu^4}{|\lambda|} + \frac{4|\mu|}{\pi}$ from λ and $-\lambda$ that are in the real axis contained in the region $\mathbf{T}$. Therefore, we conclude again that the critical values are contained in $\mathbf{T}$.

From cases (i) and (ii) we conclude that all the critical values of the family f_{λ,μ,z_0} are contained in $\mathbf{T}$. Therefore, the lemma is proved. ■

To prove the following lemma we recall the class $\mathcal{B}$ is defined to be the set of functions $f \in \mathscr{M}$ of bounded type, that is, the class $\mathcal{B}$ consists of functions $f \in \mathscr{M}$ for which all singular values are contained in a bounded set in $\mathbb{C}$.

Lemma 4.4. *The family* $f_{\lambda,\mu,z_0}(z) = \lambda\sin(z) + \frac{\mu}{z-z_0}$, $\lambda, \mu, z_0 \in \mathbb{R}\setminus\{0\}$ *is in class* $\mathcal{B}$.

Proof. First we shall show that λ and $-\lambda$ are the only accumulations points of all singular values of f_{λ,μ,z_0}. By Lemma 4.3, the singular values of the family f_{λ,μ,z_0} are contained in $\mathbf{T}$ and they are of finite type. If w is a critical point of f_{λ,μ,z_0}, then

$$f'_{\lambda,\mu,z_0}(w) = \lambda\cos(w) - \frac{\mu}{(w-z_0)^2} = 0,$$

so w satisfies the equation

$$\lambda\cos(z) = \frac{\mu}{(w-z_0)^2}.$$

Using that the function $f(z) = \cos(z)$ is periodic and defining inductively a sequence of neighborhoods $(V_i)_{i\in\mathbb{N}}$ such that every V_k has center $(\frac{(2i-1)\pi}{2}, 0)$ and radius $\delta = \frac{|\lambda|}{i}$, $i \in \mathbb{N}$, we obtain that every V_i contains a point of the form $z_i = x_i + iy_i, i \in \mathbb{N}$ that satisfies $\lambda\cos x_i = y_i = \mu/(x_i - z_0)^2$. Thus f_{λ,μ,z_0} has critical points of the form $w = \frac{(2k-1)\pi}{2} + \epsilon_k$, for $k \in \mathbb{Z}\setminus\{0\}$, with $|\epsilon_k| \to 0$ as $k \to \infty$.

Evaluating w in the function f_{λ,μ,z_0} we have

$$\begin{aligned} f_{\lambda,\mu,z_0}(w) &= \lambda\sin(w) + \frac{\mu}{w-z_0} \\ &= \sqrt{\lambda^2 - \lambda^2\cos^2(w)} + \frac{\mu}{w-z_0} \\ &= \sqrt{\lambda^2 - \frac{\mu^2}{(w-z_0)^4}} + \frac{\mu}{w-z_0}. \end{aligned}$$

Observe that for different values of w, the images f_{λ,μ,z_0} are different, as a result of consider the critical points of f_{λ,μ,z_0} of the form $w = \frac{(2k-1)\pi}{2} + \epsilon_k$ for $k \in \mathbb{Z}\setminus\{0\}$, given above. Moreover, $f_{\lambda,\mu,z_0}(w) \to \pm\lambda$ as $w \to \infty$, so, λ and $-\lambda$ are accumulation points of all singular values of the family f_{λ,μ,z_0}. Taking a bounded set $B \subset \mathbb{C}$ such that it contains $\pm\lambda$, it follows that $f_{\lambda,\mu,z_0}(z) = \lambda\sin(z) + \frac{\mu}{z-z_0}$ belongs to class $\mathcal{B}$. ■

Proof of Theorem A

The proof of Theorem 4.1.1 follows from the idea in [31], but a different approach in giving to prove that the family of functions $f_{\lambda,\mu,z_0}(z) = \lambda\sin(z) + \frac{\mu}{z-z_0}$ does not have wandering domains nor Baker domains in the Fatou set. For this, we use the following theorem proved by Zheng in [81], which can be derived from Theorem 1 in [10] due to Baker; see Pages 23 and 27 for a definition of $P(f)$ and class $\mathcal{B}$, respectively.

Theorem 1. Let f be a transcendental meromorphic function in the class $\mathcal{B}$. If $\mathfrak{J}(f) \cap (P(f))'$ is finite and $(P(f))' \cap J_\infty \setminus \{\infty\} = \emptyset$, then f has no wandering domains, where J_∞ is defined to be $J_\infty = \cup_{n=0}^{\infty} f^{-n}(\infty)$.

Proof of Theorem A. We recall the statement of Theorem A.
Theorem A. *If λ, μ, z_0 are real parameters such that $0 < |\lambda| < 1$, $\mu > 0$ sufficiently small and $|z_0| \geq |\lambda| + \frac{\pi}{2}$, then the Fatou set of the family $f_{\lambda,\mu,z_0}(z) = \lambda\sin(z) + \frac{\mu}{z-z_0}$ has an attracting completely invariant component which is multiply connected.*

Let **T** be as in Lemma 4.2. The family f_{λ,μ,z_0} is uniformly bounded in the region **T**. By Montel's Theorem, f_{λ,μ,z_0} is normal in **T** which must belongs to an invariant component U of the Fatou set. Thus, by the Schwarz's Lemma, the family f_{λ,μ,z_0} belongs to a compact set $S \subset \mathbf{T}$ in which $f^n_{\lambda,\mu,z_0} \to t_1$, where t_1 is the finite attracting fixed point of the family f_{λ,μ,z_0}. Thus, there exists an attracting invariant component U in the Fatou set, such that it contains $\mathbf{T}$. In what follows we will prove the following four results.

1. There are neither Siegel discs nor Herman rings for f_{λ,μ,z_0}.

Proof. Let $A_j = \{z \in \mathbb{C} : f^j_{\lambda,\mu,z_0}(z) \text{ is not analytic at } z\}$, where $j \in \mathbb{N}$. Thus $A_0 = \emptyset$, $A_1 = \{\infty\}$ and $A_j = \{\infty\} \cup f^{-1}_{\lambda,\mu,z_0}(\infty) \cup ... \cup f^{j-1}_{\lambda,\mu,z_0}(\infty)$. Now, consider the following sets:

(a) $P(f_{\lambda,\mu,z_0}(z)) = \bigcup_{p=1}^{\infty} S_p(f_{\lambda,\mu,z_0}(z))$.

(b) $(P(f_{\lambda,\mu,z_0})(z))' = \{$points are either accumulation points of $P(f_{\lambda,\mu,z_0}(z))$ or singularities of some branch of the nth pre-image of $f^{-n}_{\lambda,\mu,z_0}(z)\}$.

Observe that $P(f_{\lambda,\mu,z_0}(z))$ is the set of singular values of f_{λ,μ,z_0} and their images under f_{λ,μ,z_0}, except at points where f_{λ,μ,z_0} is not analytic. Thus, it consists of critical values contained in the region $\mathbf{T}$. Therefore, the family f_{λ,μ,z_0} does not have Siegel discs or Herman rings since the boundary of a Siegel disc or a Herman ring must be contained in the set $P(f_{\lambda,\mu,z_0}(z)) \cup P((f_{\lambda,\mu,z_0}(z)))'$ due to Theorems 7.1.3 and 7.1.4 proved in [45] by M. Herring. ■

2. The component U is completely invariant.

Proof. By Lemma 4.3, all the finite singular values of f_{λ,μ,z_0} are contained in $\mathbf{T} \subset U$. Now, consider a point $z_1 \in U$ and a branch g of $f^{-1}_{\lambda,\mu,z_0}(z)$ such that $g(z_1) \in U$. For any $z_2 \in U$ and for any branch h from $f_{\lambda,\mu,z_0}(z)^{-1}$ at z_2, we can reach $h(z_2)$ by analytic continuation of g along a path γ from z_1 to z_2.

The path γ is homotopic to a path γ_1 in $\mathbb{C}\backslash SV$ from z_1 to z_2, and the continuation of g along γ_1 is h into z_2. But $g(\gamma_1)$ belongs to the Fatou set of f_{λ,μ,z_0} and thus, $g(\gamma_1) \subset U$. Therefore, U is completely invariant. ■

3. There are neither wandering components nor Baker domains.

Proof. Observe that the possible constant limits of sequences f^n_{λ,μ,z_0} in the components of the Fatou set $\mathfrak{F}(f_{\lambda,\mu,z_0}(z))$ are ∞ and the attracting fixed point t_1. Therefore, if there exist other Fatou components, they must be Baker domains; see Page 24 for a definition.

We claim that the accumulation points of the post-singular set does not accumulate in a finite pre-singularity. Indeed, the family f_{λ,μ,z_0} belongs to class $\mathcal{B}$ and by Lemma 4.3 all the finite critical values are contained in the attracting component, thus, we obtain that the Julia set does not intersect the set of all limit points of $P(f_{\lambda,\mu,z_0}(z))$.

So, observe that the family f_{λ,μ,z_0} satisfies the conditions of Theorem 1. Therefore, we conclude that f_{λ,μ,z_0} has no wandering domains.

Since the Julia set does not intersect the set of all limit points of $P(f_{\lambda,\mu,z_0}(z))$ and by Theorem 4 in [81], we conclude that f_{λ,μ,z_0} has no Baker domains.

■

4. The component U is attracting and multiply connected.

Proof. As we mentioned in Page 35, the family f_{λ,μ,z_0} belongs to a compact set $S \subset \mathbf{T}$ in which $f^n_{\lambda,\mu,z_0} \to t_1$, where t_1 is the finite attracting fixed point of the family f_{λ,μ,z_0}, so U is an attracting component. Now, since the pole z_0 does not belong to the Fatou set, any closed curve γ which winds around the pole z_0, contained in the Fatou set, is not homotopic to a point in the Fatou set. Thus, U is multiply connected. ■

Thus the proof of Theorem A is completed.

In the case when $\lambda = 1$, $\mu > 0$ sufficiently small and $|z_0| \geq 1 + \frac{\pi}{2}$ we state and prove the following corollary.

Corollary 4.1. *For $\lambda = 1$, $\mu > 0$ sufficiently small and $|z_0| \geq 1 + \frac{\pi}{2}$, the Fatou set of the family $f_{1,\mu,z_0}(z) = \sin(z) + \frac{\mu}{z-z_0}$ has an attracting completely invariant component which is multiply connected.*

Proof. Consider $\lambda = 1$, $\mu > 0$ with $\mu < (1-\alpha)\rho$ sufficiently small, $|z_0| \geq 1 + \frac{\pi}{2}$ and the region $\mathbf{T}$ as in Lemmas 4.2 and 4.3. Following the idea in Lemma 4.2, it is not difficult to prove that $|f_{1,\mu,z_0}(z) - \sin(z)| < \alpha$ for $z \in \mathbf{T}$. Thus, $f_{1,\mu,z_0}(z)$ is uniformly bounded in a ball with center 1 and radius r with $0 < r < \alpha < 1$, say $B(1,r)$, such that $B(1,r) \subset \mathbf{T}$. Then it follows as in the proof of Theorem A that f_{1,μ,z_0} belongs to a compact subset of $\mathbf{T}$. Thus, $\mathbf{T}$ belongs to an invariant attracting component U of the Fatou set. Now, evaluating $\lambda = 1$ in the proof of Lemma 4.3, we also can prove the following inequalities related to the boundedness of the critical values of the family f_{1,μ,z_0}.

(i) $|f_{1,\mu,z_0}(z)| < \cosh(\alpha) + \rho\cosh(\alpha)$, if $|f'_{1,\mu,z_0}(z)| = 0$ and $|z - z_0| < \rho$.

(ii) $|f_{1,\mu,z_0}(z) \pm 1| < \frac{\mu^2}{\alpha^4} + \frac{|\mu|}{\rho}$, if $|f'_{1,\mu,z_0}(z)| = 0$, $|z - z_0| \geq \rho$ and $0 < \alpha < \frac{\pi}{4}$.

(iii) $|f_{1,\mu,z_0}(z) \pm 1| < 2\mu^2 + \frac{4|\mu|}{\pi}$, if $|f'_{1,\mu,z_0}(z)| = 0$, $|z - z_0| \geq \rho$ and $\alpha \geq \frac{\pi}{4}$.

In all the cases $f_{1,\mu z_0} \in \mathbf{T} \subset U$ provided μ was chosen sufficiently small, which means that all the finite critical values of f_{1,μ,z_0} are contained in U. From the proof of Theorem A, it follows that U is not wandering and it is the only one component of the Fatou set which is completely invariant and multiply connected. ■

4.1.2 Theorem B

In this section we will prove Theorem B mentioned in Section 1.2. The result is stated as follows:

Theorem B. *If* $\lambda \in \mathbb{C}$ *is a complex parameter with* $0 < |\lambda| < \frac{1}{e+1}$, $\mu > 0$ *and* z_0 *are real parameters such that* $\mu > 0$ *sufficiently small and* $|z_0| \geq |\Re(\lambda)| + \frac{\pi}{2}$, *then the family* $f_{\lambda,\mu,z_0}(z) = \lambda \sin(z) + \frac{\mu}{z-z_0}$, *has an attracting completely invariant component in the Fatou set, which is multiply connected.*

We are interested in determine the complex values of the parameter $\lambda \in \mathbb{C} \setminus \{0\}$ for which Theorem B is valid. In order to do that, observe that we cannot strictly proceed as in the real case of Theorem A, since for it we used some results of analysis of a real variable.

To recognize the possible dynamic behavior and the location of fixed points when the parameter $\lambda \in \mathbb{C} \setminus \{0\}$ we will use some computational techniques: Plot of a complex function and the Newton's method.

First, we use the complex function viewer designed by David Bau, which visualizes any complex function of a conformal map by assigning a color in the complex plane according to the function's value at that point. For $f_{\lambda,\mu,z_0}(z) = \lambda \sin(z) + \frac{\mu}{z-z_0}$, we shall proceed with the following steps:

Step (1). We determine a initial coloring of the complex plane; see Figure 4.3. Also, we denote the unit disc in white color as reference.

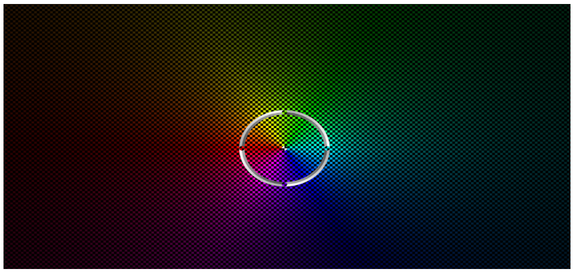

Figure 4.3: The colored complex plane

Step (2). We apply the function f_{λ,μ,z_0} and identify the modifications in the coloring of the complex plane. To apply the function f_{λ,μ,z_0} we fix the three parameters which satisfy the conditions of Theorem B only modifying the parameter $\lambda \in \mathbb{C}$ with $0 < |\lambda| < 1$. For instance, when $\lambda = 0.17 + 0.2i, \mu = 0.3$ and $z_0 = 3.2$, the graph of the function is shown in Figure 4.4.

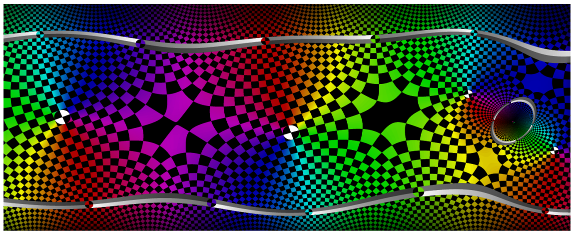

Figure 4.4: Function $f_{0.17+0.2i,0.3,3.2}(z)=(0.17+0.2i)\sin(z)+0.3/(z-3.2)$

By Newton's method we can to obtain a numerical approximation of the fixed points of $f_{\lambda,\mu,z_0}(z)=\lambda\sin(z)+\frac{\mu}{z-z_0}$ where $\lambda\in\mathbb{C}$, $0<|\lambda|<1$ and $\mu>0$, z_0 are real parameters μ with $|z_0|\geq|\lambda|+\frac{\pi}{2}$. The version of Newton's method for complex functions can be revised in [46] and [80]. It can be used to obtain that the family f_{0,μ,z_0} has two fixed points.

For instance, for the function $f_{0.17+0.2i,0.3,3.2}(z)=(02+0.5i)\sin(z)+0.3/(z-3.2)$, we obtain that $w\approx-0.0829877-0.0500925i$ is a fixed point of $f_{0.17+0.2i,0.3,3.2}$ whose modulus of derivative is less than one. Thus, we claim that there exists a point p close to 0 which is an attracting fixed point of $f_{0.17+0.2i,0.3,3.2}=(02+0.5i)\sin(z)+0.3/(z-3.2)$.

Observe that we obtained a region that is similar to the region **T** described in Lemmas 4.2 and 4.3, only need to define the new region **Q**, such that it contains all the singular values of f_{λ,μ,z_0} which accumulate in λ y $-\lambda$, for that, we need to expand the region **T** to include neighborhoods of the complex parameters λ and $-\lambda$. Moreover, the region **T** must preserve the property that the family f_{λ,μ,z_0} is locally bounded.

Let $\mathbf{Q}=\{z\in\mathbb{C}:|\Im(z)|<1+\frac{\sin(x)}{n}\}$ with n sufficiently large and take the ball with center z_0 and radius $r=\min\{1+\frac{\sin(x)}{x},\frac{|\Re(\lambda)-z_0|}{2}\}$, where $\Re(z)$ denotes the real part of z. Observe that $B(z_0,r)$ is contained in the region **Q**. As $0<r<1$ we consider $\mu>0$, such that $0<\frac{\mu}{1-\lambda(e+1)}<r<1$. It follows that $\lambda(e+1)+\frac{\mu}{r}<1$. We define the region $\mathbf{L}=\mathbf{Q}\cap\{z\in\mathbb{C}:|z-z_0|>r\}$; see Figure 4.5.

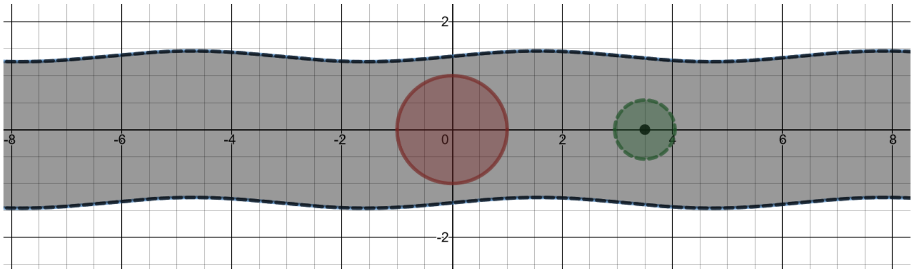

Figure 4.5: Region **L**

With the region **L** defined and the arguments given above we are ready to prove Theorem B.

Proof of Theorem B

First, we shall prove that if we consider $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \frac{1}{e+1}$, μ and $z_0 \in \mathbb{R} \setminus \{0\}$ such that $\mu > 0$ and $|z_0| \geq |\Re(\lambda)| + \frac{\pi}{2}$, where $\Re(z)$ denotes the real part of z, then the family $f_{\lambda,\mu,z_0}(z) = \lambda \sin(z) + \frac{\mu}{z-z_0}$ is uniformly bounded in the region **L**.

Let $r = \min\{1, \frac{|\Re(\lambda)-z_0|}{2}\}$, $B(z_0, r)$ a neighborhood contained in the region **Q** and $\mu > 0$ sufficiently small such that $0 < \frac{\mu}{1-\lambda(e+1)} < r < 1$. Consider the following two cases.

(a) For $z = x + iy \in \mathbf{L}$, $y = 0$, we have:
$|f_{\lambda,\mu,z_0}(z) - \lambda\sin(x)| \leq \lambda|\sin(x) - \sin(x)| + \frac{\mu}{z-z_0} = \frac{\mu}{r} < 1$, for μ sufficiently small.

(b) For $z = x + iy \in \mathbf{L}$, $y \neq 0$ and $|y| < 1 + \frac{\sin(x)}{n}$, we obtain:

$$\begin{aligned}
|f_{\lambda,\mu,z_0}(z) - \lambda\sin(x)| &= |\lambda\sin(x+iy) + \frac{\mu}{x+iy-z_0} - \lambda\sin(x)| \\
&= |\lambda\left(\sin(x)\cos(iy) + \cos(x)\sin(iy)\right) + \frac{\mu}{x+iy-z_0} - \lambda\sin(x)| \\
&= |\lambda\left(\sin(x)\cosh(y) + i\cos(x)\sinh(y)\right) + \frac{\mu}{x+iy-z_0} - \lambda\sin(x)| \\
&\leq \lambda|\sin(x)||\cosh(y)| + |\cos(x)||\sinh(y)| + \lambda|\sin(x)| + \frac{\mu}{|x+iy-z_0|} \\
&\leq \lambda\left(|\cosh(y)| + |\sinh(y)|\right) + \lambda + \frac{\mu}{|x+iy-z_0|} \\
&\leq \lambda\Big|\frac{e^y - e^{-y}}{2}\Big| + \Big|\frac{e^y - e^{-y}}{2}\Big| + \lambda + \frac{\mu}{|x+iy-z_0|} \\
&\leq \lambda(e^y + 1) + \mu\rho^{-1} < 1.
\end{aligned}$$

In both cases, we observe that $\lambda\sin(x) \in [-\lambda, \lambda]$. Since $0 < |\lambda| < \frac{1}{e+1}$, it follows that f_{λ,μ,z_0} is bounded in a ball with center in $\lambda\sin(x)$ and radius $r > 0$ with $0 < r < 1$, that is, $B(\lambda, r) \subset \mathbf{L}$, for some $n \in \mathbb{N}$ sufficiently large. Thus, the family f_{λ,μ,z_0} is uniformly bounded in **L**.

By Montel's Theorem, the family f_{λ,μ,z_0} is normal in **L**. So, the region **L** belongs to an invariant component U of the Fatou set. Thus, by the Schwarz's Lemma, the family f_{λ,μ,z_0} belongs to a compact set $K \subset \mathbf{L}$ in which $f^n_{\lambda,\mu,z_0} \to \alpha$, where α is the finite attracting fixed point of the family f_{λ,μ,z_0}. Then, there exists an attracting invariant component U in the Fatou set which contains **L**.

