

Benemérita Universidad Autónoma de Puebla
Facultad de Ciencias Físico Matemáticas
Posgrado en Ciencias Físico Matemáticas

Invariant subspaces and group invertibility of closed linear relations

T E S I S

que para obtener el grado de:

Doctor en ciencias matemáticas

presenta: Ivan Moises Roque Tlatelpa

Director de tesis:

DR. Slavisa Djordjevic

Puebla, Puebla, February 24, 2026

Para mi familia.

Acknowledgements

Introduction

The concept of linear relation was introduced by J. von Neumann [10] due to the need to consider differential operators that are not densely defined. Subsequently, M. Gromov studied differential operators, giving rise to the need to further develop the notions introduced by Neumann. Likewise, A. Favini and A. Yagi ([6]) drew upon these new theoretical results to apply methods involving the construction of linear relations in order to solve differential equations. This notion has been extensively studied in the literature due to its usefulness in several areas, including the theory of differential equations ([1], [2], [6]), degenerate semigroups of operators [3, 4] and analysis of generalized inverses ([7], [8], [9]). Interest in the theory of linear relations has increased in recent years, a several important and recent results can be find in [1], [7], [8] and [9]. We like to put out the Baskakov's important work presented in [4], the articles that largely inspire some of our postulates. The notations and definitions of some concepts may be different from ones used in the reference literature, however, mainly those are inspired by [4] and [5]. Beyond our great mathematical interest in the area, the main motivation for this work is that linear relations have the potential to be applied in many fields, such is the case of Inverse Problems and Regularization Theory. Our study will focus on showing that the use of $R-$invariant subspaces and $R-$restrictions is more natural, since it presents a better behavior in terms of spectral decomposition. We will also study the connection with the concepts used in [1], [7], [8] and [9]. We present some examples showing that both notions of invariance do not always coincide, this is $R-$invariant subspaces and $G-$invariant susbpaces not are equal; similarly, the notions of R-restriction and G-restriction are not the same either. We construct the inverse group of a linear relation and characterize its existence. Subsequently we define the notion of Drazin decomposition and show that it is equivalent to the definition of Drazin invertibility given in [1], [8] and [9]. Moreover, we will show that some results and characterizations of Mniff coincide with the notion we propose.

Contents

Acknowledgements **iii**

Introduction **v**

1 Linear relations **1**

1.1 Linear relations and basic results 1

1.2 Closed linear relations and spectral notions 7

Chapter 1

Linear relations

This chapter is devoted to the comprehensive examination of the concept of linear relation the construction of illustrative examples. The primary objective is to investigate preliminary results that arise from this notion. Furthermore, additional significant concepts will be explored, including closed linear relations, pseudoresolvents, the ascent and descent of linear relations, as well as various notions of invariance of subspaces and the restrictions of linear relations.

The symbols $X, Y, Z, \ldots$ will be utilized to denote vector spaces over the field $\mathbb{C}$ and $B(X)$ denote the space of bouden linear operators on X.

1.1 Linear relations and basic results

A relation R between two sets U and V is defined as a subset of the Cartesian product $U \times V$. The concept of a relation serves as part of the inspiration for developing the notion of a linear relation.

Definition 1.1 (By)**.** *[4, Introduction]. For given normed spaces X and Y, any linear subspace $\widehat{A}$ of $X \times Y$ is called linear relation between X and Y.*

As usual, the domain and the image of a linear relation $\widehat{A}$ are defined by

$$Dom(\widehat{A}) := \left\{ x \in X : \exists y \in Y \text{ tal que } (x, y) \in \widehat{A} \right\},$$

$$Im\widehat{A} := \left\{ y \in Y : \exists\, x \in X, (x, y) \in \widehat{A} \right\},$$

respectively.

Every linear relation is identified with the map $A : X \to 2^Y \setminus \{\varnothing\}$, $x \mapsto Ax$, where $Ax := \{y \in Y : (x, y) \in \widehat{A}\}$. Since Ax is a set, then $Ax + Ay$ represents the usual sum of sets, for all $x, y \in X$.

So, if $x \notin Dom(\widehat{A})$, we have $Ax = \varnothing$ and the other hand if $x \in Dom(\widehat{A})$,

$$Ax = \{y \in Y : (x, y) \in A\} \neq \varnothing.$$

Proposition 1.1. *[5, I.2.3 Proposition] Let $\widehat{A}$ be a linear relation. The map A associated with $\widehat{A}$ is linear in the following sense: for all $x, y \in Dom(\widehat{A})$ and for all $\alpha \in \mathbb{C} \setminus \{0\}$, $Ax + Ay = A(x + y)$ y $\alpha Ax = A(\alpha x)$.*

Proof. First, we will to show that $Ax + Ay = A(x + y)$. For any $t \in Ax + Ay$, there exists $w \in Ax$ y $z \in Ay$ such that $t = w + z$ y $(x, w), (y, z) \in \widehat{A}$. Since $\widehat{A}$ is a linear relation, it follows $(x, w) + (y, z) = (x + y, w + z) \in \mathcal{A}$, this is, $t = w + z \in A(x + y)$.

