



Benemérita Universidad Autónoma de Puebla

Facultad de Ciencias Físico-Matemáticas

Haces Bessel de orden cero y orden uno y el interferómetro de
Young de n -rendijas

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Dedicated to ...

My father who is in heaven

... because of whom it's all possible.

My dear friends

... because they motivated me to begin this story.

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Abstract

Considering the light diffraction, the spread of any light beam is affected taking place a widening of its transverse profile and with this, a decrease in light intensity. However, exist luminous beams that don't show deformation at least at a typical distance, an example of these are the Bessel beams [1]. These special beams were called nondiffracting beams or invariant beams, they have immediate application in medicine, industry and calibration systems. However, special arrangements are required to generate them. This Master's degree thesis work seeks to generate zero-order and first-order Bessel beams by the implementation of the Young interferometer of n -slits.

Keywords

Interferometer, diffraction, Bessel, Fourier transform, polar coordinates, sampling, overlapping, slits.

Chapter 1

Introduction

1.1 Background

With the laser invention and due to its intrinsic characteristics, applications in communication, medicine, industry, etc., which involved propagation of light were developed; however, a laser beam has a transverse profile mathematically described by a Gaussian function, in which the intensity decreases and the width increases as it spreads. Therefore, in all applications, diffraction is present, but an attempt has been made to reduce the widening of the beam profile, as well as improving its directionality.

In 1987 it was discovered that there are light beams (nondiffracting or invariant) that don't show diffraction in a distance range. The main characteristic of the invariant beam is that it preserves its transverse profile without diffracting as it spreads. Durnin introduced the simplest beams invariants that can be built, known as Bessel beams because its transverse profile is mathematically described by a Bessel function. Given the special beams are not naturally generated by lasers, there are experimental arrangements that start from a Gaussian beam and the interest in finding new ways to generate them has increased.

1.2 Objectives and thesis structure

GENERAL OBJECTIVE

To form zero-order and first-order Bessel beams with light by using the Young interferometer of n -slits.

PARTICULAR OBJECTIVES

- To study the continuous apertures that provide the diffraction patterns J_0 and J_1 .
- To develop the theoretical model of the n -slits configuration to produce an approximation to the desired diffraction pattern.
- To implement numerically the theoretical model in the computer.

THESIS STRUCTURE

Initially the theoretical concepts of the functions that are used throughout this work are presented. The basic properties of the Fourier transform in polar coordinates are developed. The mathematical formulation to generate zero-order and first-order Bessel beams is established using two methods, in order to visualize the results, an algorithm is implemented in Wolfram Mathematica 12 using arbitrary units.

In chapter 2 the origin of the Bessel function is mentioned as well as some properties and representations. Basic concepts of the Dirac delta function in $\mathbb{R}^n$ are presented and the particular case of polar coordinates is addressed. Finally the Hankel transform is presented with some results.

In chapter 3 the Fourier transform form in polar coordinates is presented, some examples considering known functions are shown and compared with analogs in cartesian coordinates. In general form, the Fourier transform is presented for radially symmetric functions and for functions without radial symmetry through a series development, thereby it is possible to show the Jacobi-Anger expansion. As in Cartesian coordinates, the Fourier transform of a shifted function, a functions product and a convolution, for the different cases of symmetry are treated.

In chapter 4 the problem statement is made and two methods to solve it are proposed. The first by overlapping and the second by sampling, for this fact, the sampling theorem in Cartesian coordinates is presented in order to establish this idea in polar coordinates. A Dirac impulse train in polar coordinates is proposed and its series representation for the angular part and for the radial part as angular is shown, as well as their respective Fourier transforms.

In chapter 5 the two methods to form zero-order Bessel beams are developed, using the result of each one, a numerical simulation allows to present the diffraction patterns for different values of n .

Similarly, in chapter 6 the two methods to form first-order Bessel beams are developed with simulations only for the first method.

Finally, in chapter 7 the conclusions of the work done and a future work option are presented.