Now, we claim that all the singular values of $f_{\lambda,\mu,z_0}(z) = \lambda\sin(z) + \frac{\mu}{z-z_0}$, $k \in \mathbb{Z}\setminus\{0\}$ are contained in the component U. To prove this claim, we must observe that f_{λ,μ,z_0}

does not have finite asymptotic values. Indeed, let $\lambda \in \mathbb{R}$, such that $0 < |\lambda| < 1$ and consider $\Gamma : [0,\infty) \to \widehat{\mathbb{C}}$ any path which tends to ∞. Evaluating $\lim f(\Gamma(t))$ as $\Gamma(t) \to \infty$, we have that $\lim\limits_{\Gamma(t)\to\infty} \dfrac{\mu}{\Gamma(t) - z_0} = 0$, thus $\lim\limits_{\Gamma(t)\to\infty} f_{\lambda,\mu,z_0}(\Gamma(t))$ should have a limit L if, and only if $\lambda\sin(\Gamma(t)) \xrightarrow{\Gamma\to\infty} L$, but it is satisfied only for $L = \infty$, since the family $\lambda\sin(z)$ has no finite asymptotic values.

Thus, we consider the critical values of f_{λ,μ,z_0} given by the following equation:

$$f'_{\lambda,\mu,z_0}(z) = \lambda\cos(z) - \frac{\mu}{(z-z_0)^2} = 0.$$

We have the following two cases.
Case (a): $f'_{\lambda,\mu,z_0}(z) = 0$ and $|z - z_0| > t = \frac{\pi}{4}$. With these conditions observe that

$$|\cos(z)| = \left|\frac{\mu}{\lambda(z-z_0)^2}\right| < \frac{\mu}{|\lambda| t^2} < \frac{16\mu}{|\lambda|\pi^2} < \frac{2\mu}{|\lambda|},$$

Then,

$$|\sin(z)| = |\sqrt{1-\cos(z)^2}| \leq \left|\sqrt{1 - \frac{4\mu^2}{|\lambda|^2}}\right| \leq 1 + \frac{4\mu^2}{|\lambda|^2}.$$

Therefore, $|\sin(z)| \leq 1 + \frac{4\mu^2}{\lambda^2}$. Now, taking $|f_{\lambda,\mu,z_0}(z) - \lambda\sin(z)| = |\frac{\mu}{z-z_0}| < \frac{4\mu}{\pi}$ and $|f_{\lambda,\mu,z_0}(z) - \lambda\sin(z)| \leq |f_{\lambda,\mu,z_0}(z) \pm \lambda(1 + \frac{4\mu^2}{\lambda^2})|$ we obtain

$$|f_{\lambda,\mu}(z) \pm \lambda| < \frac{2\mu}{|\lambda|} + \frac{4\mu}{\pi}.$$

Thus, for $\mu > 0$ sufficiently small, we conclude that the critical values of f_{λ,μ,z_0} are contained in in ball with center λ and radius $R = \frac{2\mu}{|\lambda|} + \frac{4\mu}{\pi}$, which belongs to $\mathbf{L} \subset U$ for some n sufficiently large. Therefore, we conclude that the critical values of f_{λ,μ,z_0} are contained in U for $f'_{\lambda,\mu,z_0}(z) = 0$ and $|z - z_0| > t = \frac{\pi}{4}$.

Case (b): $f'_{\lambda,\mu,z_0}(z) = 0$ and $|z - z_0| \leq t = \frac{\pi}{4}$. Observe that

$$|\lambda\cos(z)| = |\mu(z-z_0)^2|.$$

It follows that

$$\frac{\mu}{\lambda e^{\frac{\pi}{4}}} \leq |(z-z_0)^2| = \left|\frac{\mu}{\lambda\cos(z)}\right| \leq \sqrt{2}\frac{\mu}{\lambda},$$

since $|\cos(z)| \geq |\cos(x)\cosh(y)| \geq \frac{1}{\sqrt{2}}$ and $|\cos(z)| < e^{|y|}$. Therefore, $|(z-z_0)^2| \leq \frac{\sqrt{2}\mu}{\lambda}$ and $|z - z_0| < 2^{\frac{1}{4}}\sqrt{\frac{\mu}{\lambda}}$. Analogously, we obtain that $|\sin(z)| < 2\sqrt{\frac{\mu}{\lambda}}$ and using that $|z - z_0| \leq \frac{\pi}{4}$, we have that $\mu|z-z_0|^{-1} < \sqrt{\mu}\sqrt{\lambda}e$.

Thus, we obtain that

$$|f_{\lambda,\mu,z_0}(z)| \leq |\lambda \sin(z)| + \left|\frac{\mu}{z - z_0}\right| < 2\sqrt{\mu}\sqrt{|\lambda|} + \sqrt{\mu}|\lambda|e.$$

Therefore, we obtain that the critical values of f_{λ,μ,z_0} is contained in a ball with center in 0 and radius $R = 2\sqrt{\mu}\sqrt{|\lambda|} + \sqrt{\mu}|\lambda|e$, which belongs to $\mathbf{L} \subset U$, for μ sufficiently small. Therefore, we conclude that the critical values of f_{λ,μ,z_0} are contained in U for $f'_{\lambda,\mu,z_0}(z) = 0$ and $|z - z_0| \leq t = \frac{\pi}{4}$. Thus, from the cases (a) and (b), we conclude that the set of critical values of the family f_{λ,μ,z_0} are contained in the component U.

The rest of the proof concerning the non-existence of other types of components in the Fatou set and that the component U is completely invariant and multiply connected follows from the proof of Theorem A.

4.1.3 Some Approximations of slices of the parameter space of the family f_{λ,μ,z_0}

The family f_{λ,μ,z_0} involves three parameters, λ, μ and z_0. Thus it is not possible to draw a parameter space. So, we define an approximation of a slice of the parameter space for the family f_{λ,μ,z_0} as follows:

$$\mathbb{M}_{\lambda,(\mu,z_0)} = \{\lambda \in \mathbb{C} : |f^n_{\lambda,(\mu,z_0)}(w)| \text{ is bounded}\},$$

where w is a critical point of f_{λ,μ,z_0}.

To draw an approximation of a slice in the space of parameters for the family f_{λ,μ,z_0}, we take (μ, z_0) fixed parameters and consider the values of the third parameter, in this case $\lambda \in \mathbb{C} \setminus \{0\}$, for which the sequence of iterates of a critical point of the family f_{λ,μ,z_0} is bounded.

In order to generate an approximation of the slice of the space of parameters $\mathbb{M}_{\lambda,(\mu,z_0)}$, we use the software *FractalStream*, which is a computational tool designed by Cornell University release under the BSD License. This software allows us to have many types of graphs related to iteration of functions. To obtain an approximation of the slice $\mathbb{M}_{\lambda,(\mu,z_0)}$ is necessary a critical point of the family f_{λ,μ,z_0}. So, we consider $z \in \mathbb{C}$ which satisfies the nonlinear equation $\lambda \cos(z) - \frac{\mu}{(z-z_0)^2} = 0$. Thus, considering different real values of the parameter $0 < |\lambda| < 1$, we calculate, by Newton's method [46], the critical points of f_{λ,μ,z_0}.

The software *FractalStream* is executed with the respective functions specifying an escape condition, that is, an upper bound, say $K \in \mathbb{R}^+$, and an arbitrary number of iterations, say $N \in \mathbb{N}$. The algorithm to generate an approximation of the slice

$\mathbb{M}_{\lambda,(\mu,z_0)}$ is as follows. If for each $\lambda \in \mathbb{C}\setminus\{0\}$, the modulus of f_{λ,μ,z_0}, evaluated at the critical point is less than the upper bound K, then it passes to the next iteration and returns to compare with K, and so recursively. If after N iterations $|f_{\lambda,\mu,z_0}(w)|$ evaluated at the approximation of critical point is still lower than K, then the software assigns a specific color, in this case the color assigned is black. If in some iteration the modulus of f_{λ,μ,z_0} exceeds K, the software assigns another color, creating a color range defined from the number of iteration which it surpass the upper bound K. The values of λ assigned to the black color will correspond to the parameters for which $|f^n_{\lambda,(\mu,z_0)}(w)|$ is less than the upper bound K, and they correspond to the approximation of the slice of the space of parameters $\mathbb{M}_{\lambda,(\mu,z_0)}$ of f_{λ,μ,z_0}.

An example of an approximation of a slice of the parameter space is the following.

Example. Taking $\mu = 0.3$ and $z_0 = 3.2$ in the family f_{λ,μ,z_0}, we have the following expression:

$$f_{\lambda,(0.3),(3.2)}(z) = \lambda\sin(z) + \frac{0.3}{z-(3.2)}.$$

The set $\mathbb{M}_{\lambda,((0.3),(3.2))}$ is shown in Figure 4.6.

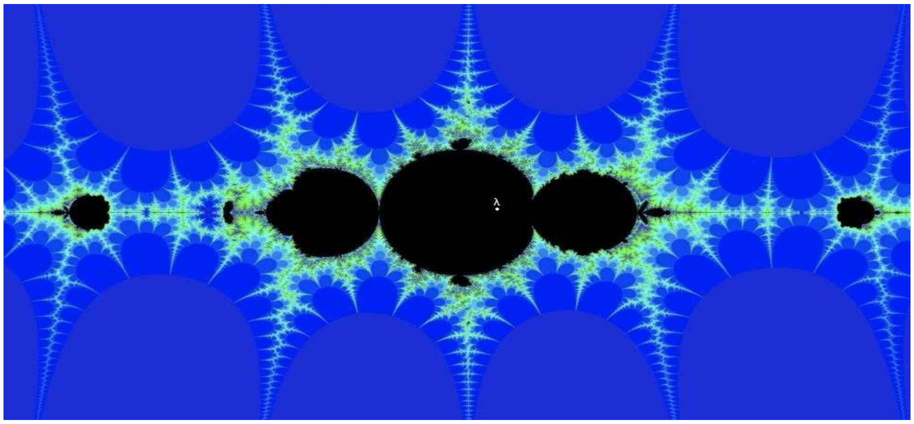

Figure 4.6: The set $\mathbb{M}_{\lambda,((0.3),(3.2))}$

From the approximation of the slice of the space of parameters and by using $|f_{\lambda,\mu,z_0}(x+iy)| = |f_{\lambda,\mu,z_0}(-x+iy)|$, we have that the set $\mathbb{M}_{\lambda,((0.3),(3.2))}$ in Figure 4.6 is symmetric with respect to the real axis.

For the fixed parameters μ and z_0 given above and $\lambda = \frac{1}{2}$, we use the Equation (3) in Lemma 4.3 to obtain an approximation of a critical point, $w \approx 4.91731$, for the family $f_{\lambda,(0.3),(3.2)}$. We recall that the family f_{λ,μ,z_0} has a countable infinite set of critical points for each parameter $\lambda \in \mathbb{C}\setminus\{0\}$. Figure 4.6 only shows an approximation of a slice of the space of parameters when the critical point is $w = 4.91731$, $\mu = 0.3$ and $z_0 = 3.2$.

There exists a dilemma related to the graph of an approximation of a slice of the space of parameters, that is, since the family f_{λ,μ,z_0} has a countable infinite set of critical points and we fix just one critical point to obtain a slice, we loose information about the whole dynamics of the family f_{λ,μ,z_0}. Observe that the slice only shows an approximate behavior of the parameters of the family around the critical point given. Therefore, we need to graph different approximations of slices of the space of parameters for different approximations of different critical points. We state some conjectures related to the properties of the approximations of the slices of the parameter space.

Conjecture 1. The set $\mathbb{M}_{\lambda,((0.3),(3.2))}$ is not bounded in the real axis.

Conjecture 2. If $0 < |\lambda| < 1$ and $0 < |\lambda'| < 1$ in $\mathbb{M}_{\lambda,((0.3),(3.2))}$, where the parameters μ and z_0 satisfy the conditions given in Theorems A and B, then the function f_{λ,μ,z_0} is quasi-conformal conjugate to f_{λ',μ,z_0}.

These conjectures are stated for the approximation of the slice $\mathbb{M}_{\lambda,((0.3),(3.2))}$, since for each set of parameters (μ, z_0) there are different $\mathbb{M}_{\lambda,((\mu),(z_0))}$. We must be careful, since we are dealing with a infinite set of approximations of slices and the results related to the dynamical planes are different. Thus, we state the following conjecture.

Conjecture 3. There exists a range of parameters (μ, z_0) with the conditions given in Theorems A and B, such that the approximations of the slices of the space of parameters are homeomorphic.

4.1.4 Example of an approximation of the Fatou and Julia sets that satisfy Theorem A

From the approximation of slice of the space of parameters generated in the Section 4.1.3, we give different values to the parameter $\lambda \in \mathbb{C} \setminus \{0\}$ and graph the approximations of the dynamical planes for the family f_{λ,μ,z_0}, that is, we plot the Fatou and Julia sets associated to a function of the family f_{λ,μ,z_0}.

Given the parameter $\lambda = \frac{1}{2}$, painted in white in the main bulb of the approximation of the slice $\mathbb{M}_{\lambda,((0.3),(3.2))}$; (see Figure 4.6), $\mu = 0.3$ and $z_0 = 3.2$, we obtain the function:

$$f_{(1/2),(0.3),(3.2)}(z) = \frac{1}{2}\sin(z) + \frac{0.3}{z - 3.2}.$$

Using the software *Fractalstream*, we graph an approximation of the Fatou and Julia sets of $f_{(0.5),(0.3),(3.2)}$; see Figures 4.7 and 4.8.

Figure 4.7: The orbit of a point in the Fatou set of the function $f_{(0.5),(0.3),(3.2)}$

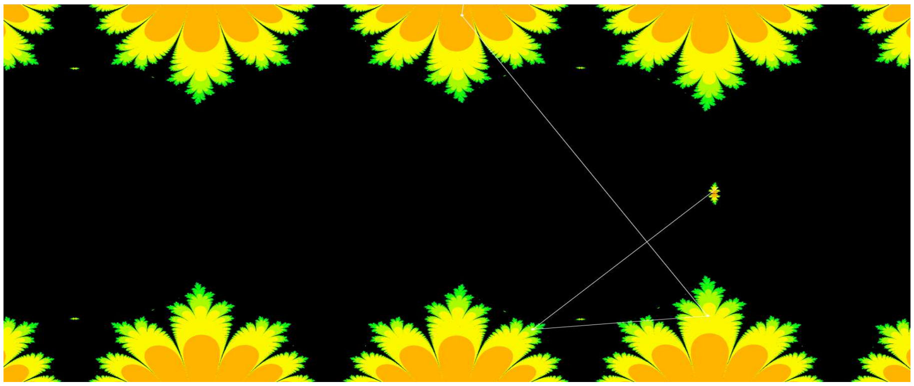

Figure 4.8: The orbit of a point in the Julia set of the function $f_{(0.5),(0.3),(3.2)}$

If we give a neighborhood of the pole z_0 with radius $\rho > 0$ such that $0 < \rho < \delta$, where $\delta = \min\{\alpha, \frac{|z_0-\lambda|}{2}\}$, $\mu < \alpha\rho(1 - |\lambda|)$ sufficiently small and the parameters λ, μ and z_0 satisfying the conditions of Theorem A. Thus, the Fatou set is a completely invariant, attracting and multiply connected component which is shown in black and the Julia set is shown in scale of yellow in Figures 4.7 and 4.8.

Figure 4.7 shows the orbit of a point, in white, which converges to the fixed point β in the Fatou set. Meanwhile, Figure 4.8 shows the orbit of a point, in white, close to the pole which vanishes in the yellow region. We recall that the Julia set is the boundary of the Fatou set, which in this case is not connected in $\mathbb{C}$.

4.1.5 Example of an approximation of the Fatou and Julia sets that satisfy Theorem B

For an example of Theorem B, we consider the complex parameters $\lambda = 0.17 + 0.2i$, $\mu = 0.3$ and $z_0 = 3.2$, which satisfy the conditions of Theorem B. Moreover, the parameter $\lambda = 0.17 + 0.2i$ is contained in the main bulb of the approximation of the slice

$\mathbb{M}_{\lambda,((0.3),(3.2))}$; see Figure 4.3. The Fatou set is shown in black and the Julia set is shown in scale of yellow. Observe that the Fatou set is a completely invariant, attracting and multiply connected component, meanwhile the Julia set is totally disconnected; see Figure 4.9.

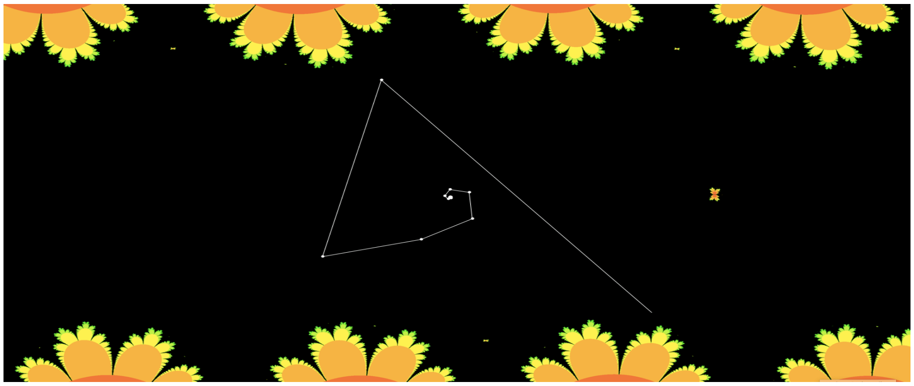

Figure 4.9: The orbit of a point, in white, in the Fatou set, in black, of $f_{(0.17+0.2i),(0.3),(3.2)}$

Conjecture 4. It is possible to extend Theorem B for different complex parameters λ, μ and z_0 with an appropriate boundedness of the parameters.

4.2 Case II: Dynamics of the family f_{λ,μ,z_0} when $z_0 = 0$

In this section we will investigate some dynamical properties of the family given in (2); see Page 6, that is,

$$f_{\lambda,\mu}(z) = \lambda \sin(z) + \frac{\mu}{z},$$

where $\lambda \in \mathbb{C} \setminus \{0\}, \mu \in \mathbb{R}^+ \setminus \{0\}$.

4.2.1 Some geometrical and dynamical properties of $f_{\lambda,\mu}$

Functions which belong to the family $f_{\lambda,\mu}$ have some symmetries due to the properties of trigonometric complex functions and the complex inversion respect to the origin. These properties are stated in the following lemma.

Lemma 4.5. *For the function* $f_{\lambda,\mu}(x+iy) = \lambda \sin(x+iy) + \frac{\mu}{x+iy}$, *with* $\lambda \in \mathbb{C} \setminus \{0\}$, $\mu \in \mathbb{R}^+ \setminus \{0\}$ *fixed, the following conditions are satisfied.*

(a) $f_{\lambda,\mu}(x-iy) + f_{\lambda,\mu}(x+iy) = 2\Re(f_{\lambda,\mu}(x+iy))$, *where* $\Re$ *is the real part of* $f_{\lambda,\mu}$.

(b) $f_{\lambda,\mu}(-x+iy) + f_{\lambda,\mu}(x+iy) = 2\Im(f_{\lambda,\mu}(x+iy))$, *where* $\Im$ *is the imaginary part of* $f_{\lambda,\mu}$.

(c) $f_{\lambda,\mu}(-x - iy) = -f_{\lambda,\mu}(x + iy)$.

(d) $|f'_{\lambda,\mu}(-x + iy)| = |f'_{\lambda,\mu}(x + iy)|$.

Proof. We shall prove only Property (c), since the proofs of the other properties are analogue. Without loss of generality, we consider $\lambda, \mu \in \mathbb{R} \setminus \{0\}$ and $z = x + iy \in \mathbb{C}$. Evaluating $z = x + iy$ in the family $f_{\lambda,\mu}$ and separating its real and imaginary parts, we obtain the fallowing:

$$\begin{aligned} f_{\lambda,\mu}(x+iy) &= \lambda \sin(x+iy) + \frac{\mu}{x+iy} \\ &= \lambda\left(\sin(x)\cos(iy) + \sin(iy)\cos(x)\right) + \frac{\mu}{x+iy} \\ &= \lambda \sin(x)\cosh(y) + i\lambda \sinh(y)\cos(x) + \frac{\mu x - i\mu y}{x^2+y^2} \\ &= \lambda \sin(x)\cosh(y) + \frac{\mu x}{x^2+y^2} + i\left(\lambda \sinh(y)\cos(x) - \frac{\mu y}{x^2+y^2}\right). \end{aligned}$$

On the other hand, evaluating $-z = -x - iy$ in $f_{\lambda,\mu}$, we have:

$$\begin{aligned} f_{\lambda,\mu}(-x-iy) &= \lambda \sin(-x-iy) + \frac{\mu}{-x-iy} \\ &= \lambda\left(\sin(-x)\cos(-iy) + \sin(-iy)\cos(-x)\right) + \frac{\mu}{-x-iy} \\ &= -\lambda \sin(x)\cosh(y) - \frac{\mu x}{x^2+y^2} + i\left(\frac{\mu y}{x^2+y^2} - \lambda \sinh(y)\cos(x) -\right). \\ &= -\left(\lambda \sin(x)\cosh(y) + \frac{\mu x}{x^2+y^2}\right) - i\left(\lambda \sinh(y)\cos(x) - \frac{\mu y}{x^2+y^2}\right) \\ &= -f_{\lambda,\mu}(x+iy). \end{aligned}$$

■

We recall that if z_0 is a periodic point of $f_{\lambda,\mu}$, its multiplier is $\lambda = (f^n_{\lambda,\mu})'(z_0)$. Observe that from (d) in Lemma 4.5, that we can reduce the study of some dynamical results of the family $f_{\lambda,\mu}$ when $z = x + iy \in \mathbb{C}$ with $\Re(z) \geq 0$.

We can consider the same property for the points obtained by reflecting with respect to the imaginary axis, since some results are based on the modulus of the multiplier of the fixed points. This property is stated in the following lemma.

Lemma 4.6. *If* $\lambda, \mu \in \mathbb{R}^+ \setminus \{0\}$ *and* μ *sufficiently small, then* $f_{\lambda,\mu}(z) = \lambda \sin(z) + \frac{\mu}{z}$ *maps a region* R *with negative real part onto the semiplane given by* $\Re(z) < 0$.