Conversely, for any $p \in A(x + y)$ there exists $(x + y, p) \in \widehat{A}$. Consider some $q \in Ax$ such that $(x + y, p) = (x, q) + (y, p - q) \in \widehat{A}$. This implies $(y, p - q) \in \mathcal{A}$, this is, $p - q \in Ay$, then $p \in q + Ay \subseteq Ax + Ay$. Therefore $Ax + Ay = A(x + y)$.

Finally we will to show that $\alpha Ax = A(\alpha x)$. For any $w \in \alpha Ax$ there exists $z \in Y$ such that $w = \alpha z$, where $(x, z) \in \widehat{A}$. So, $\alpha(x, z) = (\alpha x, \alpha z) = (\alpha x, w) \in \widehat{A}$ and therefore $w \in A(\alpha x)$. Conversely, for any $z \in A(\alpha x)$, we have that $(\alpha x, z) \in \widehat{A}$. Let $w \in Ax$ such that $(x, \alpha^{-1} z) \in \widehat{A}$, this is, $\alpha^{-1} z \in Ax$, it follows $z \in \alpha Ax$. Therefore $\alpha Ax = A(\alpha x)$. **QED**

From Proposition 1.1, the map A associated with $\widehat{A}$ is called *multivalued linear operator*.

It is already established that, given a linear relation $\widehat{A}$ betwen X and Y, one can associate a multivalued linear operator $A : X \to 2^Y \setminus \{\varnothing\}$ defined by $Ax := \{y \in Y : (x, y) \in \widehat{A}\}$. This indicates a connection between linear relations betwen X and Y and multivalued mappings from X to 2^Y.

It is natural to inquire whether this connection is bijective. The answer is affirmative. In fact, given a linear map (in the same sense of Proposition 1.1) $B : X \to 2^Y \setminus \{\varnothing\}$ such that $0 \in B(0)$, it is indeed possible to associate a linear relation with it. This assertion is demonstrated in the following proposition:

Proposition 1.2. *Let $B : X \to 2^Y$ a linear map (in the same sense of Proposition 1.1) such that $0 \in B(0)$. There exists a linear relation $\widehat{A}$ between X and Y such that for every $x \in X$, $Bx = Ax$, where $A : X \to 2^Y$ is the multivalued linear map asocciated with $\widehat{A}$.*

Proof. We define $\widehat{A} := \{(x, y) \in X \times Y : \ y \in Bx\}$ and we will to show that $\widehat{A}$ is a linear relation.

Since $0 \in B(0)$, then $(0, 0) \in \widehat{A}$. Let $(x, y), (w, z) \in \widehat{A}$, then we have that $y \in Bx$ and $z \in Bw$. From the above it follows that $y + z \in Bx + Bz = B(x + z)$, this is, $(x, y) + (w, z) = (x + z, y + z) \in \widehat{A}$. For any scalar α and since $y \in Bx$, we have that $\alpha y \in \alpha Bx = B(\alpha x)$, so $\alpha(x, y) = (\alpha x, \alpha y) \in \widehat{A}$. Therefore $\widehat{A} \in LR(X, Y)$.

Finally, for every $x \in X$, $y \in Ax$ if only if $(x, y) \in \widehat{A}$ if only if $y \in Bx$. Therefore, $Ax = Bx$. **QED**

From Proposition 1.1 and Proposition 1.2 we have the following proposition:

Proposition 1.3. *There exists a bijective association between the family of linear relations between X and Y and the family of multivalued linear operators $B : X \to 2^Y \setminus \{\varnothing\}$ with $0 \in B(0)$.*

Remark 1.1. *If $\widehat{A}$ is a linear relation between X and Y, from definition of the map A associated with $\widehat{A}$ and Proposition 1.3 it follows that $Dom(A) = Dom(\widehat{A})$ and $ImA = Im\widehat{A}$. Moreover*

$$Dom(A) = \{x \in X : Ax \neq \varnothing\},$$

$$ImA = \{y \in Y : \exists\, x \in X,\ y \in Ax\}.$$

Due to the bijective relationship between linear relations and multivalued linear operators, as established by Proposition 1.3, one can work with both objects interchangeably. More over, throughout this script, we consider the map $A : X \to 2^Y \setminus \{\varnothing\}$ as the linear relation and refer to the graph of the map A as $\widehat{A}$ and denote it by $Gr(A) := \widehat{A}$.

In the case of linear relationships, it is always possible to work with the inverse relationship, which often provides a lot of additional information. Given a linear relation, its inverse relationship is defined through its graph as shown below.

Definition 1.2. If A is a linear relation between X and Y, the inverse linear relation of A, denoted by A^{-1}, is a linear relation between Y and X and is defined by its graph as follows:

$$Gr(A^{-1}) := \{(y, x) \in Y \times X : (x, y) \in Gr(A)\}.$$

Now we will discuss sets related to linear relations, which will be important throughout the thesis.

Definition 1.3. Let A be a linear relation between X and Y and $U \subseteq X$. The image of U is defined by

$$A[U] := \bigcup \{Ax : x \in Dom(A) \cap U\}.$$

In particular, $ImA = A[X] = \bigcup \{Ax : x \in Dom(A)\}$.