Chapter 2

Bessel beams

During the 80's beams that didn't show diffraction at least at a typical distance appeared, called nondiffracting or invariant beams, Bessel beams, considered in this work, are beams of this type. The name of these beams is due to the mathematical form of their transverse distribution; therefore, it's necessary to present the origin and the most important properties of the Bessel functions that will be helpful for the development of the work.

2.1 The Bessel equation

Given the wave equation

$$\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \quad (2.1)$$

where $\psi = \psi(x_1, x_2, x_3)$ is a function that describes mathematically the profile of a wave, which depends on a variable time t and spatial variables x_1, x_2, x_3 ; v is the speed with which the wave travels and assuming a solution ψ in the form

$$\psi(\vec{r}, t) = e^{-i\omega t} U(\vec{r}), \quad (2.2)$$

with

$$U(\vec{r}) = U(x_1, x_2, x_3), \quad (2.3)$$

by replacing in the equation we have

$$\begin{aligned} e^{-i\omega t} \nabla^2 U(\vec{r}) &= \frac{1}{v^2} U(\vec{r}) (-\omega^2) e^{-i\omega t} \\ \Rightarrow e^{-i\omega t} \left[\nabla^2 U(\vec{r}) + \frac{\omega^2}{v^2} U(\vec{r}) \right] &= 0. \end{aligned} \quad (2.4)$$

For any time t must be met

$$\nabla^2 U(\vec{r}) + \frac{\omega^2}{v^2} U(\vec{r}) = 0, \quad (2.5)$$

as $v = \frac{\omega}{k}$, where k is the wave number

$$\Rightarrow [\nabla^2 + k^2] U(\vec{r}) = 0 \quad (2.6)$$

which is the Helmholtz equation.

In cylindrical coordinates, the Laplacian of $U(\rho, \varphi, z)$ is

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial U(\rho, \varphi, z)}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 U(\rho, \varphi, z)}{\partial \varphi^2} + \frac{\partial^2 U(\rho, \varphi, z)}{\partial z^2}, \quad (2.7)$$

whereby the Helmholtz equation is expressed as

$$\frac{1}{\rho} \frac{\partial U(\rho, \varphi, z)}{\partial \rho} + \frac{\partial^2 U(\rho, \varphi, z)}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2 U(\rho, \varphi, z)}{\partial \varphi^2} + \frac{\partial^2 U(\rho, \varphi, z)}{\partial z^2} + k^2 U(\vec{r}) = 0. \quad (2.8)$$

Let $U(\rho, \varphi, z) = R(\rho)\Phi(\varphi)Z(z)$, replacing in Eq.(2.8), we have

$$\frac{\Phi(\varphi)Z(z)}{\rho} \frac{dR(\rho)}{d\rho} + \Phi(\varphi)Z(z) \frac{d^2 R(\rho)}{d\rho^2} + \frac{R(\rho)Z(z)}{\rho^2} \frac{d^2 \Phi(\varphi)}{d\varphi^2} + R(\rho)\Phi(\varphi) \frac{d^2 Z(z)}{dz^2} + k^2 R(\rho)\Phi(\varphi)Z(z) = 0,$$

dividing by $U(\rho, \varphi, z)$

$$\Rightarrow \frac{1}{R(\rho)} \frac{1}{\rho} \frac{dR(\rho)}{d\rho} + \frac{1}{R(\rho)} \frac{d^2 R(\rho)}{d\rho^2} + \frac{1}{\Phi(\varphi)} \frac{1}{\rho^2} \frac{d^2 \Phi(\varphi)}{d\varphi^2} + \frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} + k^2 = 0,$$

solving for z

$$\Rightarrow \frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} = -k_z^2 - \frac{1}{R(\rho)} \frac{1}{\rho} \frac{dR(\rho)}{d\rho} - \frac{1}{R(\rho)} \frac{d^2 R(\rho)}{d\rho^2} - \frac{1}{\Phi(\varphi)} \frac{1}{\rho^2} \frac{d^2 \Phi(\varphi)}{d\varphi^2},$$

if

$$\frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} = -k_z^2 \quad (2.9)$$