Proof. Consider λ, $\mu \in \mathbb{R}^+ \setminus \{0\}$ and let $z \in R := \{z \in \mathbb{C} : -\pi < x < 0\}$. The function $f_{\lambda,\mu}(z) = \lambda \sin(z) + \frac{\mu}{z} = u(x, y) + i(x, y)$ can be written as follows:

$$\begin{aligned} f_{\lambda,\mu}(z) &= \lambda \sin(z) + \frac{\mu}{z} \\ &= \lambda \sin(x + iy) + \frac{\mu}{x + iy} \\ &= \lambda(\sin(x)\cos(iy) + \cos(x)\sin(iy)) + \frac{\mu}{x + iy} \\ &= \lambda \sin(x)\cosh(y) + i\lambda \cos(x)\sinh(y) + \frac{\mu x}{x^2 + y^2} - i\frac{\mu y}{x^2 + y^2}. \end{aligned}$$

Thus, we obtain $u(x, y) = \lambda \sin(x)\cosh(y) + \frac{\mu x}{x^2+y^2}$. Now, since $z \in R$, we have $-\pi < \Re(z) < 0$, thus both $\lambda \sin(x)\cosh(y)$ and the quotient $\frac{\mu x}{x^2+y^2}$ are negative for $\mu > 0$ sufficiently small. Therefore, $\Re(f_{\lambda,\mu}(z)) < 0$. Moreover, by using similar arguments, we obtain that if $z \in \mathbb{C}$, with $0 < \Re(z) < \pi$, then $\Re(f_{\lambda,\mu}(z)) > 0$ is satisfied. ■

The following two lemmas involve the imaginary axis. The first result states the conditions for which the imaginary axis is invariant under $f_{\lambda,\mu}$. The second result is concerning to the dynamics of imaginary axis in the family $f_{\lambda,\mu}$ by taking $\lambda \in \mathbb{C} \setminus \{0\}$ with $|\lambda| > 1$ and $\mu \in \mathbb{R}^+ \setminus \{0\}$ sufficiently small.

Lemma 4.7. *For $\lambda, \mu \in \mathbb{R} \setminus \{0\}$ the imaginary axis is invariant under f_{λ,μ,z_0}.*

Proof. Take $z = iy$, where $y \in \mathbb{R}$. Making some basic calculations we obtain:

$$f_{\lambda,\mu}(z) = f_{\lambda,\mu}(iy) = \lambda \sin(iy) + \frac{\mu}{iy} = \lambda i \sinh(y) - \frac{i\mu}{y^2} = i\left(\lambda \sinh(y) - \frac{\mu}{y^2}\right).$$

Thus $\lambda sinh(y) - \frac{\mu}{y^2}$ is a real function of y. Therefore we conclude the result. ■

Lemma 4.8. *If $\lambda \in \mathbb{C} \setminus \{0\}$ with $|\lambda| \geq 1$ and $\mu \in \mathbb{R}^+ \setminus \{0\}$ sufficiently small, then the Julia set of the family $f_{\lambda,\mu}(z) = \lambda \sin(z) + \frac{\mu}{z}$ contains the imaginary axis.*

Proof. Consider a neighborhood U with center $\zeta = iy_1$, with $y \in \mathbb{R} \setminus \{0\}$. We shall prove that in U the family $f_{\lambda,\mu}$ is not pointwise bounded. Take $w = iy_2$, $y_2 \in \mathbb{R} \setminus \{0\}$, a pure imaginary point and $\zeta \neq w$ such that $y_1 y_2 > 0$. The distance between the images $f_{\lambda,\mu}(\zeta)$ and $f_{\lambda,\mu}(w)$, we obtain:

$$\begin{aligned} |f_{\lambda,\mu}(\zeta) - f_{\lambda,\mu}(w)| &= |\lambda \sin(\zeta) + \frac{\mu}{z} - (\lambda \sin(w) + \frac{\mu}{z})| \\ &= |\lambda \sin(iy_1) + \frac{\mu}{iy_1} - (\lambda \sin(iy_2) + \frac{\mu}{iy_2})| \\ &= |i\lambda \sinh(y_1) - \frac{i\mu}{y_1} - i\lambda \sinh(y_2) + \frac{i\mu}{y_2}| \\ &= |i||\lambda(\sinh(y_1) - sinh(y_2)) + \mu(\frac{1}{y_2} - \frac{1}{y_1})| \\ &\geq |i||\lambda(\sinh(y_1) - \sinh(y_2))|. \end{aligned}$$

By the Mean Value Theorem, for the real function $g(x) = \sinh(x)$ in the interval $[y_1, y_2]$, there exists $y_3 \in (y_1, y_2)$ such that

$$|\lambda||\sinh(y_1) - \sinh(y_2)| = |\lambda|\cosh(y_3)|y_2 - y_1| > |\lambda||y_2 - y_1| > |y_2 - y_1|.$$

It follows that the family $f_{\lambda,\mu}(z) = \lambda\sin(z) + \frac{\mu}{z}$ has a repelling behavior in the imaginary axis, that is, $f_{\lambda,\mu}$ takes values of arbitrarily large magnitude under iteration, hence the family $f_{\lambda,\mu}$ is not pointwise bounded. Thus, if we take an arbitrary neighborhood U in the imaginary axis and the sets A as the positive imaginary axis and B as the negative imaginary axis, then we obtain that any point $y \in A$ under iteration tends to ∞, the same for $y \in B$. By using Theorem 2.9, we conclude that the family $f_{\lambda,\mu}$ is not normal in the imaginary axis and therefore, the Julia set of the family $f_{\lambda,\mu}(z) = \lambda\sin(z) + \frac{\mu}{z}$ contains the imaginary axis. ■

In the following section we study the points such that determines the possible dynamics in the Fatou set of the family $f_{\lambda,\mu}(z) = \lambda\sin(z) + \frac{\mu}{z}$ under certain conditions of the parameters.

4.2.2 Fixed points and singular values of $f_{\lambda,\mu}$

In this section we will describe the fixed points, the asymptotic values and the critical values of the family $f_{\lambda,\mu}(z) = \lambda\sin(z) + \frac{\mu}{z}$ for different values of the parameters $\lambda \in \mathbb{C}\setminus\{0\}$ and $\mu \in \mathbb{R}^+\setminus\{0\}$ sufficiently small.

(I) Fixed Points of $f_{\lambda,\mu}$

We recall that z_0 is called a fixed point of a function f if satisfies that $f(z_0) = z_0$. Thus, for the family $f_{\lambda,\mu}$ we need to solve the equation $\lambda\sin(z) + \frac{\mu}{z} = z$, that is:

$$\lambda\sin(z) + \frac{\mu}{z} - z = 0. \tag{5}$$

Finding the roots of the non-linear Equation (5) is not an easy task. In particular, we observe the following cases:

1. For the parameters $\lambda \in \mathbb{C}\setminus\{0\}$ and $\mu = \frac{\pi^2}{4} - \lambda\frac{\pi}{2}$, the family $f_{\lambda,\mu}$ has a fix point in $z = \frac{\pi}{2}$. Indeed, evaluating $f_{\lambda,\frac{\pi^2}{4}-\lambda\frac{\pi}{2},0}(z)$ at $z = \frac{\pi}{2}$ and we have

$$f_{\lambda,\frac{\pi^2}{4}-\lambda\frac{\pi}{2},0}\left(\frac{\pi}{2}\right) = \lambda\sin\left(\frac{\pi}{2}\right) + \frac{\frac{\pi^2}{4} - \lambda\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{\pi}{2}.$$

2. If $\lambda \in \mathbb{C}\setminus\{0\}$ and $\mu = \left(\frac{(2k-1)\pi}{2}\right)^2 - \lambda\left(\frac{(2k-1)\pi}{2}\right)$, $k \in \mathbb{Z}$, then $z = \frac{(2k-1)\pi}{2}, k \in \mathbb{Z}\setminus\{0\}$, is a fixed point of $f_{\lambda,\mu}$.

In the case when λ and μ are real parameters we state the following proposition.

Proposition 4.9. *Let $z = x \in \mathbb{R}$ and $\lambda, \mu \in \mathbb{R} \setminus \{0\}$, with $\mu > 0$ sufficiently small. The family $f_{\lambda,\mu}(z) = \lambda \sin(z) + \frac{\mu}{z}$ has two real fixed points, which are symmetric with respect to the imaginary axis.*

Proof. Let λ, μ be real fixed parameters and define

$$h(x) = f_{\lambda,\mu}(x) - x.$$

Taking $x_1 > 0$ with $x_1 \in \left(\frac{|\lambda| - \sqrt{|\lambda|^2 + 4\mu}}{2}, \frac{|\lambda| + \sqrt{|\lambda|^2 + 4\mu}}{2}\right)$, we obtain that $h(x_1) > 0$, since

$$h(x_1) = \lambda \sin(x_1) + \frac{\mu}{x_1} - x_1 \leq |\lambda| + \frac{\mu}{x_1} - x_1 = \frac{|\lambda| x_1 - \mu - {x_1}^2}{x_1}.$$

Thus, considering x_1 between the roots of the polynomial $p(x) = -x^2 + |\lambda| x - \mu$, we have that $p(x) > 0$ which implies $h(x_1) > 0$.

Taking $x_2 > 0$, with $x_2 > \frac{|\lambda| + \sqrt{|\lambda|^2 + 4\mu}}{2}$, we obtain:

$$h(x_2) = \lambda \sin(x_2) + \frac{\mu}{x_2} - x_2 \leqslant |\lambda| + \frac{\mu}{x_2} - x_2 = \frac{|\lambda| x_2 - \mu - {x_2}^2}{x_2}.$$

Thus, $p(x) < 0$ which implies that $h(x_2) < 0$. Further, by the Intermediate Value Theorem, there exists x_λ such that $f_{\lambda,\mu}(x_\lambda) = x_\lambda$. Therefore, we conclude that $x_\lambda \in \mathbb{R}^+$ is a fixed point of the family $f_{\lambda,\mu}$. If we choose $x_1, x_2 \in \mathbb{R}^-$, by analogous arguments as above, we obtain that $-x_\lambda \in \mathbb{R}^-$ is a real fixed point of $f_{\lambda,\mu}$. ■

From Proposition 4.9, we observe that there exist functions in $f_{\lambda,\mu}$ for λ, μ given, which have more than two real fixed points. Furthermore, for different conditions of the parameters λ and μ, there are functions in $f_{\lambda,\mu}$ without real fixed points. In the thesis we will focus in functions of the family $f_{\lambda,\mu}$ which have at least two real fixed points.

(II) Singular values of $f_{\lambda,\mu}$

We claim that the family $f_{\lambda,\mu}(z) = \lambda \sin(z) + \frac{\mu}{z}$, where $\lambda, \mu \in \mathbb{C} \setminus \{0\}$, has not finite asymptotic values.

Indeed, considered any path $\Gamma : [0, \infty) \to \widehat{\mathbb{C}}$ which tends to ∞ and evaluating $\lim f_{\lambda,\mu}(\Gamma(t))$ as $\Gamma(t) \to \infty$, we use the fact that $\lim\limits_{\Gamma(t) \to \infty} f_{\lambda,\mu}(\Gamma(t))$ has a limit L if, and only if $\lambda \sin(\Gamma(t)) \xrightarrow{\Gamma \to \infty} L$, but it is satisfied only for $L = \infty$, since the family $\lambda \sin(z)$ has no finite asymptotic values. Therefore, $f_{\lambda,\mu}$ has not finite asymptotic values.

To obtain critical values of $f_{\lambda,\mu}$, we take the derivative of $f_{\lambda,\mu}$, that is,

$$f'_{\lambda,\mu}(z) = \lambda\cos(z) - \frac{\mu}{z^2}, \lambda, \mu \in \mathbb{C}\setminus\{0\}.$$

We calculate the points $z \in \mathbb{C}$, such that $f'_{\lambda,\mu}(z) = 0$, that is:

$$f'_{\lambda,\mu}(z) = \lambda\cos(z) - \frac{\mu}{z^2} = 0. \tag{6}$$

Now, we need to find the roots of the non-linear Equation (6). We use numerical approximations of the critical points. Nevertheless, we obtain a characterization of the critical values of $f_{\lambda,\mu}$ when we take λ and μ non-zero real parameters.

We state the following proposition; see Page 25 for a definition of the class $\mathcal{B}$.

Proposition 4.10. *The family* $f_{\lambda,\mu}(z) = \lambda\sin(z) + \frac{\mu}{z}$, *with* $\lambda, \mu, \in \mathbb{R}\setminus\{0\}$, *belongs to class* $\mathcal{B}$.

Proof. The proof is analogue to Lemma 4.4. The family f_{λ,μ,z_0} has no finite asymptotic values, thus, the singular values of the family $f_{\lambda,\mu}$ are only finite-critical values. We recall that if w is a critical point of $f_{\lambda,\mu}$, then

$$f'_{\lambda,\mu}(w) = \lambda\cos(w) - \frac{\mu}{w^2} = 0,$$

that is, if w is a critical point of $f_{\lambda,\mu}$, then w satisfies the equation

$$\lambda\cos(w) = \frac{\mu}{w^2}.$$

Evaluating w in the function $f_{\lambda,\mu}$ we obtain

$$\begin{aligned} f_{\lambda,\mu}(w) &= \lambda\sin(w) + \frac{\mu}{w} \\ &= \sqrt{\lambda^2 - \frac{\mu^2}{w^4}} + \frac{\mu}{w}. \end{aligned}$$

The family $f_{\lambda,\mu}$ has real critical points w of the form $w = \frac{(2k+1)\pi}{2} + \epsilon_k$, for $k \in \mathbb{Z}\setminus\{0\}$, where $\epsilon_k = \frac{\mu}{z} \to 0$ as $z \to \infty$. Moreover, $f_{\lambda,\mu}(w) \to \pm\lambda$ as $w \to \infty$, so, λ and $-\lambda$ are accumulation points of all singular values of the family $f_{\lambda,\mu}$. Taking a bounded set $B \subset \mathbb{C}$ such that it contains the accumulation points $\pm\lambda$, we conclude that the family $f_{\lambda,\mu}(z) = \lambda\sin(z) + \frac{\mu}{z}$ belongs to class $\mathcal{B}$. ■

4.2.3 Attracting components in the Fatou set of $f_{\lambda,\mu}$

In this section we shall give conditions on the parameters λ and μ in order to have examples of attracting components which are in the Fatou set of $f_{\lambda,\mu}$.

Example 1. Let $\lambda = -1$ and $\mu = \pi^2$. The function $f_{-1,\pi^2}(z) = -\sin(z) + \frac{\pi^2}{z}$ has two fixed points at $\zeta_1 = \pi$ and $-\zeta_1 = -\pi$. Indeed, evaluating $z = \pi$ in the function $f_{-1,\pi^2}(z) = -\sin(z) + \frac{\pi^2}{\pi}$, we obtain

$$f_{-1,\pi^2}(\pi) = -\sin(\pi) + \frac{\pi^2}{\pi} = \pi.$$

Analogously for $-\zeta_1 = -\pi$. Now, for to determine what type of fixed point are ζ_1 and $-\zeta_1$; see Page 22; we evaluate the derivative of $f_{-1,\pi^2,0}$ at $\zeta_1 = \pi$, obtaining:

$$\left| f'_{-1,\pi^2}(\pi) \right| = \left| -\cos(\pi) - \frac{\pi^2}{\pi^2} \right| = 0 < 1.$$

Thus, $\zeta_1 = \pi$ is a super attracting fixed point of f_{-1,π^2}. From the symmetries described in Lemma 4.5, we know that $-\zeta_1 = -\pi$ is another super attracting fixed point of f_{-1,π^2}. Moreover, using Lemma 4.7, the imaginary axis is invariant onto f_{-1,π^2}.

For the case when we consider $z = x \in \mathbb{R}$, that is, $f_{-1,\pi^2}(x) = -\sin(x) + \frac{\pi^2}{x}$ a real function of one real variable, Figure 4.10 shows two orbits of points close to ζ_1 and $-\zeta_1$. We observe that the iterates converges to the supper attracting fixed points $\zeta_1 = \pi$ and $-\zeta_1 = -\pi$.

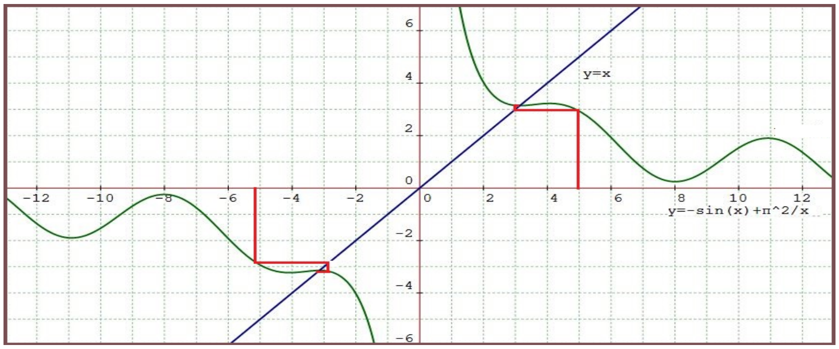

Figure 4.10: Two orbits converging to $\zeta_1 = \pi$ and $-\zeta_1 = -\pi$.

To obtain the Fatou and Julia sets of $f_{-1,\pi^2}(z) = -\lambda \sin(z) + \frac{\pi^2}{z}$, we consider Lemmas 4.6-4.8. The imaginary axis is contained in the Julia set which is completely invariant. Moreover, there exists two disjoint attracting components in the Fatou set; see Figure 4.11 where the Fatou set is shown in black and the Julia set in scale of yellow.

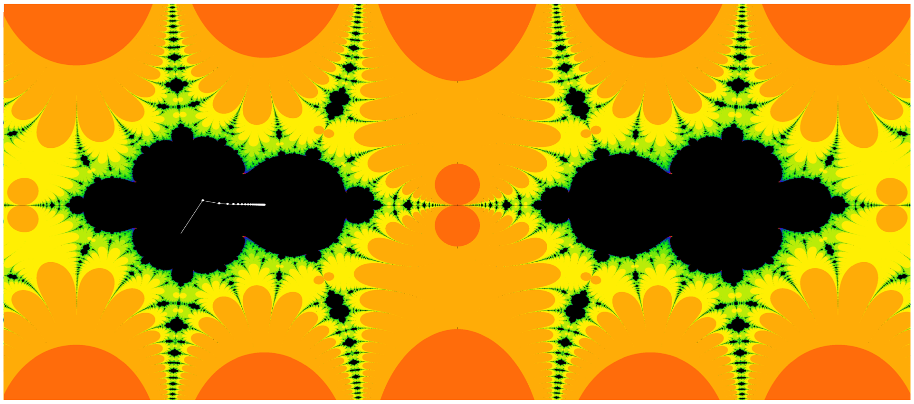

Figure 4.11: The Fatou and Julia sets of $f_{-1,\pi^2}(z) = -\sin(z) + \frac{\pi^2}{z}$.

Example 2. Let $\lambda = \sqrt{2}\pi/(4+\pi)$ and $\mu = \pi^3/(16(4+\pi))$. Observe that the function $f_{\lambda,\mu}(z) = \left(\frac{\sqrt{2}\pi}{4+\pi}\right)\sin(z) + \frac{\pi^3/(16(4+\pi))}{z}$ has two fixed points at $\zeta = \pi/4$ and $-\zeta = -\pi/4$, that is:

$$f_{\lambda,\mu}\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}\pi}{4+\pi}\right)\sin\left(\frac{\pi}{4}\right) + \frac{\pi^3/(16(4+\pi))}{\frac{\pi}{4}} = \frac{\pi}{4+\pi} + \frac{\pi^2}{4(4+\pi)} = \frac{\pi}{4}.$$

Analogously for $-\zeta = -\frac{\pi}{4}$. Moreover, the two fixed points $\frac{\pi}{4}$ and $-\frac{\pi}{4}$ are supper-attracting fixed points. Indeed, evaluating $f'_{\lambda,\mu}$ at $\zeta = \frac{\pi}{4}$, we obtain:

$$\left|f'_{\lambda,\mu}\left(\frac{\pi}{4}\right)\right| = \left|\left(\frac{\sqrt{2}\pi}{4+\pi}\right)\cos\left(\frac{\pi}{4}\right) - \frac{\pi^3/(16(4+\pi))}{\left(\frac{\pi}{4}\right)^2}\right| = \left|\frac{\pi}{4+\pi} - \frac{\pi}{4+\pi}\right| = 0 < 1.$$

Thus, $\zeta = \frac{\pi}{4}$ is a supper-attracting fixed point of $f_{\lambda,\mu}$. By using similar arguments $-\zeta = -\frac{\pi}{4}$ is another super attracting fixed point of $f_{\lambda,\mu}$.

For the case when we consider $z = x \in \mathbb{R}$, that is, $f_{\lambda,\mu}(x) = \left(\frac{\sqrt{2}\pi}{4+\pi}\right)\sin(x) + \frac{\pi^3/(16(4+\pi))}{x}$ is a real function of one real variable, Figure 4.12 shows two orbits converging to the super-attracting fixed points $\zeta = \frac{\pi}{4}$ and $-\zeta = -\frac{\pi}{4}$.

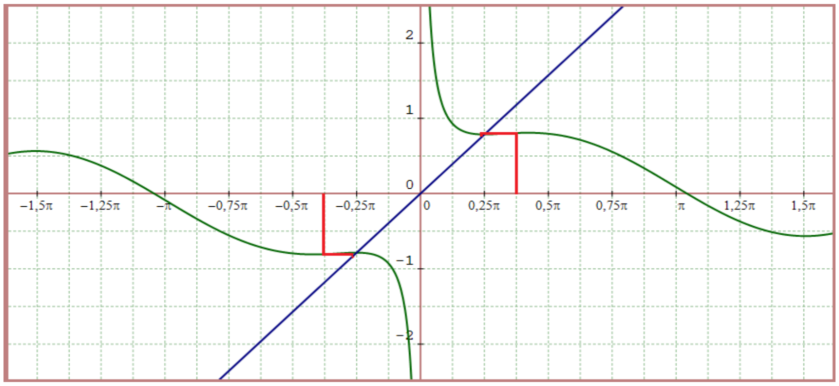

Figure 4.12: Two orbits converging to $\zeta_1 = \frac{\pi}{4}$ and $-\zeta_1 = -\frac{\pi}{4}$.

The dynamical plane of $f_{\lambda,\mu}(z) = \left(\dfrac{\sqrt{2}\pi}{4+\pi}\right)\sin(z) + \dfrac{\pi^3/(16(4+\pi))}{z}$ is shown in Figure 4.13, where the Fatou set is shown in black and the Julia set in scale of yellow. Observe that there exists two disjoint attracting components in the Fatou set and the imaginary axis is contained in the Julia set.

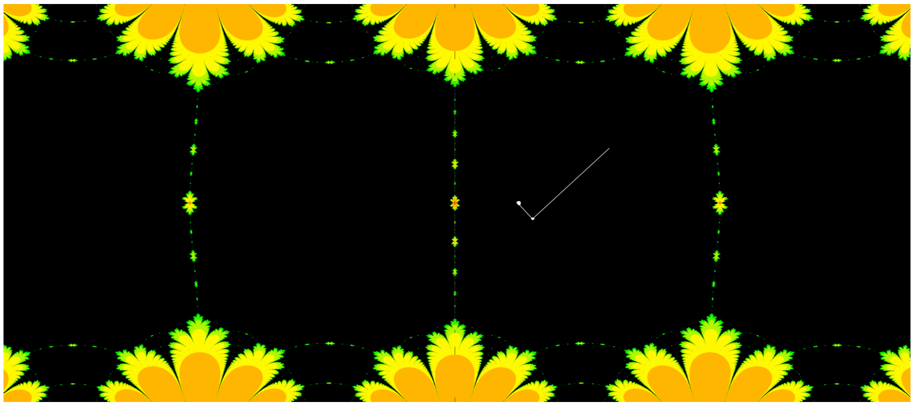

Figure 4.13: The Fatou and Julia sets of $f_{\lambda_2,\mu_2}(z) = \left(\dfrac{\sqrt{2}\pi}{4+\pi}\right)\sin(z) + \dfrac{\pi^3/(16(4+\pi))}{z}$.