Other set of utmost importance is the multivalued set of A, defined by $A(0) := \{y \in Y : (0, y) \in Gr(A)\}$; the kernel of A, defined by $KerA := A^{-1}(0)$ (the kernel of A is the multivalues of A^{-1}).

Example 1.1. *The simplest example of a linear relation is the inverse linear relation of a linear operator $T : X \to Y$. For example: Let $X = Y = \mathbb{R}^2$ and consider the matrix operator defined by*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

For every $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X$, $Ax = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$, *such that the behavior of the inverse relation is* $A^{-1}\begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + C$, *where* C *is*

$$C = \left\{ \begin{pmatrix} 0 \\ x_2 \end{pmatrix} : x_2 \in X \right\}.$$

Moreover, the graph of the inverse linear relation A^{-1} *is given by*

$$Gr(A^{-1}) = \left\{ \left(\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) : \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \right\}.$$

Remark 1.2. *By duality, from definitions of invrse linear relation it follows that*

(i) $(A^{-1})^{-1} = A$,

(ii) $Dom(A) = ImA^{-1}$ *and* $ImA = Dom(A^{-1})$,

(iii) $KerA^{-1} = A(0)$.

Proposition 1.4. *Let* A *be a linear relarion between* X *and* Y.

(i) $DomA$ *and* $KerA$ *are subspaces of* X.

(ii) ImA *and* $A(0)$ *are subspaces of* Y.

Proof. (i) First we will to show that $Dom(A)$ is a subspace of A. $Dom(A) \neq \varnothing$ since $(0,0) \in Gr(A)$, i.e., $0 \in Dom(A)$. Let $x, y \in Dom(A)$ and $\alpha \in \mathbb{C} \setminus \{0\}$. There exists $z, w \in Y$ such that $(x,z), (y,w) \in Gr(A)$. Since $Gr(A)$ is a subspace of $X \times Y$, $(x+y, z+w) \in Gr(A)$, with $z+w \in Y$, this is $x+y \in Dom(A)$. Similarly, $\alpha(x,z) = (\alpha x, \alpha z) \in A$, with $\alpha z \in Y$, then $\alpha x \in Dom(A)$. Therefore $Dom(A)$is subspace of X.

On the other hand, we have $0 \in KerA$. If we take $x, y \in KerA$ and $\lambda \in \mathbb{C} \setminus \{0\}$, then $(0,x), (0,y) \in Gr(A^{-1})$. Since $Gr(A^{-1})$ is a vectorial space, we have.

$$\alpha((0,x) + (0,y)) = \alpha(0, x+y) = (0, \alpha x + \alpha y) \in Gr(A^{-1}).$$

Therefore $\alpha x + \alpha y \in KerA$.

(ii) To prove that ImA and $A(0)$ are subspaces of Y, a similar approach of (i) is employed. **QED**

Definition 1.4. *Let* A, B *be two linear relations between* X *and* Y *and let* C *be a linear relation between* Y *and* Z. *For* $x \in X$.

(i) The sum of A and B is defined by $(A+B)x := Ax + Bx$ with $Dom(A+B) = Dom(A) \cap Dom(B)$. The graph of the sum is given by

$$Gr(A+B) = \{(x, y+z) \in X \times Y : (x,y) \in Gr(A)\ y\ (x,z) \in Gr(B)\}.$$

(ii) The composition (or product) of A and C is defined by $(CA)x := C(Ax)$, with $Dom(CA) = \{x \in Dom(A) : Ax \cap Dom(C) \neq \varnothing\}$. The graph of the composition is given by

$$Gr(BA) = \{(x,z) \in X \times Z : \exists\, y \in Dom(B) \text{ tal que}$$

$$(x,y) \in Gr(A), (y,z) \in Gr(B)\}.$$

(iii) The product of A and αA is defined by $(\alpha A)x := \alpha(Ax)$, with $Dom(\alpha A) = Dom(A)$. the graph is given by

$$Gr(\alpha A) = \{(x, \alpha y) \in X \times Y : (x,y) \in Gr(A)\}.$$

Remark 1.3. *From definition of product of two linear relations A and B and the definition of inverse linear relation, we have that $(AB)^{-1} = B^{-1}A^{-1}$.*

In [5, I.4 The Algebra of linear relations] the author shows that the family of linear relations fulfills the commutative and associative properties of addition, as well as the associative property pertaining to scalar multiplication. Additionally, [5, I.4.2 Proposition] (b) shows that the product of linear relations is associative with respect to scalar multiplication.

Given all of the above, it is natural to ask whether the family of linear relations constitutes a vector space under the defined operations of addition and multiplication. However, the answer is negative, as [5, I.4.2 Proposition] (d) provides evidence that the distributive property does not hold.

Definition 1.5. (*From [5], I.1.1 Preliminaries).* Let A be a linear relation between X and Y. A is called inyective if A^{-1} is a linear operator. A is called surjective if $ImA = Y$

Proposition 1.5. (*From [5], I.2.8 Proposition). Let A be a linear relation between X and Y.*

(i) For all $x \in Dom(A)$,

$$y \in Ax \Leftrightarrow Ax = y + A(0).$$

(ii) For all $x_1, x_2 \in Dom(A)$,

$$Ax_1 \cap Ax_2 \neq \varnothing \Leftrightarrow Ax_1 = Ax_2.$$

Proof.