$$\Rightarrow -k_z^2 = -k^2 - \frac{1}{R(\rho)} \frac{1}{\rho} \frac{dR(\rho)}{d\rho} - \frac{1}{R(\rho)} \frac{d^2 R(\rho)}{d\rho^2} - \frac{1}{\Phi(\varphi)} \frac{1}{\rho^2} \frac{d^2 \Phi(\varphi)}{d\varphi^2}$$

solving for φ

$$\Rightarrow \frac{1}{\Phi(\varphi)} \frac{d^2 \Phi(\varphi)}{d\varphi^2} = (k_z^2 - k^2)\rho^2 - \frac{\rho}{R(\rho)} \frac{dR(\rho)}{d\rho} - \frac{\rho^2}{R(\rho)} \frac{d^2 R(\rho)}{d\rho^2}$$

and if

$$\frac{1}{\Phi(\varphi)} \frac{d^2 \Phi(\varphi)}{d\varphi^2} = -p^2$$

$$\begin{aligned}
 \Rightarrow -p^2 &= (k_z^2 - k^2)\rho^2 - \frac{\rho}{R(\rho)} \frac{dR(\rho)}{d\rho} - \frac{\rho^2}{R(\rho)} \frac{d^2R(\rho)}{d\rho^2} \\
 \Rightarrow \rho \frac{dR(\rho)}{d\rho} + \rho^2 \frac{d^2R(\rho)}{d\rho^2} + [(k^2 - k_z^2)\rho^2 - p^2] R(\rho) &= 0.
 \end{aligned} \tag{2.10}$$

Eq.(2.10) is the Bessel differential equation. To write the Bessel equation in a more usual way, we change to the variable

$$\begin{aligned}
 x &= \sqrt{k^2 - k_z^2} \rho \\
 \Rightarrow \frac{dx}{d\rho} &= \sqrt{k^2 - k_z^2} \\
 \Rightarrow \frac{d}{d\rho} &= \frac{dx}{d\rho} \frac{d}{dx} = \sqrt{k^2 - k_z^2} \frac{d}{dx} \\
 \Rightarrow \rho \frac{d}{d\rho} &= x \frac{d}{dx},
 \end{aligned}$$

for some $y(x)$ we have

$$x \frac{d}{dx} y(x) + x^2 \frac{d^2}{dx^2} y(x) + [x^2 - p^2] y(x) = 0, \tag{2.11}$$

where p (not necessarily an integer) is called the order of the Bessel function (y).

Given

$$\frac{d}{dx} \left(x \frac{d}{dx} y(x) \right) = \frac{d}{dx} y(x) + x \frac{d^2}{dx^2} y(x) \tag{2.12}$$

$$x \frac{d}{dx} \left(x \frac{d}{dx} y(x) \right) = x \frac{d}{dx} y(x) + x^2 \frac{d^2}{dx^2} y(x), \tag{2.13}$$

Eq.(2.11) can be written in a more compact form

$$x \frac{d}{dx} \left(x \frac{d}{dx} y(x) \right) + [x^2 - p^2] y(x) = 0. \tag{2.14}$$

Using the generalized power series

$$y(x) = \sum_{l=0}^{\infty} a_l x^{l+s} \tag{2.15}$$

where a_l is the coefficient l th of the series, s is a number to be found to fit the problem; it may be either positive or negative and it may be a fraction (in fact, it may even be complex,

but we shall not consider this case). Then

$$\frac{d}{dx}y(x) = \sum_{l=0}^{\infty} a_l(l+s)x^{l+s-1} \quad (2.16)$$

$$x\frac{d}{dx}y(x) = \sum_{l=0}^{\infty} a_l(l+s)x^{l+s} \quad (2.17)$$

$$\frac{d}{dx}\left(x\frac{d}{dx}y(x)\right) = \sum_{l=0}^{\infty} a_l(l+s)^2x^{l+s-1} \quad (2.18)$$

$$x\frac{d}{dx}\left(x\frac{d}{dx}y(x)\right) = \sum_{l=0}^{\infty} a_l(l+s)^2x^{l+s}, \quad (2.19)$$

so that Eq.(2.14) is written as

$$\sum_{l=0}^{\infty} a_l(l+s)^2x^{l+s} + [x^2 - p^2] \sum_{l=0}^{\infty} a_lx^{l+s} = 0. \quad (2.20)$$