Example 3. Let $\lambda = (2\pi)/(3\sqrt{3}+\pi)$ and $\mu = \pi^3/(9(3\sqrt{3}+\pi))$. The function $f_{\lambda,\mu}(z) = \left(\frac{2\pi}{3\sqrt{3}+\pi}\right)\sin(z) + \frac{\pi^3/(9(3\sqrt{3}+\pi))}{z}$ has two fixed points at $\zeta = \pi/3$ and $-\zeta = -\pi/3$, Indeed, evaluating $f_{\lambda,\mu}$ at $\zeta = \pi/3$ we obtain:

$$f_{\lambda,\mu}\left(\frac{\pi}{3}\right) = \left(\frac{2\pi}{3\sqrt{3}+\pi}\right)\sin\left(\frac{\pi}{3}\right) + \frac{\pi^3/(9(3\sqrt{3}+\pi))}{\frac{\pi}{3}} = \frac{\sqrt{3}\pi}{3\sqrt{3}+\pi} + \frac{\pi^2}{3(3\sqrt{3}+\pi)} = \frac{\pi}{3}.$$

With similar calculations we obtain that $-\zeta = -\frac{\pi}{3}$ is another fixed point of f_{λ_3,μ_3}. Moreover, the two fixed points are supper-attracting. Indeed, evaluating $f'_{\lambda,\mu}$ at $\zeta = \frac{\pi}{3}$, we obtain:

$$\left|f'_{\lambda,\mu}\left(\frac{\pi}{3}\right)\right| = \left|\left(\frac{2\pi}{3\sqrt{3}+\pi}\right)\cos\left(\frac{\pi}{3}\right) - \frac{\pi^3/(9(3\sqrt{3}+\pi))}{\left(\frac{\pi}{3}\right)^2}\right| = \left|\frac{\pi}{3\sqrt{3}+\pi} - \frac{\pi}{3\sqrt{3}+\pi}\right| = 0.$$

Thus, $\zeta = \frac{\pi}{4}$ is a supper-attracting fixed point of $f_{\lambda,\mu}$. For the symmetries of $f_{\lambda,\mu}$, we obtain that $-\zeta = -\frac{\pi}{3}$ is another super attracting fixed point of $f_{\lambda,\mu}$. As in the previous examples, if take $z = x \in \mathbb{R}$, then $f_{\lambda,\mu}(x) = \left(\frac{2\pi}{3\sqrt{3}+\pi}\right)\sin(x) + \frac{\pi^3/(9(3\sqrt{3}+\pi)}{x}$ is a real function and two orbits such that converge to the super-attracting fixed points $\zeta = \frac{\pi}{3}$ and $-\zeta = -\frac{\pi}{3}$ are shown in Figure 4.14.

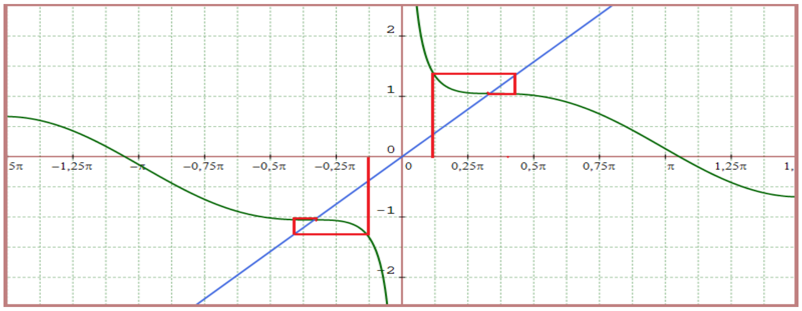

Figure 4.14: Two orbits converging to $\zeta_1 = \frac{\pi}{3}$ and $-\zeta_1 = -\frac{\pi}{3}$.

The dynamical plane of $f_{\lambda,\mu}(z) = \left(\frac{2\pi}{3\sqrt{3}+\pi}\right)\sin(z) + \frac{\pi^3/(9(3\sqrt{3}+\pi)}{z}$ is shown in Figure 4.15. Observe that there exists two disjoint attracting components in the Fatou set (in black) and the imaginary axis is contained in the Julia set colored in yellow.

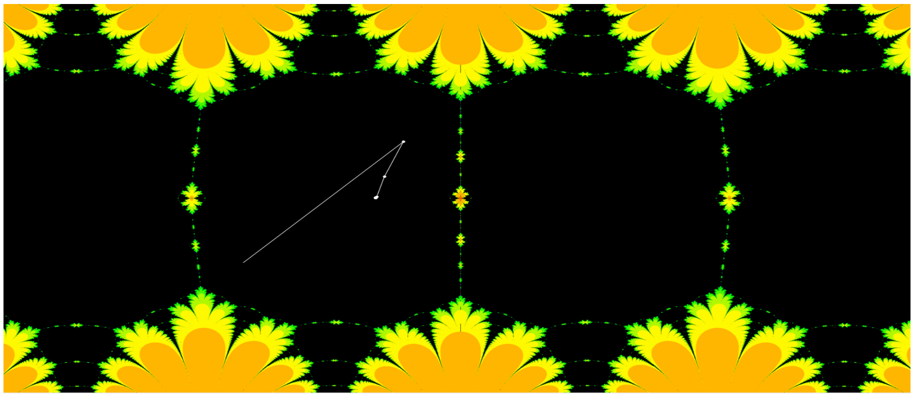

Figure 4.15: The Fatou and Julia sets of $f_{\lambda,\mu}(z) = \left(\frac{2\pi}{3\sqrt{3}+\pi}\right)\sin(z) + \frac{\pi^3/(9(3\sqrt{3}+\pi)}{z}$.

Conjecture 5. *There exists a set of parameters* $\mathrm{A} \subset \mathbb{C} \times \mathbb{C}$ *such that if* $(\lambda, \mu) \in \mathrm{A}$, *then the Fatou set of* $f_{\lambda,\mu}(z) = \lambda \sin(z) + \frac{\mu}{z}$ *contains only two invariant attracting components.*

A partial answer to the above conjecture, we identify the set of real parameters (λ, μ) such that $f_{\lambda,\mu}$ has two supper-attracting fixed points. In order to do that, we take the equation of a fixed point for $f_{\lambda,\mu}$ given by:

$$\lambda \sin(w) + \frac{\mu}{w} = w. \tag{7}$$

A super-attracting fixed point satisfies that $|f'_{\lambda,\mu}(w)| = 0$, that is,

$$\left|\lambda \cos(w) - \frac{\mu}{w^2}\right| = 0 \Leftrightarrow \lambda \cos(w) - \frac{\mu}{w^2} = 0. \tag{8}$$

Using the Equations in (7) and (8), we can express λ and μ in terms of the fixed point as follows:

$$\lambda = \frac{w}{\sin(w) + w\cos(w)}; \quad \mu = \frac{w^3 \cos(w)}{sin(w) + w\cos(w)}. \tag{9}$$

If λ and μ are as in Equation (9), then the family $f_{\lambda,\mu}$ has a super-attracting fixed point at w. Moreover, for the symmetries of $f_{\lambda,\mu}$, we obtain that $-w$ is another fixed point of $f_{\lambda,\mu}$.

Thus, we define the parametric curve $\mathcal{L}_0(w) = \left(\frac{w}{\sin(w)+w\cos(w)}, \frac{w^3\cos(w)}{sin(w)+w\cos(w)}\right)$, for $w \in \mathbb{R}$ such that $\sin(w) + w\cos(w) \neq 0$; see Figures 4.16 and 4.17. In the graphs, the horizontal axis corresponds to λ-axis and the vertical axis corresponds to μ-axis.

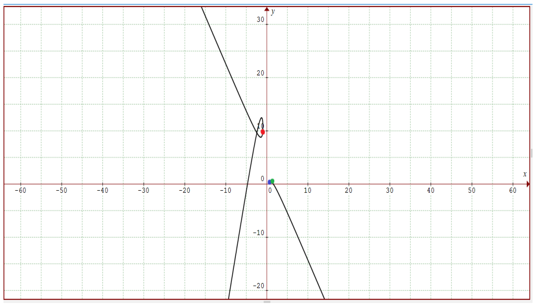

Figure 4.16: Graph of $\mathcal{L}_0$ in a (λ, μ)-plane.

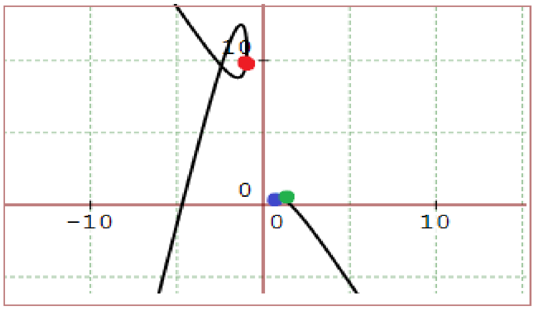

Figure 4.17: Close up to the graph $\mathcal{L}_0$.

The points in $\mathcal{L}_0$ are the set of real parameters (λ, μ) such that $f_{\lambda,\mu}$ has two supper-attracting fixed points. Nevertheless, we cannot claim that there are not other fixed points of $f_{\lambda,\mu}$. In Figure 4.17, the coordinates of the marked points are the parameters used in the examples given above. The red point $(-1, \pi^2)$ corresponds to the values of the parameters $\lambda = -1$ and $\mu = \pi^2$ used in Example 1, the green point $(\sqrt{2}\pi/(4+\pi), \pi^3/(16(4+\pi)) \approx (0.62211, 0.27135)$ are the values of the parameters $\lambda = \sqrt{2}\pi/(4+\pi)$ and $\mu = \pi^3/(16(4+\pi)$ used in Example 2 and the blue point $((2\pi)/(3\sqrt{3}+\pi), \pi^3/(9(3\sqrt{3}+\pi))) \approx (0.75358, 0.41320)$ corresponds to the values $\lambda = (2\pi)/(3\sqrt{3}+\pi)$ and $\mu = \pi^3/(9(3\sqrt{3}+\pi))$ used in Example 3.

4.2.4 Parabolic components in the Fatou set of $f_{\lambda,\mu}$

In this section we give conditions on the parameters λ and μ such that $f_{\lambda,\mu} = \lambda\sin(z)+\frac{\mu}{z}$ has parabolic components in the Fatou set. Moreover, we give examples.

Lemma 4.9. *Let* $\lambda = \dfrac{\pi}{2} - \dfrac{2\mu}{\pi}$ *and* $\mu = -\dfrac{r\pi^2}{4}$, *where r is a root of unity. The Fatou set of the family* $f_{\lambda,\mu}(z) = \lambda\sin(z) + \frac{\mu}{z}$ *contains two parabolic domains.*

Proof. We claim that for $\lambda = \frac{\pi}{2} - \frac{2\mu}{\pi}$ and $\mu = -\frac{r\pi^2}{4}$, where r is a root of unity, the family $f_{\lambda,\mu}(z) = \lambda\sin(z) + \frac{\mu}{z}$ has two fixed points at $z_1 = \frac{\pi}{2}$ and $z_2 = -\frac{\pi}{2}$.

Indeed, evaluating $z_1 = \frac{\pi}{2}$ in $f_{\lambda,\mu}$, we obtain:

$$f_{\lambda,\mu}\left(\frac{\pi}{2}\right) = \lambda\sin\left(\frac{\pi}{2}\right) + \frac{\mu}{\pi/2} = \left(\frac{\pi}{2} - \frac{2}{\pi}\left(-\frac{r\pi^2}{4}\right)\right)\sin\left(\frac{\pi}{2}\right) + \frac{-\frac{r\pi^2}{4}}{\frac{\pi}{2}} = \frac{\pi}{2}.$$

Analogously for $z_2 = -\pi/2$. Thus, z_1 and z_2 are two fixed points of $f_{\lambda,\mu}$. Now, we evaluate $z_1 = \pi/2$ in $f'_{\lambda,\mu}(z) = \lambda\cos(z) - \frac{\mu}{z}$:

$$\left|f'_{\lambda,\mu}\left(\frac{\pi}{2}\right)\right| = \left|\lambda\cos\left(\frac{\pi}{2}\right) - \frac{\mu}{\left(\frac{\pi}{2}\right)^2}\right| = \left|\left(\frac{\pi}{2} - \frac{2}{\pi}\left(-\frac{r\pi^2}{4}\right)\right)\cos\left(\frac{\pi}{2}\right) - \frac{-\frac{r\pi^2}{4}}{\left(\frac{\pi}{2}\right)^2}\right| = \left|r\right| = 1.$$

Thus $\left|f'_{\lambda,\mu}\left(\frac{\pi}{2}\right)\right| = 1$, since $|f'_{\lambda,\mu}\left(\frac{\pi}{2}\right)| = |r|$, where r is a root of unity, thus $z_1 = \frac{\pi}{2}$ is a rationally indifferent fixed point of $f_{\lambda,\mu}$. Therefore, the Fatou set of $f_{\lambda,\mu}$ contains a parabolic component. Using similar arguments, we obtain that $z_2 = -\frac{\pi}{2}$ is a rationally indifferent fixed point of $f_{\lambda,\mu}$, thus we can conclude the result. ■

The following example is an example of Lemma 4.9 when $r = 1$ as a root of unity.

Example. Let $\lambda = \pi$ and $\mu = -\frac{\pi^2}{4}$. The family:

$$f_{\pi,-\frac{\pi^2}{4}}(z) = \pi\sin(z) - \frac{\pi^2}{4z}$$

has only two indifferent rational fixed points at $\zeta = \frac{\pi}{2}$ and $-\zeta = -\frac{\pi}{2}$.

If $\lambda = \pi$, then $\mu = \frac{\pi^2}{4} - \lambda\frac{\pi}{2} = -\frac{\pi^2}{4}$, so by Lemma 4.9 we have that the function $f_{\pi,-\frac{\pi^2}{4}}(z) = \pi\sin(z) - \frac{\pi^2}{4z}$ has two rationally indifferent fixed points at $\zeta = \frac{\pi}{2}$ and $-\zeta = -\frac{\pi}{2}$.

Now, we define the real-valued polynomial $P(x) = x^2 - \pi x + \frac{\pi^2}{4}$. Observe that $P(x)$ has a root of multiplicity 2 at $x = \frac{\pi}{2}$. Thus, $P(x) > 0$ for $x \in \mathbb{R} \setminus \{\frac{\pi}{2}\}$.

For $x > 0$, $\lambda = \pi$ and $\mu = -\frac{\pi^2}{4}$ we have

$$\begin{aligned}\lambda\sin(x) + \frac{\mu}{x} \leq x \quad &\Longleftrightarrow \quad \lambda x\sin(x) + \mu < x^2\\ &\Longleftrightarrow \quad \pi x + \frac{\pi^2}{4} \leq x^2\\ &\Longleftrightarrow \quad 0 \leq x^2 - \pi x + \frac{\pi^2}{4} = P(x).\end{aligned}$$

Thus, for $x > 0$ with $x \neq \frac{\pi}{2}$, $\lambda = \pi$ and $\mu = -\frac{\pi^2}{4}$, we have that $\lambda\sin(x) + \frac{\mu}{x} < x$. Then, $f_{\pi,-\frac{\pi^2}{4}}$ has only one fixed point $\zeta \in \mathbb{R}^+$ at $\zeta = \frac{\pi}{2}$. By similar arguments, we obtain that $f_{\pi,-\frac{\pi^2}{4}}$ has only one fixed point $\zeta \in \mathbb{R}^-$ at $\zeta = -\frac{\pi}{2}$. The points are shown in the graph; see Figure 4.18, of one real variable $f_{\pi,-\frac{\pi^2}{4}}$.

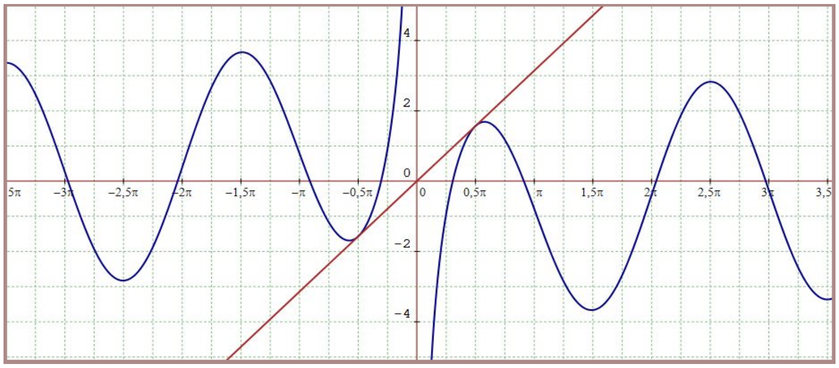

Figure 4.18: Graph of $f_{\pi,-\frac{\pi^2}{4}}$ with two indifferent rational fixed points.

Observe that for the parameters $\lambda = \pi$ and $\mu = -\frac{\pi^2}{4}$, the Fatou set of the family $f_{\pi,-\frac{\pi^2}{4}}(z) = \pi\sin(z) - \frac{\pi^2}{4z}$ contains two parabolic invariant components. As in the previous example, the imaginary axis is invariant which is contained in the Julia set; see Figure 4.19.

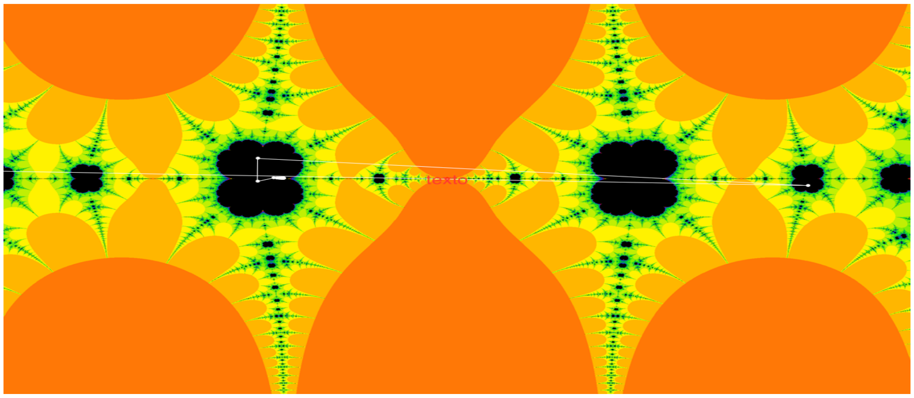

Figure 4.19: Dynamical plane of $f_{-1,\pi^2}(z)=-\sin(z)+\frac{\pi^2}{z}$ with two parabolic components

4.2.5 Siegel discs in the Fatou set of $f_{\lambda,\mu}$

In this section we shall show that the Fatou set of the family $f_{\lambda,\mu}(z)=\lambda\sin(z)+\frac{\mu}{z}$ contains two Siegel discs for some parameters. Consider $\lambda\in\mathbb{C}\setminus\{0\}$ and $\mu=\left(\frac{\pi}{2}\right)^2-\frac{\lambda\pi}{2}$, so the function

$$f_{\lambda,\left(\frac{\pi}{2}\right)^2-\frac{\lambda\pi}{2}}(z)=\lambda\sin(z)+\frac{\left(\frac{\pi}{2}\right)^2-\frac{\lambda\pi}{2}}{z}=\lambda\sin(z)+\frac{\pi^2-2\lambda\pi}{4z}.$$

Observe that $f_{\lambda,\left(\frac{\pi}{2}\right)^2-\frac{\lambda\pi}{2}}$ has two fixed points at $z_1=\frac{\pi}{2}$ and $z_2=-\frac{\pi}{2}$. Evaluating $z_1=\frac{\pi}{2}$ in $f_{\lambda,\left(\frac{\pi}{2}\right)^2-\frac{\lambda\pi}{2}}$ we obtain

$$f_{\lambda,\left(\frac{\pi}{2}\right)^2-\frac{\lambda\pi}{2}}\left(\frac{\pi}{2}\right)=\lambda\sin\left(\frac{\pi}{2}\right)+\frac{\pi^2-2\lambda\pi}{2\pi}=\frac{\pi}{2}.$$

Similar for $z_2=-\frac{\pi}{2}$. We state the following result.

Lemma 4.10. *If $\theta\in\mathbb{R}\setminus\mathbb{Q}$ is a Brujno number, λ,μ non zero are complex parameters such that $\lambda=(e^{i\theta}+1)(\pi/2)$ and $\mu=\left(\frac{\pi}{2}\right)^2-\frac{\lambda\pi}{2}$, then the Fatou set of $f_{\lambda,\mu}(z)=\lambda\sin(z)+\frac{\mu}{z}$ contains two Siegel discs.*

Proof. The points $z_1=\frac{\pi}{2}$ and $z_2=-\frac{\pi}{2}$ are fixed points for $f_{\lambda,\left(\frac{\pi}{2}\right)^2-\frac{\lambda\pi}{2}}$. Now, we determine a parameter λ such that the fixed points are irrationally indifferent.

Evaluating the derivative of $f_{\lambda,\left(\frac{\pi}{2}\right)^2-\frac{\lambda\pi}{2}}$ at $z_1=\frac{\pi}{2}$, we obtain:

$$\left|f'_{\lambda,\left(\frac{\pi}{2}\right)^2-\frac{\lambda\pi}{2}}\left(\frac{\pi}{2}\right)\right|=\left|\lambda\cos\left(\frac{\pi}{2}\right)-\frac{\left(\frac{\pi}{2}\right)^2-\frac{\lambda\pi}{2}}{(\pi/2)^2}\right|=\left|-1+\frac{2\lambda}{\pi}\right|.$$

The point z_1 is an irrationally indifferent fixed point if $\left|f'_{\lambda,\left(\frac{\pi}{2}\right)^2-\frac{\lambda\pi}{2}}(z_1)\right| = e^{i\theta}$, for some $\theta \in \mathbb{R}\setminus\mathbb{Q}$. The parameter λ must satisfy the following:

$$\left|f'_{\lambda,\left(\frac{\pi}{2}\right)^2-\frac{\lambda\pi}{2}}(z_1)\right| = e^{i\theta} \Leftrightarrow -1+\frac{2\lambda}{\pi} = e^{i\theta} \Leftrightarrow \lambda = \left(e^{i\theta}+1\right)\left(\frac{\pi}{2}\right).$$

Therefore, the Fatou set of the family $f_{\lambda,\left(\frac{\pi}{2}\right)^2-\frac{\lambda\pi}{2}}$ contains a Siegel disc with center $z_1 = \frac{\pi}{2}$. Similar for the fixed point $z_2 = -\frac{\pi}{2}$. ■

Using the *FractalStream* software, we obtain the following approximation of the Fatou and Julia sets of the family $f_{\lambda,\mu}$ associated to Lemma 4.11.