(i) ($\Rightarrow$) Let $y \in Ax$. Since A is linear, $y + A(0) \subseteq Ax + A(0) = Ax$. On the other hand, if $w \in Ax$, we have $(x, w) \in A$, where $w = y + (w - y) \in y + Ax - Ax = y + A(0)$, this imply that $Ax \subseteq y + A(0)$. Therefore $Ax = y + A(0)$.
($\Leftarrow$) Assume that $Ax = y + A(0)$. For every $t \in y + A(0)$ $t \in Ax$, in particular $y + 0 \in y + A(0)$, so $y + 0 = y \in Ax$.

(ii) ($\Rightarrow$) Assume that $Ax_1 \cap Ax_2 \neq \varnothing$. There exists $y \in Ax_1 \cap Ax_2$, this is, $y \in Ax_1$ y $y \in Ax_2$, from (i), we have $Ax_1 = y + A(0)$ and $Ax_2 = y + A(0)$, thus $Ax_1 = Ax_2$.
($\Leftarrow$) Assume now that $Ax_1 = Ax_2$. It is easy to see that $Ax_1 \cap Ax_2 \neq \varnothing$. **QED**

Corollary 1.1. *[5, I.2.9-I.2.11 Corollary] Let A be a linear relation between X and Y. The following statements holds:*

(i) A is a linear operator (univalued) if only if $A(0) = \{0\}$,

(ii) A is injective if only if $KerA = \{0\}$,

(iii) $AA^{-1}(0) = A(0)$ y $A^{-1}A(0) = A^{-1}(0)$.

(iv) $AA^{-1} = I_{ImA} + A(0)$ and $A^{-1}A = I_{Dom(A)} + A^{-1}(0)$.

Proof. (i) ($\Rightarrow$) Assume A is a linear operator. For every $x \in Dom(A)$, $Ax = \{y\}$, for some $y \in Y$; in particular, there exists $y_0 \in Y$ such that $A(0) = \{y_0\}$. Since $A(0)$ is subspace of Y, then $0 \in A(0)$, so $0 = y_0$.
($\Leftarrow$) Assume now $A(0) = \{0\}$. Let $x \in Dom(A)$, by Proposition 1.5 (i), for any $y \in Ax$, $Ax = y + A(0) = \{y\}$. Therefore, A is an univalued function.

(ii) Is an immediate consequence of (i).

(iii) Is an immediate consequence of Proposition 1.5 (ii).

(iv) It follows from (iii).

QED

From Corollary 1.1 (i), we can see that linear relations generalize to linear operators. Furthermore, the family of all linear operators is a subset of the family of linear relations.

Proposition 1.6. *Let A and B be two linear relations onto X such that $B(0) \subseteq KerA$, then $AB(0) = A(0)$.*

Proof. Let $a \in AB(0)$, by definition of multivalued set, $(0, a) \in Gr(AB)$, this is, $(0, y) \in Gr(B)$ and $(y, a) \in Gr(A)$ for some $y \in Dom(A)$. So, $y \in B(0) \subseteq KerA$ and $a \in Ay$. From the above, we ha ve that $0 \in Ay$ or equivalently $A(0) = Ay$. Thus, $a \in A(0)$.

Conversely, for any $b \in A(0)$ we have $(0, b) \in Gr(A)$. Since $0 \in Dom(A) \subseteq ImB$, then there exists $y \in Dom(B)$ such that $0 \in By$, or equivalently $By = B(0)$. Moreover $(y, 0) \in Gr(B)$. So, $(y, b) \in Gr(AB)$, i.e., $b \in ABy = AB(0)$.
Therefore $A(0) = AB(0)$. **QED**

1.2 Closed linear relations and spectral notions

We know that the spectrum of a bounded linear operator is a nonempty compact subset of the complex plane, however the spectrum of a linear unbounded closed operator is an unbound set in the complex plane. A similar situation occurs when analyzing linear relations. In fact, the most interesting examples come from linear relations that are not necessarily bounded. This thesis will primarily focus on closed linear relations.

This section will introduce the notion of closed linear relation, the concept of the spectrum of a linear relation, as well as the resolvent set and the resolvent function. In the following, X, Y, Z, ... denote complex Banach spaces over the field $\mathbb{C}$. The product space $X \times Y$ will be equipped with the following norm: for $x \in X$ and $y \in Y$

$$||(x, y)|| = \max \{||x||_X, ||y||_Y\} .$$

And let us consider the extended complex plane $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ with the Alexandroff topology given by

$$\tau := \tau_e \cup \{A \subseteq \mathbb{C}_\infty : \mathbb{C} \setminus A \text{ is compact}\}$$

where τ_e is the usual topology of $\mathbb{C}$. With this topology, $\mathbb{C}_\infty$ is compact.

Definition 1.6. *If $Gr(A)$ is a closed subspace of $X \times Y$, we said that A is a closed linear relation.*

$LC(X, Y)$ denote the family of all closed linear relations between X and Y, if $X = Y$, we simplify $LC(X, X) := LC(X)$.