Tabulating the coefficients

Table 2.1: Coefficients					
	x^s	x^{s+1}	x^{s+2}	$\dots$	x^{s+l}
$x\frac{d}{dx}\left(x\frac{d}{dx}y(x)\right)$	a_0s^2	$a_1(1+s)^2$	$a_2(2+s)^2$	$\dots$	$a_l(l+s)^2$
$x^2y(x)$			a_0	$\dots$	a_{l-2}
$p^2y(x)$	$-p^2a_0$	$-p^2a_1$	$-p^2a_2$	$\dots$	$-p^2a_l$

where the total coefficients of each power of x must be zero, the coefficient of x^s gives the indicial equation and the s values

$$s^2 - p^2 = 0, \quad s = \pm p. \quad (2.21)$$

For x^{s+1}

$$\begin{aligned} [(1+s)^2 - p^2]a_1 = 0 &\Leftrightarrow [(1+2s+s^2-p^2)]a_1 = 0 \\ &\Leftrightarrow [(1\pm 2p)]a_1 = 0 \\ &\Leftrightarrow a_1 = 0. \end{aligned} \quad (2.22)$$

In general for x^{s+l}

$$[(l+s)^2 - p^2]a_l + a_{l-2} = 0 \Leftrightarrow a_l = \frac{-a_{l-2}}{[(l+s)^2 - p^2]}. \quad (2.23)$$

To find all solutions, from here on out, we solve two separate problems, one when $s = p$, and

another when $s = -p$.

Case 1 $s = p$

For this case Eq.(2.23) becomes

$$a_l = \frac{-a_{l-2}}{l(l+2p)},$$

since $a_1 = 0$, all odd a 's are zero. For even a 's it's convenient to replace l by $2l$; then

$$a_{2l} = \frac{-a_{2l-2}}{2l(2l+2p)} = \frac{-a_{2l-2}}{2^2 l(l+p)}. \quad (2.24)$$

Using the Gamma function

$$\Gamma(p) = (p-1)! \quad (2.25)$$

$$\begin{aligned} l=1 &\Rightarrow a_2 = \frac{-a_0 \Gamma(p+1)}{2^2 \Gamma(p+2)} \\ l=2 &\Rightarrow a_4 = \frac{-a_2 \Gamma(p+2)}{2^3 \Gamma(p+3)} = \frac{a_0 \Gamma(p+1) \Gamma(p+2)}{2! 2^4 \Gamma(p+3) \Gamma(p+2)} = \frac{a_0 \Gamma(p+1)}{2! 2^4 \Gamma(p+3)} \\ l=3 &\Rightarrow a_6 = \frac{-a_4 \Gamma(p+3)}{2^2 3 \Gamma(p+4)} = \frac{-a_0 \Gamma(p+1) \Gamma(p+3)}{3! 2^6 \Gamma(p+3) \Gamma(p+4)} = \frac{-a_0 \Gamma(p+1)}{3! 2^6 \Gamma(p+4)}, \end{aligned}$$

then Eq.(2.20) can be rewritten as

$$\begin{aligned} y(x) &= \sum_{l=0}^{\infty} a_{2l} x^{2l+p} = a_0 x^p + a_2 x^{2+p} + a_4 x^{4+p} + \dots \\ &= a_0 x^p \Gamma(p+1) \left[\frac{1}{\Gamma(p+1)} - \frac{1}{\Gamma(p+2)} \left(\frac{x}{2}\right)^2 + \frac{1}{2! \Gamma(p+3)} \left(\frac{x}{2}\right)^4 + \dots \right]. \end{aligned}$$

If $a_0 = \frac{1}{2^p \Gamma(p+1)}$

$$\Rightarrow y(x) = \left[\frac{1}{\Gamma(1) \Gamma(p+1)} \left(\frac{x}{2}\right)^p - \frac{1}{\Gamma(2) \Gamma(p+2)} \left(\frac{x}{2}\right)^{p+2} + \frac{1}{\Gamma(3) \Gamma(p+3)} \left(\frac{x}{2}\right)^{p+4} + \dots \right]$$

and the following solution is obtained

$$y(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{\Gamma(l+1) \Gamma(l+1+p)} \left(\frac{x}{2}\right)^{2l+p} = J_p(x) \quad (2.26)$$

which is the so-called Bessel function of first type of order p .