Taking: $\theta = \frac{1+\sqrt{5}}{2}$, $\lambda = (e^{\frac{i(1+\sqrt{5})}{2}}+1)(\frac{\pi}{2})$ and $\mu = \left(\frac{\pi}{2}\right)^2 - \frac{\lambda\pi}{2} = \left(\frac{\pi}{2}\right)^2 - \frac{(e^{\frac{i(1+\sqrt{5})}{2}}+1)(\frac{\pi}{2})\pi}{2}$, that is, we plot an approximaton of the Fatou and Julia sets of

$$f_{\lambda,\left(\frac{\pi}{2}\right)^2-\frac{\lambda\pi}{2}}(z) = \lambda\sin(z) + \tfrac{\pi^2-2\lambda\pi}{4z}, \text{ for } \lambda = (e^{\frac{i(1+\sqrt{5})}{2}}+1)(\tfrac{\pi}{2});$$

see Figure 4.23.

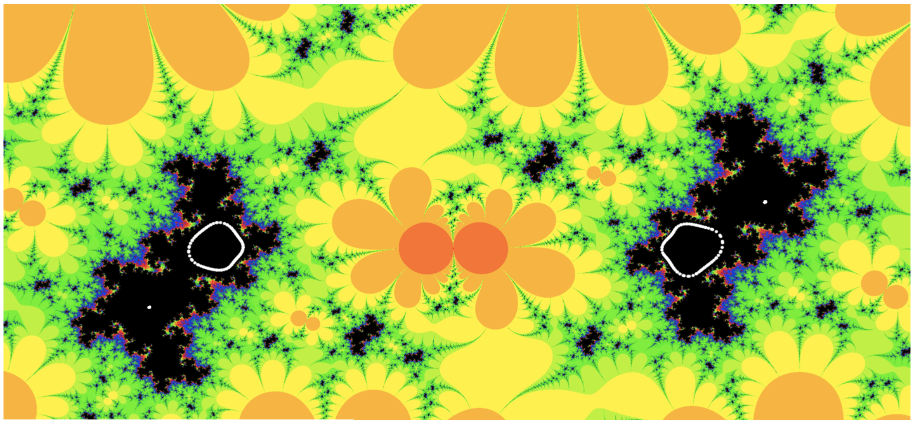

Figure 4.20: Siegel discs of $f_{\lambda,\left(\frac{\pi}{2}\right)^2-\frac{\lambda\pi}{2}}(z) = \lambda\sin(z) + \frac{\pi^2-2\lambda\pi}{4z}$, for $\lambda = (e^{\frac{i(1+\sqrt{5})}{2}}+1)(\frac{\pi}{2})$

Figure 4.17 shows the orbits of two points, in white, inside each Siegel disc, in black, where the orbits rotates points on invariant loops around the each fixed point and the Julia set is shown on scale of yellow.

4.2.6 The Fatou set with two types of components

The Fatou set can have different types of components in the same dynamical plane, an example of this case is given as follows.

Example: Dynamical plane with attracting and parabolic components.

Let $\lambda = -2$ and $\mu = \pi^2$. The function $f_{-1,\pi^2}(z) = -\sin(z) + \frac{\pi^2}{z}$ has two rational indifferent fixed points in $\zeta_1 = \pi$ and $\zeta_2 = -\pi$. Observe that for the parameters $\lambda = -2$ and $\mu = \pi^2$ the function has two extra fixed points which can be obtained by numerical methods. These points are approximately $z_3 \approx 4.919317$ and $z_4 \approx -4.919317$.

Evaluating both points in $|f'_{-2,\pi^2}(z)|$ we obtain that they are attracting fixed points. Considering $z = x \in \mathbb{R}$ and f_{-1,π^2} a real function of one real variable, Figures 4.21 and 4.22 show the four fixed points of f_{-1,π^2}.

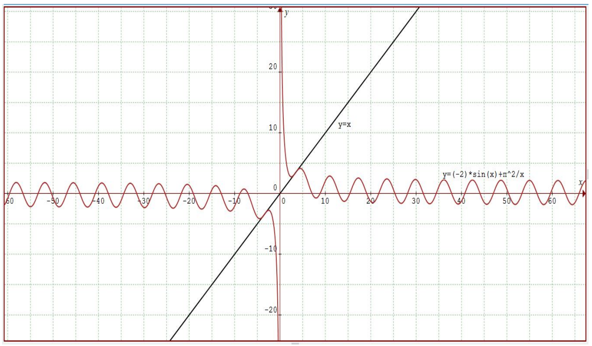

Figure 4.21: The function f_{-2,π^2}.

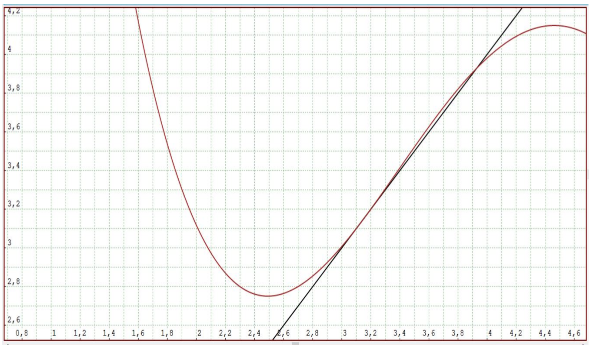

Figure 4.22: Close up to the function f_{-2,π^2}.

Using the *FractalStream* software, we obtain the following approximation of the dynamical plane of the family $f_{\lambda,\mu}$ related to the conditions mentioned above. Figure 4.22 shows in white the orbit of a point in the Fatou set (in black) of $f_{-2,\pi^2}(z) = -2\sin(z) + \frac{\pi^2}{z}$ and in scale of yellow the Julia set.

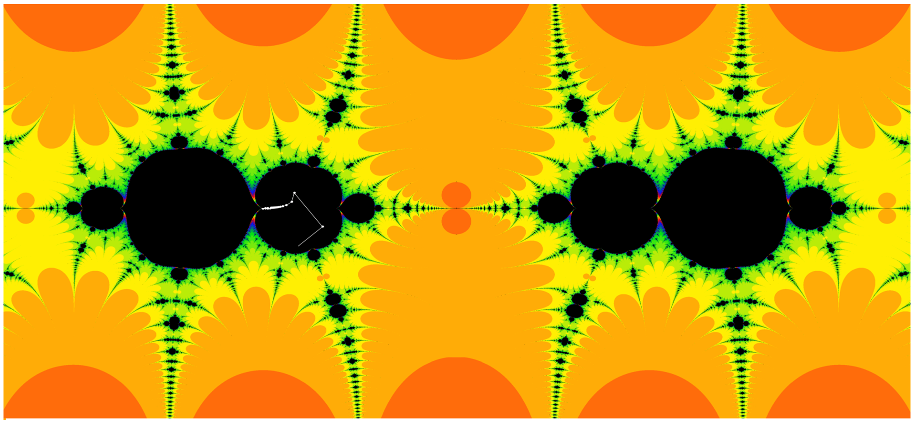

Figure 4.23: The Fatou and Julia sets of $f_{-2,\pi^2,0}(z) = -2\sin(z) + \dfrac{\pi^2}{z}$ with attracting and parabolic components.

Chapter 5

Residual Julia set for transcendental meromorphic functions with a $p-$cycle of Herman rings

We recall that a p-periodic component H of the Fatou set is a *Herman ring* of $f \in \mathscr{M}$, if f^p is conformally conjugate to an irrational rotation on a concentric annulus, that is, there exist $0 < r_0 < 1$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and a conformal mapping $h : H \to H_{r_0,1}$ such that $f^p \circ h = h \circ R_\theta$, where R_θ is the rigid rotation of angle $2\pi\theta$ and $H_{r_0,1}$ is the standard annulus of inner radius r_0 and outer radius 1. A Herman ring is a doubly connected component of the Fatou set, so its complement consists of two components, one bounded and the other unbounded.

We say that a p-cycle of Herman rings $H_1, H_2, H_3, ...H_p$, $p \in \mathbb{Z}$, is *nested* if H_i lies in the bounded component of the complement of H_{i+1}, with $i = 1, \ldots, p-1$ and it is *almost nested* if all the Herman rings in the p cycle are nested except for one Herman ring, which lies outside the unbounded component of the complement of all nested Herman rings.

The *Residual Julia set* for a function in class $\mathscr{M}$, denoted by $J_r(f)$, is defined as the set of those points of the Julia set which do not belong to the boundary of any component of the Fatou set. A point of the residual Julia set is called *buried point* and a component of the residual Julia set is called a *buried component*. This concept was first introduced in the context of Kleinian groups by Abikoff in [1] and [2]. McMullen in [55] defined and gave an example of a buried component for a family of rational functions. Later, Beardon in [15] gave conditions for the existence of buried components for rational functions. After these results several mathematicians have studied buried components for different classes of functions; see for instance [7], [29], [33], [59], [60], [66] and [67].

We are interested in functions in class $\mathscr{M}$, so we shall mention some properties of the Residual Julia set for this class; see for a proof [7] and [33]:

(a) If the Fatou set has a completely invariant component, then the residual Julia set $J_r(f)$ is empty.

(b) If there exists a buried component of the Julia set, then the Julia set is disconnected.

(c) If $J_r(f) \neq \emptyset$, then $J_r(f)$ is completely invariant, dense in $\mathcal{J}(f)$ and unaccountably infinite.

For functions in class $\mathscr{M}$ the existence of buried singleton components in the Julia set is given by the following theorems below; see [7] and [31].

Theorem 5.1. *If f is meromorphic in $\mathbb{C}$ with no wandering domains if $\mathcal{J}(f)$ is not connected and if $\mathcal{F}(f)$ has no completely invariant component, then $J_r(f) \neq \emptyset$*

Theorem 5.2. *Suppose that $f \in \mathscr{M}$, $\mathcal{F}(f)$ has no completely invariant components and $\mathcal{F}(f)$ has three doubly-connected components U_i, $1 \leq i \leq 3$, such that one of the following conditions holds*

(i) Each U_i lies in the unbounded component of the complement of the other two.

(ii) Two of the components U_1, U_2 lie in the bounded component of U_3^c but U_1 lies in the unbounded component of U_2^c and U_2 lies in the unbounded component of U_1^c.

Then $\mathcal{J}(f)$ contains singleton components which are dense and buried.

Theorem 5.2 does not cover the cases when the doubly connected components U_i, $1 \leq i \leq 3$, are nested or almost nested, because the proof uses the concept of an island over a domain taken from the Ahlfors' theory of covering surfaces. By the time Theorem 5.2 was prove the author did not know if examples of it could exist. Later, in [7] an example with doubly connected components was constructed, by using complex approximation, satisfying assumption (i). Also in [32] an example was given satisfying part of the assumption (i), by generalizing Shisikura's surgery construction for rational functions [73], and obtaining unbounded continua of buried points. There is still the question: Are there examples of $f \in \mathscr{M}$ which satisfy assumption (ii) of Theorem 5.2?

In [74] and [75] Shishikura studies the configuration of a p-cycle of Herman rings and defines an associated abstract tree , for the configuration given, he shows that it is possible to construct a rational function with a p-cycle of Herman rings realizing such tree. Fagella and Peter in [40] show that every configuration of Herman rings that is realizable by a rational function is also realizable by a transcendental meromorphic function. The authors add an essential singularity to a rational function with a p-cycle of Herman rings by using quasi-conformal surgery, to obtain a transcendental meromorphic function with exactly the same configuration of Herman rings as the rational function.

By using the results of Shishikura [75] and Fagella and Peter [40] it is possible to give examples which satisfy assumptions (i) and (ii) of Theorem 5.2. Unfortunately the construction do not cover the case when the doubly connected components are no Herman rings. Nevertheless, the constructions allows us to give examples and a "kind" of generalization of Theorem 5.2, when the doubly connected components in the Fatou set are in a p-cycle of Herman rings with almost any configuration. The result is stated as follows.

Theorem C. *There exist $g \in \mathscr{M}$ which satisfies that the Fatou set of g has a p-cycle of Herman rings H_1, H_2, …, H_p, $p \geq 3$, removing the cases when*

(a) the p-cycle of Herman rings is nested and

(b) the p-cycle of Herman rings is almost nested,

such that the Julia set of g contains singleton components which are dense and buried.

Observe that in Theorem C the smallest cycle is a 3-cycle of Herman rings which is already included in Theorem 5.2. For $p > 3$ the configurations of the p-cycle of Herman rings could be very diverse. In the proof of Theorem C the concept of an island over a domain will be used, since we still do not know how to prove Theorem C without using the concept of an island, which is the main reason to remove cases (a) and (b) in Theorem C.

In the proof of Theorem C we will observe that the function g has no wandering domains in the Fatou set, but here are cases where wandering domains certainly occur with residual Julia set not empty. For instance, it is known that for functions in class $\mathscr{M}$ there are examples of wandering components of any prescribe connectivity that can be bounded or unbounded, which were constructed by using complex approximation; see [13]. We state the following result.

Theorem D. *There exist $h \in \mathscr{M}$ which satisfies that the Fatou set has a bounded wandering component which is neither simply connected nor nested multiply connected, such that the Julia set contains singleton components which are dense and buried.*

Observe that in Theorems C and D the Julia set is not connected. There is an interesting question whether the residual Julia set is non-empty when Julia set is connected? The following theorem is a partial answer to the question, but we do not know if there are examples of it.

Theorem E. *Suppose that f is a transcendental meromorphic function, the Fatou set has only two attracting simply connected invariant bounded components U_1 and U_2, such that:*

(a) $\partial U_1 \cap \partial U_2 = \{0\}$, where zero is a non-omitted pole of f,

(b) U_1 and U_2 are symmetric with respect to the imaginary axes, and

(c) the imaginary axes is in the Julia set, where $f^n(iy) \to \infty$, with $y \in \mathbb{R}$.

Then the imaginary axis is a buried component, except for zero.

This chapter is divided in four sections. In Section 5.1 we include a brief summary of the results related to the realization of a configuration of Herman rings for a rational function through a construction of a tern given in [74, 75]. In Section 5.2 we state some results by Fagella and Peter in [40] and give two examples of a transcendental meromorphic function which satisfies Theorem C, by using the results in Section 5.1 and in [40]. Finally, in Section 5.3 we prove Theorems C, D and E. The results in Chapter 5 are under review; see [22].

5.1 Shishikura's model for a configuration of a cycle of Herman rings

In this section we are particularly focused in a specific p-periodic rotation domain H of a rational function f, namely a Herman ring H.

Shishikura in [74] defines a *configuration* $\mathcal{C}$ as a cyclically order collection of disjoint, oriented Jordan Curves $\gamma_0, \gamma_1, ..., \gamma_{p-1}$ in $\widehat{\mathbb{C}}$, up to orientation preserving homomorphism of $\widehat{\mathbb{C}}$ and simultaneous change of orientation of all curves γ_j. In the case of a $p-$cycle of Herman rings a configuration $\mathcal{C}$ is defined as follows: fix r such that $r_0 < r < 1$ and define $\gamma_0 = h^{-1}(\{z \in \mathbb{C} : \mid z \mid = r\})$ with a determinate orientation and $\gamma_j = f^j(\gamma_0)$, thus f induces equivariant orientations of γ_j, so $\{\gamma_j\}$ defines a configuration, which is independent of the choice of r and the orientation of γ_0.

With the definition given above Shishikura investigates the conditions when a finite arrangement of topological annuli $\mathcal{D} := \{H_0, ..., H_{p-1}\}$ corresponds to a configuration $\mathcal{C}$ of a p-cycle of Herman rings, where the elements in $\mathcal{D}$ satisfy either (i) $H_i \subset H_j$, for some i, j, (ii) $H_i \cap H_j = \emptyset$, for every i, j, or (iii) the combination of the two cases (i) and (ii). In order to do that, Shishikura constructs three objects: an abstract tree T, a linear metric d on $\hat{\mathbb{C}}$ and a continuous function F onto T, where the three objects conform a *tern*, which we call *Shishikura's tern* denoted by (T, d, F). Then Shishikura defines an *orbit model* on (T, d, F) to obtain a rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and a collection of Herman rings $\{H_0, H_1, ..., H_{p-1}\}$, which is a $p-$periodic cycle of Herman rings of f, where $f(H_i) = H_j$, for some $i, j \in \{0, ..., p-1\}$. When the conditions above are satisfied, it is said that f *realizes* the configuration $\mathcal{C}$.

In the following section we review briefly what is known for the construction of the *Shishikura's tern* (T, d, F) and the *orbit model* on (T, d, F); see [74] and [75]

5.1.1 Shishikura's tern (T, d, F)

To construct the Shishikura's tern (T, d, F) we will need a topological tree T, a linear metric $d \in \widehat{\mathbb{C}}$ and a continuous functions F onto T, but first we will define an abstract tree.

I. Abstract tree

We recall that an annulus $H \subset \widehat{\mathbb{C}}$ is a connected open set. If H^c has two components, neither of them is a point. The following concepts are related to an annulus; see [74].

(i) An annulus H *separates* a single set $X \in \widehat{\mathbb{C}}$, if X intersects both components of H^c.

(ii) If H is an annulus, there exists $r_H \in \mathbb{R}$ with $0 < r_H < 1$ and a conformal map ϕ_H such that $\phi_H : H \mapsto \{z \in \mathbb{C} : r_H <\mid z \mid< 1\}$.

(iii) The *modulus* of H is defined as $m(H) = -\frac{1}{2\pi} \log r_H$.

Let H be an annulus. Define the following sets as in [74]:

$$H[z] = \phi_H^{-1}(\{\psi : \, |\psi| = |\phi H(z)|\}), \text{ for } z \in H$$

,

$$H(x, y) = \{z \in H : H[z] \text{ separates } x \text{ and } y\}.$$

(1) $H[z]$ and $H(x, y)$ does not depend of the choice of ϕ_H;
(2) $H[z]$ is a simple closed curve in H which contains z and separates ∂H;
(3) $H(x, y)$ is either empty or an annulus separating x and y.

Condition D. Consider $\mathcal{D} := \{H_0, ..., H_{(p-1)}\}$ a collection of disjoint topological annuli in $\widehat{\mathbb{C}}$ and assumed that there exists a non-empty closed set $\mathcal{B} \subset \widehat{\mathbb{C}}$ which consists of a finite number of connected components $B_1, ..., B_n$, where none of them is a point and each $H_i \in \mathcal{D}$ is disjoint with $\mathcal{B}$ and separates $\mathcal{B}$; see [75] for more details.

For $x, y \in \hat{\mathbb{C}}$ and for some $A = H_i \in \mathcal{D}$, $0 \leq i \leq p-1$, define

$$d(x, y) = \sum_{A \in \mathcal{D}} m(A(x, y)).$$

It is proved in [74] that d is a pseudo-metric on $\hat{\mathbb{C}}$, $d(x, y) < \infty$ and $d(\cdot, \cdot)$ is continuous on $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$. Using d, it is possible to construct an *abstract tree associated to the collection* $\mathcal{D}$ as follows. Define an equivalence relation $\sim_d$ on $\widehat{\mathbb{C}}$ given by $x \sim_d y$ if and only if $d(x, y) = 0$. The quotient space

$$T(\mathcal{D}) := \hat{\mathbb{C}} / \sim_d$$

is called the *abstract tree associated to the configuration* $\mathcal{D}$ with the following properties; see [74].

(i) $T(\mathcal{D})$ is indeed a topological finite tree, that is, a one-dimensional finite simplicial complex which is connected and without loops.

(ii) $\overline{d} : T(\mathcal{D}) \to T(\mathcal{D})$ defined by $\overline{d}(\overline{x}, \overline{y}) = d(x, y)$, where $\overline{x} \in \pi^{-1}(x)$, $\overline{xy} \in \pi^{-1}(y)$ and π is the natural projection.

(iii) The projections $\pi(H_j)$ are dense in $T(\mathcal{D})$, for all $H_j \in \mathcal{D}$, $0 \leq i \leq p-1$.

(iv) For all i we have that $\pi(H_i)$ is isometric to an open interval.

(v) $T(\mathcal{D})$ is the union of specific arcs with determined end points.

It is possible to construct $T(\mathcal{D})$ in the following primitive way. Let $\mathcal{D}$ be a collection of disjoint topological annuli as before. A graph Γ is obtained if we consider that the vertices of Γ are the connected components of the complement of the annuli and the edges are the annuli. Moreover, two vertices are connected by an edge, if and only if the corresponding two vertices have a common annulus whose boundary is contained in them; see Figure 5.1 for an example.

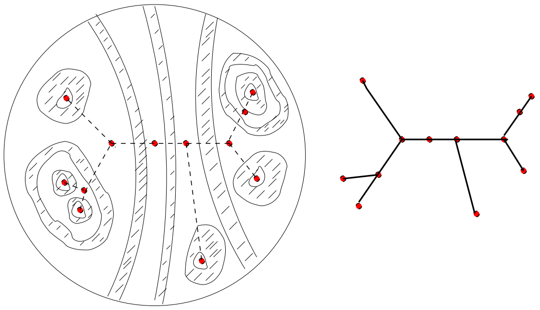

Figure 5.1: A configuration of Herman Rings (left) and its abstract tree associated (right)

We shall state some definitions of certain elements of the tree $T(\mathcal{D})$ given in [74] and [75].

(a) For $x \in T(\mathcal{D})$ a *branch at* x is a component of $T(\mathcal{D}) \setminus \{x\}$.
(b) The collection of all the branches at x is denoted by B_x.
(c) x is an *end point* if $\#B_x = 1$ and a *branch point* if $\#B_x \geq 3$, where $\#$ denotes the cardinality of a set.

The original pseudo-metric d on $\hat{\mathbb{C}}$ is projected by π to a metric $\hat{d}$ on $T(\mathcal{D})$, that is, if $x, y \in T(\mathcal{D})$, then $\hat{d}(x, y) = d(\hat{x}, \hat{y})$, where $\hat{x} \in \pi^{-1}(x)$, $\hat{y} \in \pi^{-1}(y)$ and π denote the natural projection from $\hat{\mathbb{C}}$ to $T(\mathcal{D})$. Thus, $T(\mathcal{D})$ is the union of specific arcs with determined end points and satisfies that the projections $\pi(H_i)$ are dense in $T(\mathcal{D})$ and are isometric to open intervals; see [74] for a proof.