Remark 1.4. *(i) If $A \in LC(X, Y)$, then $A^{-1} \in LC(Y, X)$.*

(ii) If $T \in B(X)$, then by Closed Graph Theorem, $Gr(T)$ is closed, therefore $A := T^{-1} \in LC(X)$.

Proposition 1.7. *If $A \in LC(X, Y)$, then $A(0)$ is a closed subset of ImA.*

Proof. Let $v \in \overline{A(0)}$. There exists a sequence $\{u_n\}$ in $A(0)$, such that $\lim_n u_n = v$. So, for every $n \in \mathbb{N}$, $(0, u_n) \in Gr(A)$. Moreover $\{(0, u_n)\}$ is a sequence in $Gr(A)$ such that $\lim_n (0, u_n) = (0, v)$ and since $Gr(A)$ is a closed set, then $(0, v) \in Gr(A)$, thus $v \in A(0)$ and therefore $\overline{A(0)} \subseteq A(0)$. **QED**

Proposition 1.8. *Let $A \in LC(X,Y)$. $KerA$ is a closed subspace of X.*

Proof. Is an immediate consequence of the fact $KerA = A^{-1}(0)$ and Proposition 1.7. **QED**

Remark 1.5. *(i) If $A, B \in LC(X,Y)$, in general, $A+B$ can be not closed, since $(A+B)(0) = A(0)+B(0)$ is not necessary closed.*

(ii) Similarly, if $B \in LC(X,Y)$ and $A \in LC(Y,Z)$, is possible that $AB \notin LC(X,Z)$ since $AB(0)$ is not necessary closed.

(iii) By [5, II.5.16 Exercise], $I\lambda - A := \lambda - A$ is a closed linear relation for every $\lambda \in \mathbb{C}$.

Definition 1.7. *[4, Definition 1.2] A linear relation $A \in LC(X,Y)$ is said invertible if is both injective and surjective.*

It is important to mention that if $A \in LC(X,Y)$ is injective and surjective, then $ImA = Y$ and $KerA = A^{-1}(0) = \{0\}$, this is, A^{-1} is a closed linear operator with $Dom(A^{-1}) = Y$. By Closed graph theorem, we have that $A^{-1} \in B(Y,X)$. Therefore, if A is a closed invertible linear relation, then A^{-1} is a bounded linear operator.

Spectral concepts will now be introduced. The spectrum and the resolvent set of a linear relation are similar to those for linear operators, but with differences that are important for understanding their essence.

Definition 1.8. *[4, Definition 1.3] The resolvent set of $A \in LC(X)$ is defined by*

$$\rho(A) := \left\{\lambda \in \mathbb{C} : (\lambda - A)^{-1} \in B(X)\right\}.$$

The spectrum of A is defined by $\sigma(A) := \mathbb{C} \setminus \rho(A)$.

Proposition 1.9. *[5, VI.1.3 Proposition] $\rho(A)$ is an open set.*

Proof. Let $\lambda \in \rho(A)$. We will to show that $B(\lambda, r) \subseteq \rho(A)$, where $B(\lambda, r)$ is the open ball with radius $r = \|(\lambda - A)^{-1}\|^{-1}$.

Let $\mu \in B(\lambda, r)$. Since

$$\left\|(\lambda-\mu)(\lambda-A)^{-1}\right\| = |\lambda-\mu| \left\|(\lambda-A)^{-1}\right\| \leq$$

$$\leq \left\|(\lambda-A)^{-1}\right\|^{-1} \left\|(\lambda-A)^{-1}\right\| = 1,$$

the operator $I - (\lambda-\mu)(\lambda-A)^{-1}$ is invertible. Moreover, for all $x \in Dom(A)$, we have

$$(\lambda-A)\left[I-(\lambda-\mu)(\lambda-A)^{-1}\right]x = (\lambda-A)\left[x-(\lambda-\mu)(\lambda-A)^{-1}x\right] =$$

$$= (\lambda-A)x - (\lambda-\mu)(\lambda-A)(\lambda-A)^{-1}x =$$

$$= (\lambda - A)x - (\lambda - \mu)\left[x + (\lambda - A)(0)\right] = (\lambda - A)x - (\lambda - \mu)\left[x + A(0)\right] =$$
$$= (\lambda - A)x - (\lambda - \mu)x - (\lambda - \mu)A(0) = (\lambda - A)x - (\lambda - \mu)x - A(0) =$$
$$= \lambda x - Ax - \lambda x + \mu x - A(0) = \mu x - A(x - 0) =$$
$$= \mu x - Ax = (\mu - A)x.$$

So, $(\lambda - A)\left[I - (\lambda - \mu)(\lambda - A)^{-1}\right] = (\mu - A)$. Furthermore $(\mu - A)$ is an invertible closed linear relation since $(\lambda - A)$, $I - (\lambda - \mu)(\lambda - A)^{-1}$ are invertible. Therefore $(\mu - A)^{-1} \in B(X)$ and $\mu \in \rho(A)$. **QED**

From the above Proposition it follows that $\sigma(A)$ is a closed subset of $\mathbb{C}$.