Case 2 $s = -p$

Similarly a second solution is

$$J_{-p}(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{\Gamma(l+1)\Gamma(l+1-p)} \left(\frac{x}{2}\right)^{2l-p}, \quad (2.27)$$

using the wronskian can be verified that for p (not an integer), the general solution of the Bessel equation is

$$X_p(x) = aJ_p(x) + bJ_{-p}(x). \quad (2.28)$$

If p is integer

$$J_{-p}(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!(l-p)!} \left(\frac{x}{2}\right)^{2l-p} \quad (2.29)$$

where $(l-p)! = \Gamma(l+1-p) \rightarrow \infty$ for $l = 0, 1, \dots, p-1$, so that, for those values of l , the terms of the series are canceled, therefore the sum begins with $l = p$; i.e.,

$$J_{-p}(x) = \sum_{l=p}^{\infty} \frac{(-1)^l}{l!(l-p)!} \left(\frac{x}{2}\right)^{2l-p} \quad (2.30)$$

but

$$\sum_{l=p}^{\infty} f(l) = \sum_{l=0}^{\infty} f(l+p) \quad (2.31)$$

for any function

$$\begin{aligned} \Rightarrow J_{-p}(x) &= \sum_{l=0}^{\infty} \frac{(-1)^{(l+p)}}{(l+p)!(l+p-p)!} \left(\frac{x}{2}\right)^{2(l+p)-p} \\ &= \sum_{l=0}^{\infty} \frac{(-1)^{(l+p)}}{(l+p)! l!} \left(\frac{x}{2}\right)^{2l+p} \\ &= (-1)^p \sum_{l=0}^{\infty} \frac{(-1)^l}{l!(l+p)!} \left(\frac{x}{2}\right)^{2l+p} \end{aligned} \quad (2.32)$$

$$i.e., J_{-p}(x) = (-1)^p J_p(x) \quad \text{for } p \in \mathbb{Z} \quad (2.33)$$

If p is integer, the general solution of the Bessel equation is the Neumann function

$$N_p(x) = \frac{1}{\pi} \left[\frac{\partial J_p(x)}{\partial p} - (-1)^p \frac{\partial J_{-p}(x)}{\partial p} \right]. \quad (2.34)$$

2.2 Bessel function properties

Below are just some of the properties of the bessel functions that will be used in this work.

2.2.1 Zeros of Bessel functions

The zeros of the Bessel functions don't occur at regular intervals, they have to be computed numerically, the values can find by computer or in tables. The difference between two successive zeros becomes approximately π when x is large, between two zeros of $J_p(x)$ there is a zero of $J_{p+1}(x)$ and vice versa. A property that is evident in the graphs. Except for $J_0(x)$, all the J_p 's start at the origin and oscillate with decreasing amplitude, $J_0(x)$ is equal to 1 at $x = 0$, this can be observed in the graph bellow