II. Shishikura's tree

With base to the previous definition of an abstract tree we are able to define a Shishikura's tree for a configuration of a p-cycle of Herman rings; see [74]and [75] for details.

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ a rational function and f^n the n-iterate of f. Take f such that it has a p-cycle of Herman rings $\{H_0, ..., H_{p-1}\}$ and define the following sets:

(i) $\mathcal{C} = \{f^n(z) |\ z$ is *critical point* of f, $n \geq 0\}$;

(ii) $\mathcal{A}_0 = \{connected\ components\ of\ (H_0 \setminus closure(H_0 \cap \mathcal{C})\}$;

(iii) $\mathcal{A}' = \{connected\ components\ of\ f^{-n}(A) : A \in \mathcal{A}_0,\ n \geq 0\}$;

(iv) $B = \cup \partial H_i$, where ∂H_i denotes the boundaries of the Herman ring H_i, for $0 \leq i \leq p-1$;

(v) $\mathcal{A} = \{A \in \mathcal{A}' : A\ is\ essential\}$, where an annulus $A \in \mathcal{A}'$ is *essential,* if $f^n(A)$ separates B for all $n \geq 0$.

The sets $\mathcal{A}_0$, $\mathcal{A}'$ and $\mathcal{A}$ are collections of disjoint annuli. Taking $\mathcal{A}$ as $\mathcal{D}$ and B as $\mathcal{B}$ as in **I** above, we can observe that $T(\mathcal{A})$ is a metric space and d can be projected to a linear metric.

If $\mathcal{A}$ satisfies **Condition D** in **II** with the B above and according to **II** we can defined the tree $T_f = T(\mathcal{A})$, which is called the *tree associated with the Herman rings* of f, we will called the *Shishikura's tree.*

III. Shishikura's tern

Let $T_f = T$ be a Shishikura's tree as in **II** and $F : T \to T$ be defined by

$$F(x) = \pi \circ (f \circ \partial(\pi^{-1}(x)),$$

where π is the canonical projection, $\partial(\pi^{-1}(x))$ is the boundary of $\pi^{-1}(x)$ in $\hat{\mathbb{C}}$ and f is the rational function given in **II**. The function F is well-defined and continuous; see [73] for details.

The following lemma summarizes some properties of the tern (T, d, F), which we call *Shishikura's tern*; see [75] for a proof.

Lemma 5.1. *Let (T, d, F) be a Shishikura's tern. Then the following statements are true.*

1. *(T, d) is a tree with a linear metric.*

2. *$F : T \to T$ is a continuous.*

3. *There exists a function* $DF : T \setminus Sing(T, F) \to \mathbb{N}$, *which is constant on each component, and a finite subset* $Sing(T, F)$ *of* T, *where*

$$Sing(T,F) := \{\textit{end points, branch points of } T\} \bigcup \pi(\{\textit{critical points of } f\}),$$

   such that if $T^{'}$ *is a connected component of* $T \setminus Sing(T, F)$, *then*

   (a) $F_{|_{T'}} : T^{'} \to F(T^{'})$ *is a homeomorphism.*

   (b) *d(F(x),F(y))=k d(x,y) for* $x, y \in T^{'}$, *where* $DF_{|_{T'}} = k$.

4. *There exist subarcs* $I_{ij}(i = 1, \ldots, l, j \in \mathbb{Z}/p_i \in \mathbb{Z})$, *of* T *such that* I_{ij} *have disjoint interiors and do not contain branch points except at its end points,* $F(I_{ij}) = I_{ij+1}$ *and* $F^{p_i} = Id$ *on* I_{ij}. *Here* l *is the number of cycles of Herman rings and* p_i *the periods.*

5. *There exists* $N \geq 0$ *such that* $T = \bigcup_{n=0}^{N} F^{-n}(S)$, *where* $S := \cup_{ij} I_{ij} \cup \{x : DF(x, b) \geq 2, for\ some\ b \in \mathcal{B}_x\}$.

6. *Any one-sided neighborhood of a folding point of F, a point where F is not locally injective, intersects* $\cup_j F^{-n}(intI_{ij})$.

7. *Every end point of* T *is an end point of an* I_{ij}.

Remark 5.1. It follows from the construction of T and the properties of F that F is a piecewise linear function that acts as a permutation on the elements of T and the arcs I_j can be oriented in such way that f respects the orientations, that is, considering $\partial^- I_j$ and $\partial^+ I_j$ the beginning and the end of the arc I_j, then $\partial I_j = \{\partial^- I_j, \partial^+ I_j\}$ and $F(\partial^- I_j) = \partial^- I_{j+1},\ F(\partial^+ I_j) = \partial^+ I_{j+1}$. Thus, to represent T we need a finite set of arrowed segments, denoted by $x_1, ..., x_n$, and their respective lengths as well as a periodic arc, x_{n+1}, with its end points.

5.1.2 Local model

In this subsection, we will mention how Shishikura in [75] gets a rational function that realizes a configuration of a p-cycle Herman rings H_i, $0 \leq i \leq p-1$, for that the concept of *local model for singular orbits* is needed.

First, thicken all the segments of the tree to tubes with a small common radius. Second, blow up all the singular points to balls. Gluing the balls and the tubes, a topological sphere is obtained. Next, define a function on the tubes so that if $x, F(x) \notin Sing(T, F)$, it is a covering of degree $DF(x)$ from the circle corresponding to x to that corresponding to $F(x)$. Thus, find a suitable mapping on each ball, so that one can get a rational function with the desired configuration of Herman rings; see Figure 5.2.

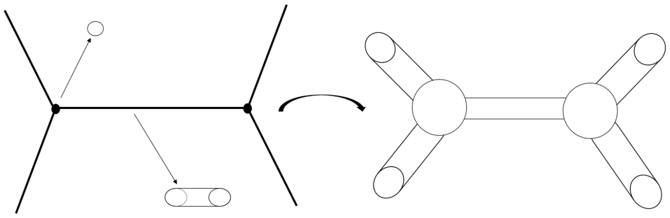

Figure 5.2: The balls and the tubes

Since every complementary component of each H_i is collapsed to a point, the dynamics is lost on this component, nevertheless, it can be recovered by defining local models for the periodic points on the tree T via surgery, that is, cutting the sphere along circles sufficiently close to each periodic point of T, and replace the map $F : T \to T$ given above by a simpler one outside the corresponding domains, it is obtained a cycle of rational maps, using surgery, which have marked points corresponding to the branch points of T. The following definition in [74] shows how to obtain the map on the balls.

Definition 5.1. Let $X_1 = Sing(T, F)$, $X = X_1 \cup F(X_1)$ and $X_* = \{x \in X_1 : x \text{ has a } pre-periodic \text{ orbit in } X_1\}$. Consider $X \times \hat{\mathbb{C}}$ and define $\hat{\mathbb{C}}_x = x \times \hat{\mathbb{C}}$. A *local model* for singular points of (T, d, F) is a tern $(X, g, \{p_\beta\})$ satisfying:

(a) X is a finite subset of T containing to $Sing(T, F)$, end points, branch points of F and ∂I_j.

(b) $\{p_b\} := \{p_b : b \in \mathcal{B}_x,\ x \in F(X) \cup X\}$ is a set of points of $(X \cup F(X)) \times \hat{\mathbb{C}}$, such that p_b are distinct and $p_b \in \hat{\mathbb{C}}_x := x \times \hat{\mathbb{C}}$ if $x \in \mathcal{B}_x$.

(c) $g : X_1 \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ is an analytic map such that $g(\hat{\mathbb{C}}_x) \subset \hat{\mathbb{C}}_{F(x)}$.

(d) If x is an end point of an I_{ij} and $\beta \in B_x$ contains I_{ij}, then p_β is the centre of a Siegel disc of g with rotation number θ_β.

Remark 5.2. From (d), it is possible to obtained a rational function with a cycle of Siegel discs, which via quasi-conformal surgery can be transformed into Herman rings; see [73] for a proof.

The following theorem relates a Shishikura's tern and its local model; see [74].

Theorem 5.3. *Suppose that (T, d, F) is a Shishikura's tern which has a model. Then (T, d, F) is realizable by a rational function f.*

5.2 Examples of Theorem C

To construct examples of Theorem C we will need two main ingredients: the realization of a specific configuration of Herman rings by a rational function f in Section 5.1

[74, 75], and the results given by Peter and Fagella in [40], where the authors show that every configuration that is realizable by f is also realizable by a transcendental meromorphic function.

Following [40], a configuration $\mathcal{C}_{\hat{\mathbb{C}}} = (\Gamma, \sigma)_{\hat{\mathbb{C}}}$ is a finite collection $\Gamma = \{\gamma_1, \cdots, \gamma_p\}$ of disjoint oriented simple closed curves in $\hat{\mathbb{C}}$, (respectively $\mathbb{C}$) together with a permutation σ of $\{1, \ldots, p\}$. By Section 5.1, a rational function$f$ with a p-cycle of Herman rings $H_1, H_2..., H_p$ generates a configuration, denoted by $\mathcal{C}_{\hat{\mathbb{C}}}(f)$, where the permutation σ is f itself and γ_i is the invariant curve contained in H_i, for all i. Moreover, each γ_i is a simple closed curve, so it follows from the Jordan Curve Theorem that $\mathbb{C} \setminus \gamma_i$, for each i, is a union of two disjoint connected components, one of them is bounded, which is called the interior of γ_i, denoted by $int(\gamma_i)$, and the other is not bounded which is called the exterior of γ_i, denoted by $ext(\gamma_i)$.

Two configurations $\mathcal{C}_{\hat{\mathbb{C}}} = (\Gamma, \sigma)_{\hat{\mathbb{C}}}$ and $\mathcal{C}'_{\hat{\mathbb{C}}} = (\Gamma', \sigma')_{\hat{\mathbb{C}}}$ are considered equivalent, denoted by $\mathcal{C}_{\hat{\mathbb{C}}} \sim \mathcal{C}'_{\hat{\mathbb{C}}}$, if one of the following conditions is satisfied:
(i) $\sigma = \sigma'$ and there exists a homeomorphism ψ of $\widehat{\mathbb{C}}$ such that $int(\psi(\gamma_i)) = int(\gamma'_i)$,
(ii) $\sigma = \sigma'$ and the orientations of γ_i and γ'_i are reversed for all i that form one or more cycles of σ.
(iii) there exists a permutation τ of $\{1, ..., p\}$,such that $\tau^{-1} \circ \sigma \circ \tau = \sigma'$ and $int(\gamma_{\tau(i)}) = int(\gamma'_i)$.

Definition 5.2. The configuration $\mathcal{C}_{\hat{\mathbb{C}}} = (\Gamma, \sigma)_{\hat{\mathbb{C}}}$ is realizable by f if $\mathcal{C}_{\hat{\mathbb{C}}}(f) \sim \mathcal{C}_{\hat{\mathbb{C}}}$

The main theorem in [40, Theorem D] is stated as follows.

Theorem 5.4. *Let f be a rational function with Herman rings. Let $\mathcal{C}_{\hat{\mathbb{C}}}(f)$ be the configuration associated to f and $\mathcal{C}_{\hat{\mathbb{C}}}$ a configuration with $\mathcal{C}_{\hat{\mathbb{C}}} \sim \mathcal{C}_{\hat{\mathbb{C}}}(f)$. Then there exists a transcendental meromorphic function g with $\mathcal{C}_{\hat{\mathbb{C}}} \sim \mathcal{C}_{\hat{\mathbb{C}}}(g)$, i.e., g realizes $\mathcal{C}_{\hat{\mathbb{C}}}$.*

With the above information, we shall construct two examples with a specific configuration of the p-cycle of Herman rings realizable by a transcendental meromorphic function which satisfy Theorem C.

Example 1. *There exists $g \in \mathcal{M}$ such that g has 4-cycle of Herman rings H_1, H_2, H_3, V_1, such that H_i are nested Herman rings, $1 \leq i \leq 3$ and V_1 is Herman rings between H_i and H_{i+1}, for $i = 1$.*

In Figure 5.3, we show the configuration $\mathcal{D}$ associated to the 4-cycle of Herman rings H_1, H_2, H_3, V_1. We will proceed to prove that $\mathcal{D}$ is realizable by a rational function f and therefore by a transcendental meromorphic function g.

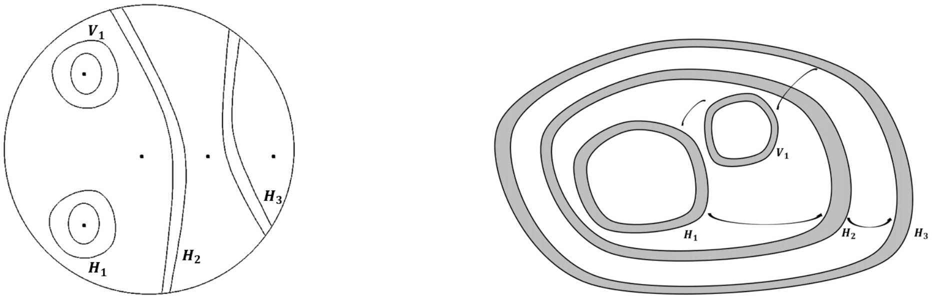

Figure 5.3: Configuration of annuli $\mathcal{D}$: In the sphere (left) and in the plane (right)

Using the results from Section 5.1, we consider the abstract tree associate to $\mathcal{D}$, see **I** in Section 5.1 and Figure 5.4.

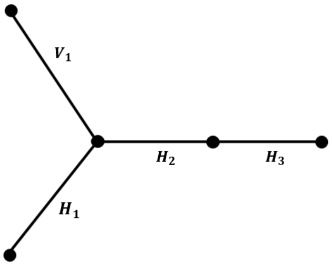

Figure 5.4: Abstract tree associated to the configuration $\mathcal{D}$

Now, we need to construct the Shishikura's tree T and define the function F for the configuration given. In order to do that, by assuming that the degree of each must be 2 for segments x_1 and x_3, it is necessary to determinate the length of the edges of T in term of the periodic arc x_5, for which we will consider the following arrowed tree; see Figure 5.5.

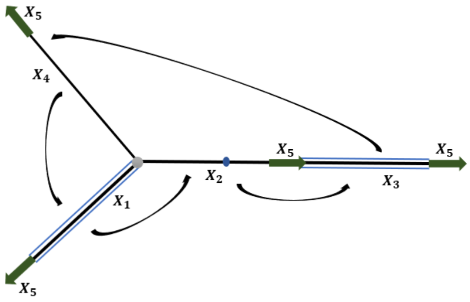

Figure 5.5: Arrowed tree associated to the configuration $\mathcal{D}$

For this case, let us determine the lengths of the segments that form the abstract tree, in order to do it, we consider the following system of equations.

$$\begin{cases} 2x_1 = x_4 + x_5; \\ x_2 = x_3 + x_5; \\ 2x_3 = x_4; \\ x_4 = x_5 + x_1. \end{cases} \tag{1}$$

Solving the above system of equations, we obtain $x_1 = 2x_5$, $x_2 = \frac{5}{2}x_5$, $x_3 = \frac{3}{2}x_5$, $x_4 = 3x_5$, with $x_5 > 0$. Therefore, it is possible to assign finite lengths to each segment and construct a finite topological tree.

Using the configuration $\mathcal{D}$, we can associate it to an arrowed tree with 8 symbols, say $0_-, 0_+, 1_-, 1_+, 2_-, 2_+, 3_-, 3_+$, according where the corresponding segment starts and finishes, see Figure 5.6, these symbols are used in Table 5.1.

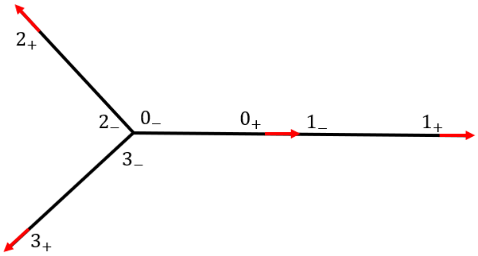

Figure 5.6: Labelled tree associated to the configuration $\mathcal{D}$

Now, we shall construct a local model for singular points in the given tree. In Table 5.1, we present the function g at each point of X, where the expression $x \to \varphi$ with $x \in X$, means that $p_\beta = \varphi$, for the branch β at x containing the segment x. Considering that the degree of each must be 2 for some functions given that $DF = 2$, for some segments, and each function has a Siegel disc and by using quasi-conformal surgery, we have a p-cycle of Herman rings.

Point x	$g_{\vert\widehat{\mathbb{C}}_x} : \widehat{\mathbb{C}}_x \to \widehat{\mathbb{C}}_{F(x)}$	$x \to \varphi$
$1_+, 3_+, 0_+$	$z \mapsto e^{2\pi i\theta} z(1-z)$	$x_2 \to 1, x_1 \to 1, x_5 \to \infty$
$0_-, 2_-, 3_-$	$z \mapsto z$	$x_5 \to \infty,\ x_4 \to 0$
$1_-, 2_+$	$z \mapsto e^{\pi i\theta} z(1-z)$	$x_5 \to \infty, x_3 \to 2, x_2 \to 0$

Tabla 5.1: The local model for singular orbits of T

By Theorem 5.3 the configuration is realizable by the rational function f. Moreover, by Theorem 5.4 there exists a transcendental meromorphic function g which has the same configuration of the p-cycle of Herman rings as f.

Using similar arguments of the example given above, it is possible to give a configuration of a p-cycles of Herman rings, removing H_3, that satisfies assumption (ii) of Theorem 5.2.

Example 2. *There exists* $g \in \mathcal{M}$ *such that* g *has* 7*-cycle of Herman rings* $H_1, ..., H_4$, V_1, V_2, V_3, *such that* H_i *are nested Herman rings,* $1 \leq i \leq 4$ *and* V_i *are Herman rings between* H_i *and* H_{i+1}, $1 \leq i \leq 3$.

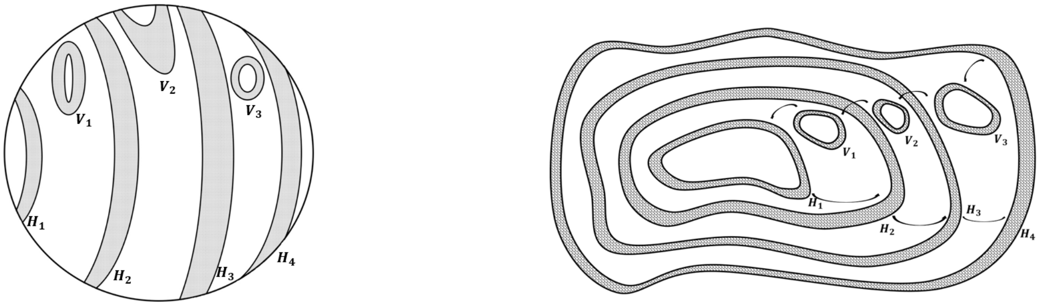

Figure 5.7: Configuration of annuli $\mathcal{D}$: In the sphere (left) and in the plane (right)

In Figure 5.7, we show the configuration $\mathcal{D}$ associated to the 7-cycle of Herman rings $H_1, ..., H_4$, V_1, V_2, V_3. We will proceed to prove that $\mathcal{D}$ is realizable by a rational function f and therefore by a transcendental meromorphic function g.

Using the results from Section 5.1, first consider the abstract tree associate to $\mathcal{D}$, see **I** in Section 5.1 and Figure 5.8.

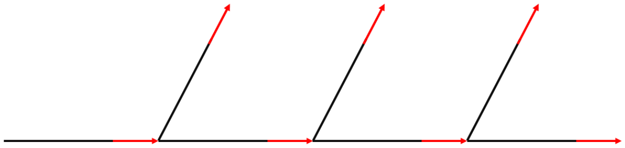

Figure 5.8: Abstract tree for the configuration $\mathcal{D}$ in the complex plane

Second, to construct the Shishikura's tree T and define the function F for the configuration given we know from Lemma 5.1 and Remark 5.1 that it is only necessary to determine the length of the edges of T in term of the periodic arc x_8 and use the fact that F permutes cyclically such segments. For our case we label T with the symbols $x_1, x_2, x_3, x_4, x_5, x_6$ and x_7 whose orientation in $\mathcal{D}$ is given in Figure 5.9. Thus we observe the following facts:

(i) $F(x_i) = x_{(i+1) mod\, 7}$, for $1 < i < 7$.

(ii) Using the definition in Lemma 5.1 we have that $DF = 2$ on the segment x_1, x_2 and x_3 represented by the blue lines in Figure 5.9.

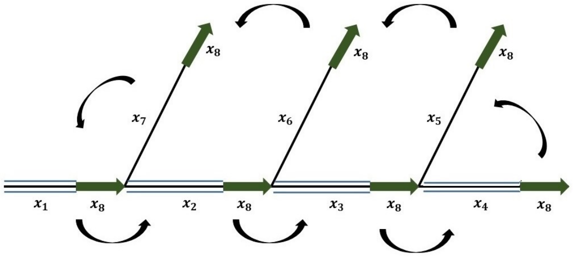

Figure 5.9: Shishikura's tree for the configuration $\mathcal{D}$

Since we assumed that F maps x_1 onto x_2, x_2 onto x_3 and so on, this allows us to establish the following system of equations:

$$\begin{cases} 2x_1 = x_7 + x_8; \\ 2x_2 = x_6 + x_8; \\ 2x_3 = x_5 + x_8; \\ 2x_4 = x_5; \\ x_5 = x_3 + x_8; \\ x_6 = x_2 + x_8; \\ x_7 = x_1 + x_8. \end{cases} \tag{2}$$

The solution of the system is: $x_1 = 2x_8, x_2 = 2x_8, x_3 = 2x_8, x_4 = \frac{3}{2}x_8, x_5 = 3x_8, x_6 = 3x_8$ and $x_7 = 3x_8$ with $x_8 > 0$ arbitrary. Thus, the function F and therefore T satisfies the five properties of Lemma 5.1.

Finally, we shall establish the model of singular points for T, so we need to determinate the functions $g_{|_{\hat{\mathbb{C}}_x}}$ and the set p_β.

From Definition 5.1 $p_\beta = \{\beta \in \mathcal{B}_x : x \in X \cup F(X)\}$, where X is a sub-finite set of T containing $Sing(T, F) = \{\text{end points, branch points of } T\} \cup \pi\{\text{critical points of } f\}$ and $\partial I_j = \{\pi(\overline{H}) : H$ is a Herman ring of $f\}$. Now, let us define the maps on the singular points of T. Since T has an arrangement of seven topological annuli, thus T has seven arrowed segments each one with two end points. The segments of T are labeled with the numbers from 0 to 6. Thus, T posses 14 end points, denoted by $0_+, 0_-, 1_+, 1_-, ..., 6_+, 6_-$, which correspond to the singular points of T; see Figure 5.10.