Proposition 1.10. *Let $A \in LC(X)$. $Ker(\lambda - A)^{-1} = A(0)$ for any $\lambda \in \rho(A)$.*

Proof. Let $x \in Ker(\lambda - A)^{-1}$. By definition of kernel, $(\lambda - A)^{-1}x = 0$. From Corollary 1.1 (iv)

$$(\lambda - A)(0) = (\lambda - A)(\lambda - A)^{-1}x = x + (\lambda - A)(0),$$

this is, $A(0) = x + A(0)$. By Proposition 1.5 (i), $x \in A(0)$.

Conversely, if we take $x \in A(0)$, by Corollary 1.1 (iii),

$$(\lambda - A)^{-1}x \in (\lambda - A)^{-1}A(0) = (\lambda - A)^{-1}(0) = \{0\}.$$

Thus, $(\lambda - A)^{-1}x = 0$ and therefore $x \in Ker(\lambda - A)^{-1}$. **QED**

Proposition 1.11. *Let $A \in LC(X)$. $Im(\lambda - A)^{-1} = Dom(A)$ for any $\lambda \in \rho(A)$.*

Proof. Let $\lambda \in \rho(A)$. We have $Im(\lambda - A)^{-1} = Dom(\lambda - A) = Dom(I) \cap Dom(A) = Dom(A)$, since $Dom(A) \subseteq Dom(I)$. **QED**

Proposition 1.12 (Hilbert's identity)**.** *Let $A \in LC(X)$. For every $\lambda_1, \lambda_2 \in \rho(A)$ the following equality holds:*

$$(\lambda_1 - A)^{-1} - (\lambda_2 - A)^{-1} = (\lambda_2 - \lambda_1)(\lambda_1 - A)^{-1}(\lambda_2 - A)^{-1}.$$

Proof. Let $\lambda_1, \lambda_2 \in \rho(A)$ and $x \in X$.

$$(\lambda_2 - \lambda_1)(\lambda_1 - A)^{-1}(\lambda_2 - A)^{-1}x = (\lambda_1 - A)^{-1}(\lambda_2 - \lambda_1)(\lambda_2 - A)^{-1}x.$$

Nótese

$$(\lambda_1 - A)^{-1}\left[\lambda_2 - \lambda_1 + A - A\right](\lambda_2 - A)^{-1}x =$$
$$= (\lambda_1 - A)^{-1}\left[(\lambda_2 - \lambda_1)(\lambda_2 - A)^{-1}x + (A - A)(\lambda_2 - A)^{-1}x\right] =$$
$$= (\lambda_1 - A)^{-1}\left[(\lambda_2 - \lambda_1)(\lambda_2 - A)^{-1}x + A(0)\right] =$$

$$= (\lambda_1 - A)^{-1}(\lambda_2 - \lambda_1)(\lambda_2 - A)^{-1}x + (\lambda_1 - A)^{-1}A(0) =$$
$$= (\lambda_1 - A)^{-1}(\lambda_2 - \lambda_1)(\lambda_2 - A)^{-1}x + 0 =$$
$$= (\lambda_1 - A)^{-1}(\lambda_2 - \lambda_1)(\lambda_2 - A)^{-1}x.$$

Así,

$$(\lambda_1 - A)^{-1}(\lambda_2 - \lambda_1)(\lambda_2 - A)^{-1}x =$$
$$= (\lambda_1 - A)^{-1}\left[\lambda_2 - \lambda_1 + A - A\right](\lambda_2 - A)^{-1}x =$$
$$= (\lambda_1 - A)^{-1}[(\lambda_2 - A) - (\lambda_1 - A)](\lambda_2 - A)^{-1}x =$$
$$= (\lambda_1 - A)^{-1}[(\lambda_2 - A)(\lambda_2 - A)^{-1}x - (\lambda_1 - A)(\lambda_2 - A)^{-1}x] =$$
$$= (\lambda_1 - A)^{-1}[x + (\lambda_2 - A)(0)] - (\lambda_1 - A)^{-1}(\lambda_1 - A)(\lambda_2 - A)^{-1}x =$$
$$= (\lambda_1 - A)^{-1}x + (\lambda_1 - A)^{-1}(\lambda_2 - A)(0) - (\lambda_2 - A)^{-1}x =$$
$$= (\lambda_1 - A)^{-1}x - (\lambda_2 - A)^{-1}x = [(\lambda_1 - A)^{-1} - (\lambda_1 - A)^{-2}]x.$$

QED

Corollary 1.2. *Let $A \in LC(X)$. For every $\lambda, \mu \in \rho(A)$, $(\mu - A)^{-1}(\lambda - A)^{-1} = (\lambda - A)^{-1}(\mu - A)^{-1}$.*

Proof. Let $\lambda, \mu \in \rho(A)$. By Hilbert's identity,

$$(\mu - A)^{-1}(\lambda - A)^{-1} - (\lambda - A)^{-1}(\mu - A)^{-1} =$$
$$= \frac{(\mu - A)^{-1} - (\lambda - A)^{-1}}{\lambda - \mu} - \frac{(\lambda - A)^{-1} - (\mu - A)^{-1}}{\mu - \lambda} =$$
$$= \frac{(\mu - A)^{-1} - (\lambda - A)^{-1} + (\lambda - A)^{-1} - (\mu - A)^{-1}}{\lambda - \mu} = 0.$$