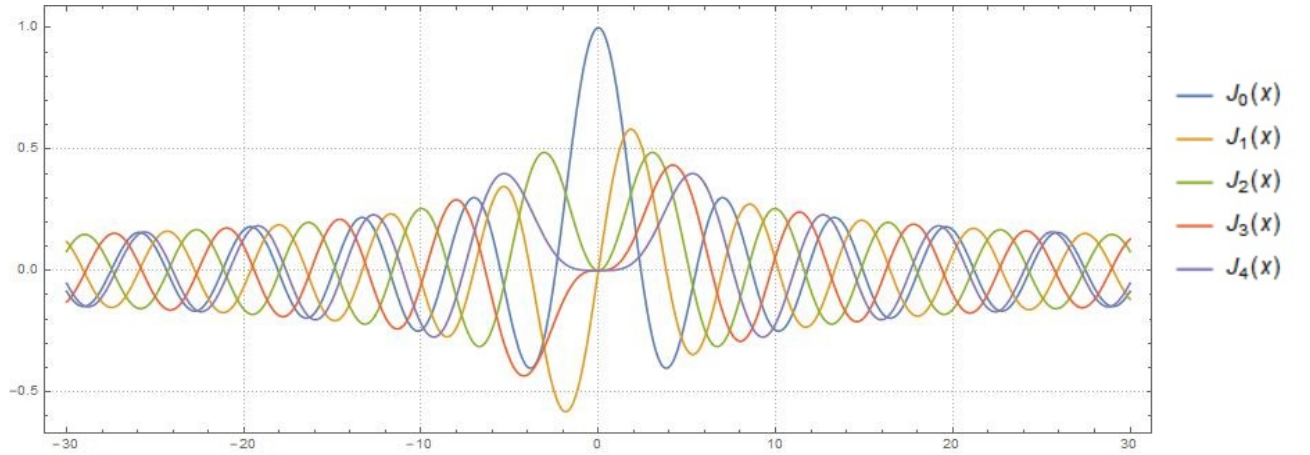


Figure 2.1: Bessel functions.

For very large values of x

$$J_0(x) \cong \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right) \quad (2.35)$$

$$J_p(x) \cong \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{p\pi}{2} - \frac{\pi}{4}\right). \quad (2.36)$$

In a two-dimensional system that shows circular symmetry, the Bessel function at the origin can be expressed as

$$J_p(r) \text{ with } r^2 = x^2 + y^2, \quad (2.37)$$

and the graphic representation of a couple of them is

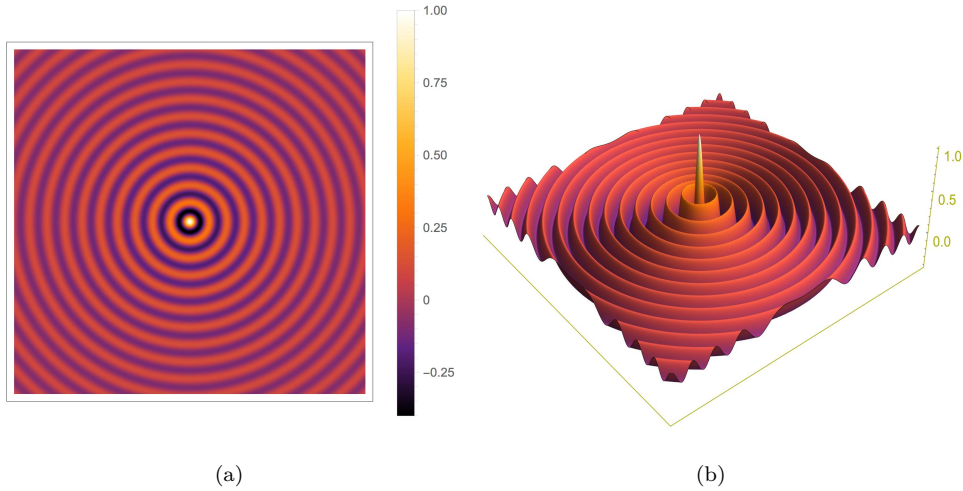


Figure 2.2: Zero-order Bessel function representation in (a) 2D and (b) 3D.