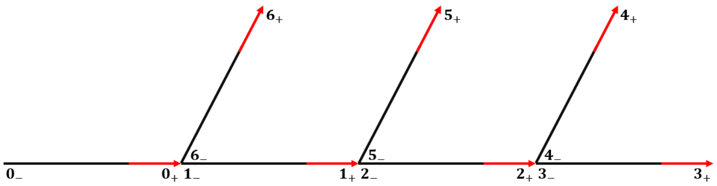

Figure 5.10: The 14 end points in the Shishikura's tree

The local model for T is presented in Table 5.2, where $x \to \varphi$ (x is a singular point) means that $p_\beta = \varphi$, for the branch β at x containing the segment x. By Definition 5.1 and the fact that $DF = 2$ on the segments x_i, $1 \le i \le 3$, the function g defined on the respective singular points must be a rational function of degree 2 with a Siegel disc.

x= singular point	$g_{\|\hat{\mathbb{C}}_x} : \hat{\mathbb{C}}_x \to \hat{\mathbb{C}}_{F(x)}$	$x \to \varphi$
$0_-, 1_+, 1_-, 3_+, 3_-, 4_+, 6_+$	$z \mapsto z$	$x_8 \to 0$
$0_+, 2_+$	$z \mapsto e^{i\pi\theta} z(1-z)$	$x_2, x_3, x_7 \to \infty,\ x_8 \to 0$
$5_+, 5_-$	$z \mapsto z$	$x_6 \to \infty,\ x_8 \to 0$
$2_-, 4_-$	$z \mapsto e^{i(\pi-\theta)} z(1-z)$	$x_6, x_1, x_3 \to \infty,\ x_8 \to 0$

Tabla 5.2: The local model for T

By Theorem 5.3 the configuration is realizable by the rational function f. Moreover, by Theorem 5.4 there exists a transcendental meromorphic function g which has the same configuration of Herman rings as f.

Using similar arguments of the example given above, it is possible to give different examples with different configurations of p-cycles of Herman rings that satisfies the hypothesis of Theorem C; see Appendix A.

5.3 Proofs of Theorems C, D and E

If a function f is a meromorphic function in $\mathbb{C}$ and D is a simply-connected domain in $\widehat{\mathbb{C}}$, whose boundary is a sectionally analytic Jordan curve γ, then an *island* (with respect to f) over D is a bounded component I of $f^{-1}(D)$ so that $f(I) = D$ and the map $f : I \to D$ is a finite branched cover. Thus we have $f(\partial I) = \gamma$.

To prove Theorems A and B we shall need the following result which can be found in [44, Theorem 5.5].

Lemma 5.2. *If f is a transcendental and meromorphic function in $\mathbf{C}$, then given any three simply-connected domains D_i, $1 \le i \le 3$, with sectionally analytic boundaries, such that $\overline{D_i}$ are mutually disjoint, there is at least one value of i such that f has infinitely many simply-connected islands over D_i.*

5.3.1 Proof of Theorem C

By the results given in Section 5.1, it is possible to have a configuration of a p-cycle of Herman rings $H_1, H_2, \ldots, H_p$, $p \ge 3$, realizable by some rational function f removing the cases when the p-cycle of Herman rings is nested and almost nested. Then by Theorem 5.4, there exists g a function in class $\mathscr{M}$ that realizes the configuration of the

p-cycle of Herman rings of f. By the surgery constriction of g in [40], the function g does not have wanderings domains nor Baker domains, since the set of critical values $CV$is finite, and there are no completely invariant domains. Thus by Theorem 5.1 the residual Julia set of g is not empty.

We shall prove that the residual Julia set of g consists of singleton buried components which are dense. We separate the proof in the following cases: (i) each Herman ring H_i in the p-cycle lies in the unbounded component of the complement of H_j, for each i, $j = 1, \ldots, p$ and (ii) the complement of (i) avoiding cases (a) and (b).

(i) Since each Herman ring H_i in the p-cycle lies in the unbounded component of the complement of H_j, for each i and j, then the bounded components of the complement of each Herman ring H_i of the p-cycle are disjoint. Now take in each Herman ring of the p cycle sectionally analytic curves γ_i, $1 \leq i \leq p$, such that $\gamma_i \subset H_i$, the curves γ_i cannot be deformed to a point in H_i and $\gamma_i \cap \gamma_j = \emptyset$, $i \neq j$. Now, let Δ_i be the bounded component of γ_i^c, that is, Δ_i is $int(\gamma_i)$. We can take Δ_i, $1 \leq i \leq p$, by the configuration of the p-cycle of Herman rings we have that $\Delta_i \cap \Delta_j = \emptyset$, $i \neq j$, and $\overline{\Delta_i} \cap \overline{\Delta_j} = \emptyset$, where each Δ_i contains points of the Julia set.

From here the proof follows by using the concept of an island and Lemma 5.2, as in Theorem 5.2. We shall write it to complete the proof.

Pick a point p_1 in the Julia set and any open neighborhood U_1 which contains the point p_1. Then there is some $k \in \mathbb{N}$ such that the k-iterate of g, that is g^k, has a pole or pre-pole β in U_1, since $g \in \mathscr{M}$. We recall that functions in class $\mathscr{M}$ have at lest one non-omitted pole, which means that $g{-}n(\infty)$ is infinite.

Take a a neighborhood U of β such that $\overline{U} \subset U_1$ which is mapped by g^k locally equivalently (except perhaps at β) onto a neighborhood N of ∞. It follows from Lemma 5.2 that g has a simply connected island $I \subset N \cap \mathbb{C}$ which lies over one of the Δ_i, say Δ_2. The branches of $(g^k)^{-1}$ which takes values in U for $z \in I$ are each univalent in I. Take one of such branch, say h, which maps equivalently onto a simply connected component $U_2 \subset U \setminus \beta$. Thus g^{k+1} maps U_2 onto Δ_2, so U_2 contains points in the Julia set since $\Delta_2 \cap \mathcal{J}(g) \neq \emptyset$. Therefore, U_2 contains a point $p_2 \in \mathcal{J}(g)$ and $\partial U_2 \subset \mathcal{F}(g)$.

Now, we can replace U_1 by U_2, p_1 by p_2 and do the same argument above, so we obtain U_3 and $p_3 \in \mathcal{J}(f)$ and $\partial U_3 \subset \mathcal{F}(g)$. Continuing this process inductively we obtain a sequence of nested simply-connected domains U_n, each of them contains a point $p_n \in \mathcal{J}(g)$. The sets $\overline{U_n}$ shrink to a single point $p = lim_{n \to \infty} p_n$ which is in the Julia set. Further, ∂U_n is in the Fatou set, so p is a singleton buried component in the Julia set. Since we start with a point p_1 in the Julia set and any arbitrary neighborhood of p_1, then we can assure that the singleton buried components are dense in the Julia set.

(ii) For this case the configuration of the p-cycle of Herman rings $H_1, H_2, \ldots, H_p$, with $p \geq 3$, where (a) and (b) do not occur, can be very diverse. The idea of the proof is similar to (i), take sectionally analytic curves γ_i in the p-cycle of Herman rings such that $\gamma_i \subset H_i$, $1 \leq i \leq p$, $\gamma_i \cap \gamma_j = \emptyset$, $i \neq j$ and the curves γ_i cannot be deformed to a point in H_i, $1 \leq i \leq p$. Let Δ_i be either the bounded component of γ_i^c or the unbounded component of γ_i^c, $1 \leq i \leq p$. Since the cases (a) and (b) do not occur in the configuration of the p-cycle of Herman rings, we can always take three Δ_i, $1 \leq i \leq 3$, such that $\Delta_i \cap \Delta_j = \emptyset$, $i \neq j$, and $\overline{\Delta_i} \cap \overline{\Delta_j} = \emptyset$. Since each Δ_i contains points of the Julia set, the result follows from the same arguments as in (i) above. Thus $\mathcal{J}(g)$ contains singleton components which are dense and buried. ■

Observe that in the proof of Theorem C, it is important to get the three domains Δ_i, $1 \leq i \leq 3$, in order to use the definition of an island and Lemma 5.2, so the configuration of the p-cycle of Herman rings of g must avoid cases (a) and (b) in Theorem C.

Corollary 5.1. *There exists g in class $\mathscr{M}$ with no wandering domains, no completely invariant Fatou components components, $\mathcal{J}(g)$ is disconnected in such a way that g satisfies Theorem C and the residual Julia set of g contains buried singleton components of $\mathcal{J}(g)$.*

5.3.2 Proof of Theorem D

By the result given in [13, Theorem (i) and (iii), page 268] there exist $h \in \mathscr{M}$, such that the Fatou set has a bounded wandering component, say W, which is not simply connected (the connectivity can be finite or infinite) and no nested, thus we have that $h^k(W_i) = W_{i+1}$ for $k, i \in \mathbb{N}$. The construction does not allow completely invariant components in the Fatou set.

Now take γ_i sectionally analytic curves $\gamma_i \subset W_i \subset \mathcal{F}(h)$, for $i \in \mathbb{N}$, such that the curves γ_i cannot be deformed to a point in W_i. Denote by Δ_i the bounded component of of the complement of γ_i, that is, Δ_i is the $int(\gamma_i)$. Since $\gamma_i \cap \gamma_j = \emptyset$, $i \neq j$, thus Δ_i are disjoint, so $\overline{\Delta_i}$. Taking three Δ_i, $1 \leq i \leq 3$, which contains points of the Julia set, the proof follows from the same arguments as in Theorem C.

Thus $\mathcal{J}(h)$ contains singleton components which are dense and buried. ■

5.3.3 Proof of Theorem E

Suppose that there are no buried components not buried points in the Julia set. Take U some component of the Fatou set. Thus we have either $f^n(U) = U_1$ or $f^n(U_2) = U_2$, since U_1 and U_2 are the only two invariant components in the Fatou set.

If $iy \in \partial U \in \mathcal{J}(f)$, $y \in \mathbb{R}$, then for large n, $f^n(iy) \in f^n(U) = U_1$ (or U_2), which is a simply connected bounded component.

(i) $f^n(iy) \notin U_1$ (or U_2), for large n, because U_1 is simply connected component in the Fatou set.

(ii) $f^n(iy) \notin \partial U_1$ (or U_2), for large n, because U_1 is bounded, so $f^n(iy) \nrightarrow \infty$.

Thus all points in the imaginary axes are buried except for zero, since it is in the intersection of the boundaries of U_1 and U_2.

Thus there are buried components iy when $y \in R^+$ and $y \in \mathbb{R}^-$. ■

Chapter 6

Conclusions

In this thesis we investigate some dynamical properties of the family:

$$f_{\lambda,\mu,z_0}(z) = \lambda \sin(z) + \frac{\mu}{z - z_0}, \text{ where } \lambda \in \mathbb{C} \setminus \{0\}, \mu \in \mathbb{R}^+ \setminus \{0\} \text{ and } z_0 \in \mathbb{R}.$$

A. When $z_0 \neq 0$ we have two cases:

(i) The real parameters λ, μ, z_0 such that $0 < |\lambda| < 1$, $\mu \in \mathbb{R}^+ \setminus \{0\}$ sufficiently small and $|z_0| \geq |\lambda| + \frac{\pi}{2}$.

(ii) The complex parameter $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \frac{1}{e+1}$ and the real parameters $\mu \in \mathbb{R}^+ \setminus \{0\}$ sufficiently small and $|z_0| \geq |\Re(\lambda)| + \frac{\pi}{2}$, where $\Re$ denotes the real part of z.

B. When $z_0 = 0$ and the parameters $\lambda \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{R}^+ \setminus \{0\}$.

In **A.**(i) and **A.**(ii) we give conditions on the parameters λ, μ and z_0 for the existence of an attracting multiply connected completely invariant component in the Fatou set of f_{λ,μ,z_0}. In **B** we study some geometric properties of functions in the family $f_{\lambda,\mu}$ and give some conditions on the parameters λ and μ for the existence of different types of components in the Fatou set of $f_{\lambda,\mu}$, such as: attracting, parabolic, Siegel discs and combination of two components. Also, we investigate some approximations of slices of the parameter space of the family f_{λ,μ,z_0} and show some examples of Fatou and Julia sets of the family f_{λ,μ,z_0}.

C. We investigate the residual Julia set for transcendental meromorphic functions with a $p-$cycle of Herman rings and we prove some results related to the existence of buried singleton components in the Julia set under certain conditions.

The main results are described in the next subsection.

6.1 Contributions of the thesis

A. In the investigation of the family $f_{\lambda,\mu,z_0}(z) = \lambda\sin(z) + \frac{\mu}{z-z_0}$, where $\lambda \in \mathbb{C} \setminus \{0\}$, $\mu \in \mathbb{R}^+ \setminus \{0\}$ and $z_0 \in \mathbb{R}$, we obtained the following results:

(I) **Theorem A**. *If λ, μ, z_0 are real parameters such that $0 < |\lambda| < 1$, $\mu > 0$ sufficiently small and $|z_0| \geq |\lambda| + \frac{\pi}{2}$, then the family $f_{\lambda,\mu,z_0}(z) = \lambda\sin(z) + \frac{\mu}{z-z_0}$ has an attracting completely invariant component in the Fatou set, which is multiply connected.*

(II) **Theorem B.** *If $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \frac{1}{e+1}$, $\mu > 0$ and z_0 are real parameters such that $\mu > 0$ sufficiently small and $|z_0| \geq |\Re(\lambda)| + \frac{\pi}{2}$, then the family $f_{\lambda,\mu,z_0}(z) = \lambda\sin(z) + \frac{\mu}{z-z_0}$, has an attracting completely invariant component in the Fatou set, which is multiply connected, where $\Re(\lambda)$ denotes the real part of the parameter λ.*

(III) Approximation of a slice of the parameter space and the dynamical planes (the Fatou and Julia sets) of the family f_{λ,μ,z_0}, where the parameters λ, μ and z_0 satisfy the conditions given in Theorems A and B.

B. In the investigation of the family $f_{\lambda,\mu}(z) = \lambda\sin(z) + \frac{\mu}{z}$, where $\lambda \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{R}^+ \setminus \{0\}$, we obtained the following results:

(IV) For the the family $f_{\lambda,\mu}(z) = \lambda\sin(z) + \frac{\mu}{z}$ in the case when the pole z_0 is the origin, we obtain: (i) Some geometric properties of $f_{\lambda,\mu}$. (ii) The fixed points and singular values of $f_{\lambda,\mu}$. (iii) Some conditions under the parameters λ and μ and give examples of different types of components in the Fatou set of $f_{\lambda,\mu}$, such as: (1) attracting components, (2) parabolic components, (3) Siegel discs and (4) Fatou set with more than one type of components. (iv) We plot approximations of the Fatou and Julia sets for $f_{\lambda,\mu}(z) = \lambda\sin(z) + \frac{\mu}{z}$ in the four cases given in (iii).

C. In the investigation of the residual Julia set for transcendental meromorphic functions with a $p-$cycle of Herman rings, we prove the following results:

(V) **Theorem C.** *There exist $g \in \mathscr{M}$ which satisfies that the Fatou set of g has a p-cycle of Herman rings $H_1, H_2, \ldots, H_p$, $p \geq 3$, removing the cases when*

(a) the p-cycle of Herman rings is nested and

(b) the p-cycle of Herman rings is almost nested,

such that the Julia set of g contains singleton components which are dense and buried.

(VI) **Theorem D.** *There exist $h \in \mathscr{M}$ which satisfies that the Fatou set has a bounded wandering component which is neither simply connected nor nested multiply connected, such that the Julia set contains singleton components which are dense and buried.*

(VII) **Theorem E.** *Suppose that f is a transcendental meromorphic function, the Fatou set has only two attracting simply connected invariant bounded components U_1 and U_2, such that:*

(a) $\partial U_1 \cap \partial U_2 = \{0\}$, where zero is a non-omitted pole of f,

(b) U_1 and U_2 are symmetric with respect to the imaginary axes, and

(c) the imaginary axes is in the Julia set, where $f^n(iy) \to \infty$, with $y \in \mathbb{R}$.

Then the imaginary axes is a buried component, except for zero.

The investigation in this thesis has been published in the following Journals:

(i) Theorem 4.1 in *Discontinuity, Nonlinearity and Complexity* (2016); see [36].

(ii) Theorem A with an approximation of a slice of the parameter spaces and dynamical planes for the family f_{λ,μ,z_0} in *Actas 1, Paphiros, IMATE UNAM* (2019); see [35].

(iii) Chapter 5 is under review in *Israel Journal of Mathematics* (2020); see [22].

(iv) Appendix A in *Revista Integración, temas de matemáticas* (2020); see [21].

(v) Section 4.2 is in preparation to be sent to a journal (2020).

6.2 Future research

In the thesis we stated some conjectures related to the study of the dynamics of the family $f_{\lambda,\mu,z_0}(z) = \lambda \sin(z) + \frac{\mu}{z-z_0}$, with $\lambda \in \mathbb{C} \setminus \{0\}$, $\mu \in \mathbb{R}^+ \setminus \{0\}$ and $z_0 \in \mathbb{R}$, and the residual Julia set of transcendental meromorphic functions with a $p-$cycle of Herman rings. We state some of them below.

Conjecture 1. The slice of the parameter space $\mathbb{M}_{\lambda,((0.3),(3.2))}$ is not bounded in the real axis.

Conjecture 2. If $0 < |\lambda| < 1$ and $0 < |\lambda'| < 1$ in $\mathbb{M}_{\lambda,((0.3),(3.2))}$, where the parameters μ and z_0 satisfy the conditions given in Theorems A and B, then the function f_{λ,μ,z_0} is quasi-conformal conjugate to f_{λ',μ,z_0}.

Conjecture 3. There exists a range of parameters (μ, z_0) with the conditions given in Theorems A and B, such that the approximations of slices of the space of parameters are homeomorphic.

Conjecture 4. It is possible to extend Theorem B for different complex parameters λ, μ and z_0 with an appropriate boundedness of the parameters.

Conjecture 5. There exists a set of parameters $\mathrm{A} \subset \mathbb{C}\times\mathbb{C}$ such that if $(\lambda, \mu) \in \mathrm{A}$, $\lambda, \mu \neq 0$, then the Fatou set of $f_{\lambda,\mu}(z) = \lambda \sin(z) + \frac{\mu}{z}$ only contains two attracting components.

Conjecture 6. It is possible to prove Theorem C if the Fatou set has a cycle of nested Herman rings without using the concept of island.

Appendix A

Examples of realizable configurations for transcendental meromorphic functions

In Appendix A we shall construct examples of specific configurations of Herman rings realizable by a rational function f by using the results in Chapter 5. Moreover, using the main theorem in [40], we obtain that in every example there exists a transcendental meromorphic function g which has the same configuration of the p-cycle of Herman rings as f; see also [21].

Example 1. Consider the configuration $\mathcal{C}_1$ of the cyclic annuli given in Figure A.1.

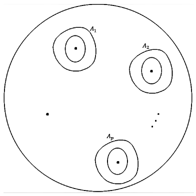

Figure A.1: Configuration $\mathcal{C}_1$ of a p-cycle of annuli

This configuration was proposed by Shishikura in [74]. The abstract tree associated to the configuration $\mathcal{C}_1$ is given in Figure A.2.

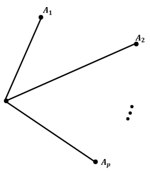

Figure A.2: Abstract tree associated to the configuration $\mathcal{C}_1$

As we mention in Section 5.1, it is necessary to construct an associated arrowed tree; see Figure A.3.

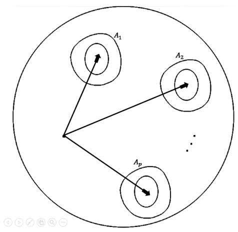

Figure A.3: Associated arrowed tree

By [74] the configuration $\mathcal{C}_1$ is realizable by a rational and by using [40] there exists a transcendental meromorphic function g for which the the configuration $\mathcal{C}_1$ is realizable.

Example 2. Consider the following configuration $\mathcal{C}_2$ of a p-cycle of annuli given in Figure A.4.

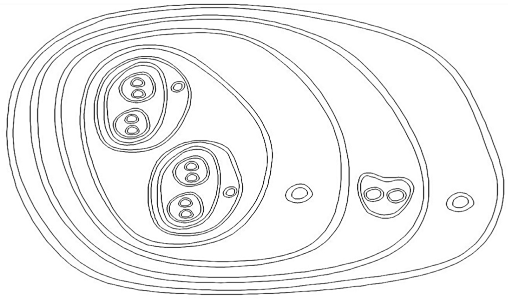

Figure A.4: Configuration $\mathcal{C}_2$ of a p-cycle of annuli.

Following the algorithm described in Section 5.1, we obtain the tree associated to the configuration $\mathcal{C}_2$; see Figure A.5.

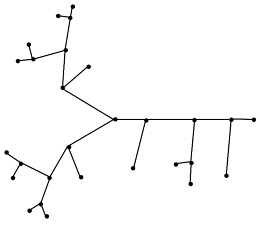

Figure A.5: Tree associated to the configuration $\mathcal{C}_2$

As in the examples in Section 5.2, we associate an arrowed tree associated to the abstract tree; see Figure A.6.