Therefore, $(\mu - A)^{-1}(\lambda - A)^{-1} = (\lambda - A)^{-1}(\mu - A)^{-1}$. **QED**

Proposition 1.13. *Let $A \in LC(X)$. The resolvent function of A is a holomorphic function.*

Proof. Let $\lambda \in \rho(A)$. Since $\rho(A)$ is an open set, we have $B(\lambda, r) \subseteq \rho(A)$, where $r = \|(\lambda - A)^{-1}\|^{-1}$. Set $\mu \in B(\lambda, r)$ such that $\mu \neq \lambda$. By Hilbert's equation,

$$((\lambda - A)^{-1} - (\mu - A)^{-1})(\lambda - A)^{-1} = (\lambda - A)^{-2} - (\mu - A)^{-1}(\lambda - A)^{-1} =$$
$$= (\lambda - A)^{-2} - \frac{(\mu - A)^{-1} - (\lambda - A)^{-1}}{\lambda - \mu} =$$
$$= (\lambda - A)^{-2} + \frac{(\mu - A)^{-1} - (\lambda - A)^{-1}}{\mu - \lambda}.$$

Thus,

$$\left\|(\lambda - A)^{-2} + \frac{(\lambda - A)^{-1} - (\mu - A)^{-1}}{\mu - \lambda}\right\| =$$
$$\left\|((\lambda - A)^{-1} - (\mu - A)^{-1})(\lambda - A)^{-1}\right\| \leq$$
$$\leq \left\|(\lambda - A)^{-1} - (\mu - A)^{-1}\right\| \left\|(\lambda - A)^{-1}\right\|$$

On the other hand,

$$\left\|(\mu - \lambda)(\lambda - A)^{-1}\right\| = |\mu - \lambda| \left\|(\lambda - A)^{-1}\right\| \leq$$
$$\leq \left\|(\lambda - A)^{-1}\right\|^{-1} \left\|(\lambda - A)^{-1}\right\| = 1.$$

Thus, $I - (\mu - \lambda)(\lambda - A)^{-1}$ is an invertible linear with inverse is given by

$$(I - (\mu - \lambda)(\lambda - A)^{-1})^{-1} = \sum_{n=0}^{\infty} (\mu - \lambda)^n (\lambda - A)^{-n}. \tag{1.1}$$

From proof of Proposition 1.9 it follows

$$(\lambda - A)\left[I - (\mu - \lambda)(\lambda - A)^{-1}\right] = (\mu - A).$$

So

$$(\mu - A)^{-1} - (\lambda - A)^{-1} = \left(\sum_{n=0}^{\infty} (\mu - \lambda)^n (\lambda - A)^{-(n+1)}\right) - (\lambda - A)^{-1}$$
$$= \sum_{n=1}^{\infty} (\mu - \lambda)^n (\lambda - A)^{-(n+1)} = (\lambda - A)^{-1} \sum_{n=1}^{\infty} (\mu - \lambda)^n (\lambda - A)^{-n}.$$

Now, we have the following inequality:

$$\left\|(\mu - A)^{-1} - (\lambda - A)^{-1}\right\| \leq$$
$$\leq \left\|(\lambda - A)^{-1}\right\| \sum_{n=1}^{\infty} |(\mu - \lambda)|^n \left\|(\lambda - A)^{-1}\right\|^n.$$

Inasmuch,

$$\lim_{\mu \to \lambda} \left\|(\lambda - A)^{-2} + \frac{(\mu - A)^{-1} - (\lambda - A)^{-1}}{\mu - \lambda}\right\| \leq$$
$$\leq \lim_{\mu \to \lambda} \left\|(\lambda - A)^{-1} - (\mu - A)^{-1}\right\| \left\|(\lambda - A)^{-1}\right\| \leq$$
$$\leq \lim_{\mu \to \lambda} \left\|(\lambda - A)^{-1}\right\| \sum_{n=1}^{\infty} |(\mu - \lambda)|^n \left\|(\lambda - A)^{-1}\right\|^n \left\|(\lambda - A)^{-1}\right\|$$

$$= \lim_{\mu\to\lambda} \left\|(\lambda - A)^{-1}\right\|^2 \sum_{n=1}^{\infty} |(\mu - \lambda)|^n \left\|(\lambda - A)^{-1}\right\|^{n+1} = 0$$

Therefore

$$\lim_{\mu\to\lambda} \frac{(\mu - A)^{-1} - (\lambda - A)^{-1}}{\mu - \lambda} = -[(\lambda - A)^{-1}]^2. \qquad \textbf{QED}$$

Definition 1.9. *[4, Definition 1.4] The spectrum of A is defined by $\sigma(A) := \mathbb{C} \setminus \rho(A)$. The map $R(\cdot, A) : \rho(A) \to B(X)$, defined by $R(\lambda, A) = (\lambda - A)^{-1}$, denote the resolvent function of A.*

From [5, VI.3.3 Theorem], we know that the spectrum of a closed bounded linear relation is non-empty (see [7] for more information about boundeness of linear relations). However, in general the spectrum of a linear relation can be empty or entire complex plane. For example let $B \in B(X)$ be a quasinilpotent operator and $A := B^{-1} \in LC(X)$. Then $\sigma(B) = \{0\}$, and, by the spectral mapping theorem, $\sigma(A) = \varnothing$.

To prevent the scenario in which the spectrum of a linear relation is an empty set, it is imperative to establish the concept of the extended spectrum of a linear relation. This is particularly crucial, as future chapters will focus on the analysis of specific isolated points within that set.

Definition 1.10. *[4, Definition 1.5] The extended spectrum of $A \in LC(X)$ is the set $\widetilde{\sigma}(A) \subseteq \mathbb{C}_\infty$ that is equal to $\rho(A)$ if the following three conditions are satisfied:*

(i) A is a closed linear operator;

(ii) the resolvent function $R(\cdot, A)$ can be analytically continued to ∞;

(iii) $\lim_{|\lambda|\to\infty} R(\lambda, A) = 0.$

Otherwise, $\widetilde{\sigma}(A) = \sigma(A) \cup \{\infty\}$. The set $\widetilde{\rho}(A) := \mathbb{C}_\infty \setminus \widetilde{\sigma}(A)$ is called the extended resolvent set of A.

The following proposition shows that the extended spectrum of a closed linear relation is, in fact, non-empty.

Proposition 1.14. *If X is not the zero space, for any $A \in LC(X)$, $\widetilde{\sigma}(A) \neq \varnothing$.*

Proof. Suppose that $\widetilde{\sigma}(A) = \varnothing$. This is equivalent to $\widetilde{\rho}(A) = \mathbb{C}_\infty$, so that $R(\cdot, A)$ is an analytic function on whole complex plane $\mathbb{C}$. By Liouville's theorem, $R(\cdot, A)$ is a constant function; in addition, we know that $R(\lambda, A)$ is a bounded operator for all λ, so that for every $\lambda \in \mathbb{C}$, $R(\lambda, A) = R \in B(X)$. Since $R = R(0, A) = A^{-1}$, then for each λ, $(\lambda - A)^{-1} = A^{-1}$ wich is false. **QED**

The extended spectrum of closed linear relations possesses significant topological characteristics, for example, it has been established in [5, VI.4.3 Corollary] that $\widetilde{\sigma}(A)$ is a compact subset of $\mathbb{C}_\infty$.

At this point, a relevant question emerges: Is there a relationship between the extended spectrum of A and the extended spectrum of A^{-1}? The response to this inquiry is affirmative, from [4, Theorem 2.4] we have the following relationship: $\widetilde{\sigma}(A^{-1}) = \{\lambda^{-1} : \lambda \in \widetilde{\sigma}(A)\}$. Moreover in [4, Lemma 2.2], Baskakov shows that $A \in LC(X)$ is a bounded linear operator if only if $\infty \notin \widetilde{\sigma}(A)$. It is important to note that this characterization coincides with the definition of extended spectrum given in [5, VI.4.1 Definition] since $\infty \notin \widetilde{\sigma}(A)$ occurs if only if $0 \notin \sigma(A^{-1})$ and this is equivalent to $A \in B(X)$.

Definition 1.11. *Let $A \in LC(X)$. A is group invertible if there exists a linear operator $B \in B(X)$ such that*

(i) $ABA = A$,

(ii) $BAB = B$,

(iii) $AB = BA + A(0)$.

Bibliography

[1] A. Ghorbel, M. M. Drazin inverse of multivalued operators and its applications. *Monatshefte für Mathematik 189 (2)* (2019), 273–293.

[2] Alrazi, A., and Trung Dinh, T. Singular differential equations and g-Drazin invertible operators. *International Journal of Analysis 2016* (2016), 5.

[3] Baskakov, A. G., and Chernyshov, K. I. On some questions of the spectral theory of linear relations. *Vesnik VGU: Serii Fisics, Mathematics (1)* (2002), 81–94.

[4] Baskakov, A. G., and Chernyshov, K. I. Spectral analysis of linear relations and degenerate operator semigroups. *Sbornik: Mathematics 193 (11)* (2002), 1573–1610.

[5] Cross, R. *Multivalued Linear Operators*, 1st ed. Marcel Dekker, Inc., 1998.

[6] Favini, A., and Yagi, A. Multivalued linear operators and degenerate evolution equations. *Annali di Matematica pura ed applicata 163* (1993), 353–384.

[7] Lajnef, M., and Mnif, M. Isolated spectral points of a linear relation. *Monatshefte für Mathematik 191* (2020), 595–614.

[8] Lajnef, M., and Mnif, M. On generalized Drazin invertible linear relations. *Rocky Mountain J. 50 (24)* (2020), 1387–1408.

[9] Mbekhta, M. Opérateurs pseudo-Fredholm I: Résolvant généralisé. *J. Operator Theory 24* (1990), 255–276.

[10] von Neumann, J. Functional operators II. The Geometry of Orthogonal Spaces. *Annals of Mathematics Studies, Princeton University Press, Princeton, NJ 22* (1950).