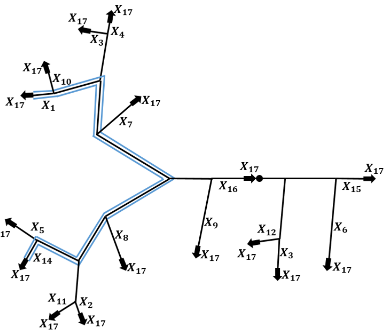

Figure A.6: Arrowed tree associated to the configuration $\mathcal{C}_2$

From the above tree we can determinate the length of the segments solving the following system of equations associated to the arrowed tree:

$$\begin{cases}
2x_1 = x_5 + x_{11} + 2x_{14} + x_{17}; \\
x_2 = x_8 + x_9 + x_12 + x_{16} + x_{17}; \\
x_3 = x_6 + x_7 + x_{15} + x_{16} + x_{17}; \\
x_4 = x_{13} + x_{10} + x_1 + x_{17}; \\
x_5 = 2x_{14} + x_{11} + x_2 + x_8 + x_9 + x_{16} + x_{12} + x_3 + x_{17}; \\
x_6 = x_{15} + x_{16} + x_{17}; \\
x_7 = x_4 + x_{13} + x_{10} + 2x_1 + x_5 + 2x_{14} + x_{11} + x_2 + x_{17}; \\
x_8 = x_{17}; \\
x_9 = x_{16} + x_{12} + x_3 + x_6 + x_{15} + x_7 + x_4 + x_{13} + x_7; \\
x_{10} = 2x_1 + x_5 + 2x_{14} + x_{17}; \\
x_{11} = x_2 + x_8 + x_{16} + x_9 + x_{17}; \\
x_{12} = x_3 + x_6 + x_{15} + x_7 + x_4 + x_{17}; \\
x_{13} = x_{10} + 2x_1 + x_5 + x_{17}; \\
2x_{14} = x_{11} + x_2 + x_8 + x_9 + x_{16} + x_{12} + x_3 + x_6 + x_{17}; \\
x_{15} = x_{17}; \\
x_{16} = x_{17}.
\end{cases} \tag{1}$$

The above system of equations has solution as follows:
$x_1 = \frac{-614}{1289}x_{17}, x_2 = \frac{-1696}{1289}x_{17}, x_3 = \frac{1783}{1289}x_{17}, x_4 = \frac{-3527}{1289}x_{17}, x_5 = \frac{-2063}{1289}x_{17}, x_6 = -3x_{17}, x_7 = \frac{-5951}{1289}x_{17}, x_8 = x_{17}, x_9 = \frac{-4313}{1289}x_{17}, x_{10} = \frac{-1100}{1289}x_{17}, x_{11} = \frac{-1356}{1289}x_{17}, x_{12} = \frac{-1250}{1289}x_{17}, x_{13} = \frac{-3102}{1289}x_{17}, x_{14} = \frac{451}{1289}x_{17}, x_{15} = x_{17}, x_{16} = x_{17}.$

We associate an arrowed tree with the symbols $0_-, 0_+, 1_-, 1_+, ..., 16_-, 16_+$ according where the corresponding segment starts and finishes. With these symbols, we present the local model for singular orbits of the abstract tree associated to the configuration $\mathcal{C}_2$; see Table A.1.

Point x	$g_{\vert_{\widehat{\mathbb{C}}_x}} : \widehat{\mathbb{C}}_x \to \widehat{\mathbb{C}}_{F(x)}$	$x \to \varphi$
0_+	$z \mapsto e^{\pi i\theta} z(1-z)$	$x_5 \to 0, x_{14}, x_2 \to 1$
$8_+, 9_+, 16_+, 12_+$	$z \mapsto z$	$x_5 \to 0$
$6_-, 15_-$	$z \mapsto e^{2\pi i\theta} z(1-z)$	$x_6, x_{15} \to 0,$
$13_-, 10_-, 1_-, 7_-$	$z \mapsto z e^{2\pi i\theta} z(1-z)$	$x_4 \to 0$
$5_-, 14_-, 1_- 2_-, 9_- x_{16}, x_3$	$z \mapsto z$	$x_5 \to 0$
$6_-, 15_+,$	$z \mapsto e^{\pi i\theta} z(1-z)$	$x_{16}, x_{15} \to \infty,$
$7_+, 4_+, 13_- 10_+, 1_-, 5_+, 14_+, 11_+, 2_+,$	$z \mapsto e^{2\pi i\theta} z(1-z)$	$x_{16}, \to 2,$
8_+	$z \mapsto z$	$x_{16} \to 0,$
$9_+, 12_+, 3_+ 6_+, 15_+, 4_+ 13_+$	$z \mapsto e^{2\pi i\theta} z(1-z)$	$x_{16} \to 3,$
$10_+, 1_+, 5_+, 14_+$	$z \mapsto e^{2\pi i\theta} z(1-z)$	$x_{16} \to 2$
$2_+, 8_+, 9_+, 16_+$	$z \mapsto e^{\pi i\theta} z(1-z)$	$x_{16} \to 4$
$3_+, 6_+, 15_+$	$z \mapsto e^{2\pi i\theta} z(1-z)$	$x_{16} \to 2$
$10_+, 1_+, 5_+$	$z \mapsto e^{2\pi i\theta} z(1-z)$	$x_{16} \to 2$
$11_+, 2_+, 8_+, 9_+, 12_+, 3_+ 6_+$	$z \mapsto e^{2\pi i\theta} z(1-z)$	$x_{16} \to 4$
15_+	$z \mapsto e^{\frac{\pi}{2} i\theta} z(1-z)$	$x_{16} \to 5$

Tabla A.1: The local model for singular orbits of T

By Theorem 5.3 and 5.4, we conclude that there exists a transcendental meromorphic function g such that g has a cycle of Herman rings of period 15.

Now, we will give an example of a non realizable configuration for to illustrate the importance of the local degree.

Example 3. Consider the configuration $\mathcal{C}_3$ of a p-cycle of annuli given in Figure A.7.

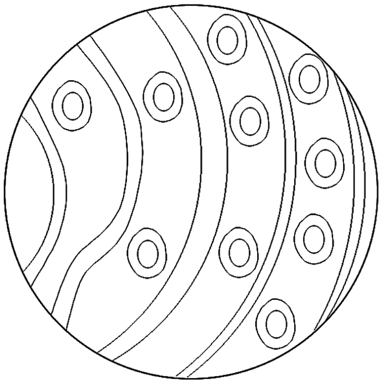

Figure A.7: Configuration $\mathcal{C}_3$ of a p-cycle of annuli

As the previous two examples, we need to determinate the lengths of the segments that form the tree, so first we construct the abstract tree associated to the given configuration; see Figure A.8.

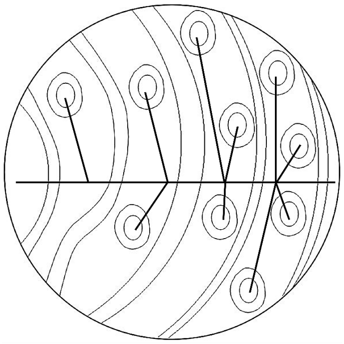

Figure A.8: Abstract tree associated to the configuration $\mathcal{C}_3$

We take the arrowed tree associated to this abstract tree; see Figure A.9.

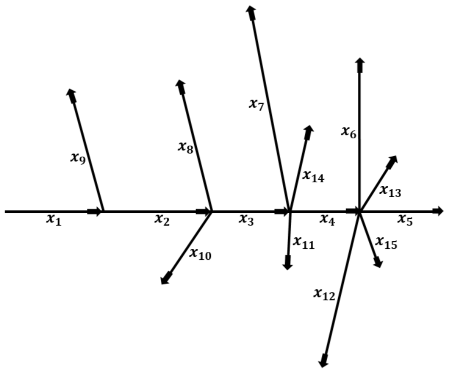

Figure A.9: Arrowed tree associated to the configuration $\mathcal{C}_4$

The tree associated determines a linear system of equations given as follows.

$$\begin{cases} -x_1 + x_2 + x_3 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{13} + x_{14} + 9x_16 = 0 - x_2 + x_9 + x_{16} = 0; \\ -x_3 + x_8 + x_{10} + 2x_6 = 0; \\ -x_4 + x_7 + x_{11} + x_{14} + 3x_{16} = 0; \\ -x_5 + x_6 + x_{12} + x_{13} + x_{15} + 4x_{16} = 0; \\ -x_6 + x_13 + x_16 = 0; \\ x_4 - x_7 + +x_{14} + 2_x16 = 0; \\ x_3 - x_8 + x_16 = 0; \\ x_2 - x_9 + x_{16} = 0; \\ x_1 - x_{10} + x_{16} = 0; \\ x_3 - x_{11} + x_{16} = 0; \\ x_4 - x_{12} + x_16 = 0; \\ x_5 - x_{13} + x_15 + 2x_{16} = 0; \\ x_1 + x_2 + x_3 + x_4 + x_7 + x_8 + x_9 + x_{10} + x_{11} - x_{14} + 2x_{16} = 0; \\ x_1 + x_2 + x_3 + x_4 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} - x_{15} + 10x_{16} = 0. \end{cases} \tag{2}$$

The coefficients of the x_i's correspond to the local degree of the desired function. The solution of the system of equations is:

$x_1 = 0, x_2 = 0, x_3 = 4x_{15}, x_4 = -3x_{15}, x_5 = 0, x_6 = x_15, x_7 = -5x_{15}, x_8 = 4x_{15}, x_9 = 0, x_{10} = 0, x_{11} = 4x_{15}, x_{12} = 0, x_{13} = 0, x_{14} = 4x_{15}$, where x_{15} denotes the length of the periodic arc. Thus, this configuration is non realizable, since we obtain segments whose length is zero.

Now, if we consider other conditions under the local degree on some segments, in this case three, two and two on the segments x_1, x_2 and x_3, respectively, we obtain a new arrowed tree associated to the configuration $\mathcal{C}_4$; see Figure A.10.

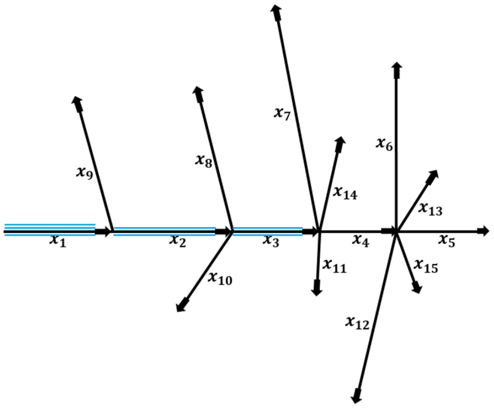

Figure A.10: Second arrowed tree associated to the configuration $\mathcal{C}_3$

The solution of the system of equations is as follows: $x_1 = -\frac{249}{65}x_{16}, x_2 = 2x_{16}, x_3 = \frac{11}{65}x_{16}, x_4 = -\frac{239}{65}x_{16}, x_5 = -\frac{269}{65}x_{16}, x_6 = -\frac{211}{65}x_{16}, x_7 = -\frac{33}{5}x_{16}, x_8 = \frac{76}{65}x_{16}, x_9 = 3x_{16}, x_{10} = -\frac{184}{65}x_{16}, x_{11} = \frac{76}{65}x_{16}, x_{12} = -\frac{174}{65}x_{16}, x_{13} = -\frac{276}{65}x_{16}, x_{14} = -\frac{-64}{13}x_{16}, x_{15} = -\frac{137}{65}x_{16}$; where $x_{16} > 0$ denotes the length of the periodic arc.

By Theorem 5.3 and 5.4, it follows that there exists a transcendental meromorphic function g such that g realizes the configuration $\mathcal{C}_3$.

Bibliography

[1] Abikoff, W. (1970). Some remarks on Kleinian groups. *Ann. of Math. Studies*, **66**, 1–7.

[2] Abikoff, W. (1973). The residual limit sets of Kleinian groups. *Acta Math.*, **130**, 127–144.

[3] Ahlfors, L. V. (1984). *Complex Analysis.* McGraw Hill.

[4] Alexander, D. S. (1995). Gaston Darboux and the History of Complex Dynamics. *Historia Mathematica,* **22**, 179–185.

[5] Alexander, D. S., Iavernaro, F., and Rosa, A. (2012). *Early days in Complex Dynamics: A history of Complex Dynamics in one variable during 1906-1942*, American Mathematical Society.

[6] Baker, I. (1968). Repulsive fixpoints of entire functions. *Mathematische Zeitschrift*, **104**, 252–256.

[7] Baker, I. and Domínguez, P. (2000). Residual Julia sets. *J. of Analysis*, **8**, 121–137.

[8] Baker, I. N. (1975). The domains of normality of an entire function. *Ann. Acad. Sci. Fenn. Ser. AI Math*, **1**, 277–283.

[9] Baker, I. N. (1976). An entire function which has wandering domains. *J. of the Australian Mathematical Society*, **22**(2), 173–176.

[10] Baker, I. N. (2002). Limit functions in wandering domains of meromorphic functions. *Ann. Acad. Sc. Fenn. Math.*, **27**, 499–505.

[11] Baker, I. N., Kotus, J., and Yinian, L. (1990). Iterates of meromorphic functions II: Examples of wandering domains. *J. London. Math. Soc.*, **42**(2), 267–278.

[12] Baker, I. N., Kotus, J., and Yinian, L. (1991a). Iterates of meromorphic functions I. *Ergodic Theory and Dynamical Systems*, **11**(2), 241–248.

[13] Baker, I. N., Kotus, J., and Yinian, L. (1991b). Iterates of meromorphic functions III: Preperiodic domains. *Ergodic Theory and Dynamical Systems*, **11**(4), 603–618.

[14] Baker, I. N., Kotus, J., and Yinian, L. (1992). Iterates of meromorphic functions IV: Critical finite functions. *Results. Math.*, **22**, 651–656.

[15] Beardon, A. (1991). The components of a Julia set. *Ann. Acad. Sci. Fenn. Ser. A, I. Math*, **16**, 173–177.

[16] Bergweiler, W. (1993). Iteration of meromorphic functions. *Bulletin (New Series) American Mathematical Society*, **29**(2), 151–188.

[17] Bergweiler, W. (1995). An introduction to Complex Dynamics. *Textos de matemática, Universidade de Coimbra,* Série B **6** , 1–38.

[18] Bergweiler, W. and Eremenko, A. (1995). On the singularities of the inverse to a meromorphic function of finite order. *Revista Matemática Iberoamericana*, **11**(2), 355–373.

[19] Bhattacharya, P. (1969). *Iteration of analytic functions.* Ph.D. thesis, Imperial College London.

[20] Bolsch, A. (1997). *Iteration of meromorphic functions with countably many singularities.* Dissertation, Berlin.

[21] Cano, L., Domínguez, P. and Vázquez, J. (2020). Examples of codification of the dynamics of a rational function into a topological tree. *Revista Integración, temas de matemáticas*, **38** (1), 1–14.

[22] Cano, L., Domínguez, P and Vázquez, J. (2020). Examples of residual Julia set of transcendental meromorphic functions. *Israel Journal of Mathematics*, under review.

[23] Cannizzo, J. (2011). *A course in Complex Analysis: From basic results to advanced topics.* Springer Science and Business Media.

[24] Cayley, A. (1879). Applications of the newton-fourier method to an imaginary root of an equation. *Quart. J. of Pure and Applied Math*, **16**, 179–185.

[25] Churchill R.V. and Brown, J.W. (2014). *Complex variables and applications.* McGraw-Hill.

[26] Conway, J. B. (1978). *Functions one Complex Variable I.* Springer Verlag.

[27] Cremer, H. (1932). Über die schrödersche funktionalgleichung und das schwarzsche eckenabbildungsproblem. *Leipziger Berichte*, **84**, 291–324.

[28] Croom, F. H. (2016). *Principles of Topology.* Cengage Learning.

[29] Curry, C. P. and Mayer, J. C. (2010). Buried points in Julia sets. *J. of Difference Equations and Applications*, **16**(5-6), 435–441.

[30] Devaney, R. L. (1991). e^z: Dynamics and bifurcations. *Int. J. of Bifurcation and Chaos*, **1**(2), 287–308.

[31] Domínguez, P. (1998). Dynamics of transcendental meromorphic functions. *Ann. Acad. Sc. Fenn. Math.*, **23**, 225–250.

[32] Domínguez P. and Fagella N., Existence of Herman Rings for Meromorphic functions, *Complex Variables Theory Appl.* **49** (12) (2004), 851-870.

[33] Dominguez, P. and Fagella, N. (2008). Residual Julia sets of rational and transcendental functions. *Transcendental Dynamics and Complex Analysis* (London Math. Soc. Lecture Note Series). Cambridge University Press, 138–164.

[34] Domínguez, P. and Sienra, G. (2002). A study of the dynamics of λsin z. *International Journal of Bifurcation and Chaos*, **12**(12), 2869–2883.

[35] Domínguez, P. and Vázquez J. (2019). Attracting multiply connected components in the Fatou set for the family $f_{\lambda,\mu,z_0}(z) = \lambda\sin(z) + \frac{\mu}{z-z_0}$, where $\lambda, \mu, z_0 \in \mathbb{R} \setminus \{0\}$. *Actas 1, Papirhos*, IMATE UNAM. *Proceedings of the 2018 Workshop on Holomorphic Dynamics*, 133–150.

[36] Domínguez, P., Vázquez, J., and Montes de Oca, M. A. (2016). A study of the dynamics of the family $f_{\lambda,\mu} = \lambda sin(z) + \frac{\mu}{z-k\pi}, where \lambda, \mu \in \mathbb{R}\setminus\{0\}$ and $k \in \mathbb{Z}\setminus\{0\}$. *Discontinuity, Nonlinearity and Complexity.* **5**(4), 407–414.

[37] Epstein, A. L. (1993). *Towers of finite type complex analytic maps.* Ph.D. thesis, City University of New York.

[38] Eremenko, A. and Lyubich, M. Y. (1992). Dynamical properties of some classes of entire functions. *Annales de l'institut Fourier*, **42**(4), 989–1020.

[39] Eremenko, A. E. and Lyubich, M. Y. (1990). The dynamics of analytic transformations. *Leningrad Math. J*, **1**(3), 563–634.

[40] Fagella, N. and Peter, J. (2012). On the configuration of Herman rings of meromorphic functions. *J. of Mathematical Analysis and Applications*, **394**(2), 458–467.

[41] Fatou, P. (1919). Sur les equations fonctionnelles (deuxiéme mémoire). *Bull. Sci. Math. France*, **47**, 161–271.

[42] Fatou, P. (1926). Sur l'itération des fonctions transcendentes entierés. *Acta Math.*, **47**, 337–370.

[43] Gong, S. and Gong Y. (2007). *Concise Complex Analysis: Revised.* World Scientific Publishing Company.

[44] Hayman, W. K. (1964). *Meromorphic functions.* Oxford Clarendon Press.

[45] Herring, M. (1994). *An extension of the Julia-Fatou theory of the iteration.* Ph.D. thesis. University of London.

[46] Hubbard, J., Schleicher, D., and Sutherland, S. (2001). How to find all roots of complex polynomials by Newton's method. *Inventiones mathematicae*, **146**, 1–33.

[47] Iversen, F. (1914). *Recherches sur les fonctions inverses des fonctions méromorphes.* Imprimerie de la Société de littérature finnoise.

[48] Julia, G. (1918). Sur l'itération des fonctions rationelles. *J. Math. Pures Appl.*, **7**, 47–245.

[49] Kisaka, M. and Shishikura, M. (2008). On multiply connected wandering domains of entire functions. *Transcendental Dynamics and Complex Analysis*, **348**, 217–250.

[50] Koebe, P. (1907). Über die uniformisierung beliebiger analytischer kurven. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, **177**, 191–210.

[51] Köenigs, G. (1883). Recherches sur les substitutions uniformes. *Bulletin des sciences mathiêmatiques et astronomiques*, **2**(7), 340–357.

[52] Lascurain, A. (2007). *Curso basico de variable compleja.* Prensas de Ciencias. UNAM, Facultad de Ciencias.

[53] Markushevich, A. I. (1977). *Theory of functions of a complex variable.* Edit. Mir.

[54] Marsden, J. and Hoffman, M. (1999). *Basic Complex Analysis.* W. H. Freeman.

[55] McMullen, C. (1988). Automorphisms of rational maps. In holomorphic functions and moduli I. *Mathematical Sciences Research Institute Publications*, **10**, 31–60.

[56] McMullen, C. T. (1994). *Complex Dynamics and Renormalization.* Annals of Mathematics Studies. Princeton University Press.

[57] Milnor, J. (2006). *Dynamics in One Complex Variable.* Annals of Mathematics Studies. Princeton University Press.

[58] Montel, P. (1912). Sur les familles de fonctions analytiques qui admettent des valeurs exceptionnelles dans un domaine. *Ann. Sci. de l'École Normale Supérieure.* **29**, 487–535.

[59] Morosawa, S. (1997). On the residual Julia sets of rational functions. *Ergodic Theory and Dynamical Systems*, **17**(1), 205–210.

[60] Morosawa, S., Nishimura, Y., Taniguchi, M. and Ueda, T. (2000). *Holomorphic dynamics.* Cambridge University Press.

[61] Munkres, J. R. (2000). *Topology.* Prentice Hall.

[62] Nayak, T. and Zheng, J.-H. (2011). Omitted values and dynamics of meromorphic functions. *J. of the London Math. Soc.*, **83**, 121–136.

[63] Newman, M. H. A. (1939). *Elements of the Topology of Plane Sets of Points.* Cambridge University Press.

[64] Noguchi, J. (1998). *Introduction to Complex Analysis.* American Mathematical Society.

[65] Palka, B. P. (2012). *An Introduction to Complex Theory.* Springer Verlag.

[66] Qiao, J. (1994). The Julia set of the function z→ z exp (z+ μ). *Chn. Sci. Bull.*, **39**, 529–533.

[67] Qiao, J. (1995). The buried points on the Julia sets of rational and entire functions. *Science in China (Scientia Sinica) Series A*, **12**(38), 1409–1419.

[68] Rippon, P. and Stallard, G. (1999). Iteration of a class of hyperbolic meromorphic functions. *Proceedings of the American Mathematical Society*, **127**(11), 3251–3258.

[69] Rippon, P. J. and Stallard, G. M. (2008). On multiply connected wandering domains of meromorphic functions. *J. of the London Math. Soc.*, **77**(2), 405–423.

[70] Schröder, E. (1870). About infinitely many algorithms for the solution of equations. (Über unendlich viele Algorithmen zur Auflösung der Gleichungen.). *Math. Ann.*, **2**, 317–365.

[71] Schröder, E. (1871). Uber iterite funcktionen. *Math. Ann.*, **3**, 296–322.

[72] Shiff, J. L. (1993). *Normal Families.* Springer Verlag.

[73] Shishikura, M. (1987). On the quasiconformal surgery of rational functions. *Ann. Sci. de l'École Normale Supérieure* **20**(1), 1–29.

[74] Shishikura, M. (1989). Trees associated with the configuration of Herman rings. *Ergodic Theory and Dynamical Systems*, **9**(3), 543–560.

[75] Shishikura, M. (2002). A new tree associated with Herman rings. *Department of Mathematics, Kyoto University*, **1269**, 74–92.

[76] Siegel, C. L. (1942). Iteration of analytic functions. *Ann. of Mathematics*, **43**(4), 607–612.

[77] Sullivan, D. (1985). Quasiconformal homeomorphisms and dynamics I. Solution of the Fatou-Julia problem on wandering domains. *Ann. of Mathematics*, **122**(2), 401–418.

[78] Tsuji, M. (1959). *Potential theory in modern function theory. Chelsea Publishing Company.*

[79] Wilde, I.F. (2006). *Lecture notes on Complex Analysis.* Imperial College Press.

[80] Yau, L. and Ben-Israel, A. (1998). The Newton and Halley methods for complex roots. *The American Mathematical Monthly*, **105**(9), 806–818.

[81] Zheng, J.-H. (2003). Singularities and limit functions in iteration of meromorphic functions. *J. of the London Math. Soc.*, **67**(1), 195–207.