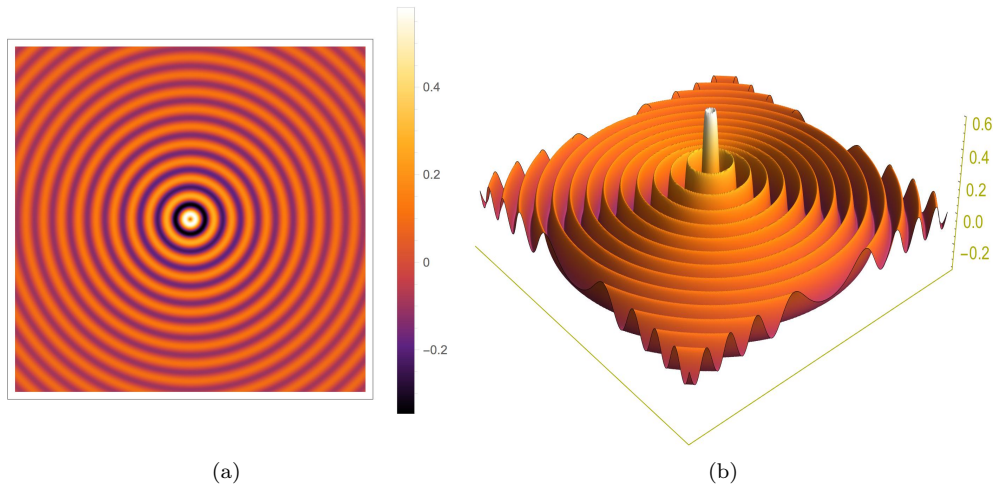


Figure 2.3: First-order Bessel function representation in (a) 2D and (b) 3D.

2.2.2 Recurrence relations

The following useful relations hold among Bessel functions and their derivatives.

$$J_p(x) = (-1)^p J_p(-x) \quad (2.38)$$

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x) \quad (2.39)$$

$$J_{p-1}(x) - J_{p+1}(x) = 2J'_p(x) \quad (2.40)$$

Adding Eq.(2.39) and (2.40), we have

$$\begin{aligned} 2J_{p-1}(x) &= 2J'_p(x) + \frac{2p}{x} J_p(x) \\ \Rightarrow J_{p-1}(x) &= J'_p(x) + \frac{p}{x} J_p(x) \\ \Rightarrow x^p J_{p-1}(x) &= x^p J'_p(x) + p x^{p-1} J_p(x) \\ \Rightarrow \frac{d}{dx} [x^p J_p(x)] &= x^p J_{p-1}(x). \end{aligned} \quad (2.41)$$

Subtracting Eq.(2.40) from (2.39) yields

$$\begin{aligned} -2J_{p+1}(x) &= 2J'_p(x) - \frac{2p}{x} J_p(x) \\ \Rightarrow -J_{p+1}(x) &= J'_p(x) - \frac{p}{x} J_p(x) \\ \Rightarrow -x^{-p} J_{p+1}(x) &= x^{-p} J'_p(x) - p x^{-p-1} J_p(x) \\ \Rightarrow \frac{d}{dx} [x^{-p} J_p(x)] &= -x^{-p} J_{p+1}(x). \end{aligned} \quad (2.42)$$

$$J_p(x) = (-1)^p x^p \left(\frac{1}{x} \frac{d}{dx} \right)^p J_0(x). \quad (2.43)$$

2.2.3 Integral representation

$$J_p(x) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha e^{ix \sin(\alpha) - p\alpha} = \frac{i^{-p}}{2\pi} \int_0^{2\pi} d\alpha e^{ix \cos(\alpha) + p\alpha} \quad (2.44)$$

$$\frac{\chi}{b} J_1(bx) = \int_0^x x dx J_0(bx) \quad (2.45)$$

2.2.4 Continuum form

For operations with a continuum of Bessel functions, $J_p(\alpha x)$, a key relation is the Bessel function closure equation,

$$\int_0^\infty x dx J_p(\alpha x) J_p(\alpha' x) = \frac{1}{\alpha} \delta(\alpha - \alpha') \quad p > -\frac{1}{2}, \quad (2.46)$$

where δ is the Dirac delta function. This may be proved by the use of Hankel transforms.

2.2.5 Graf's addition theorem

Let $\vec{x}$ and $\vec{y}$ be two-dimensional vectors, θ_{x-y} the angle between $\vec{x} - \vec{y}$ and the x axis, $\mathcal{B}$ that denotes the functions $J, Y, H^{(1)}, H^{(2)}$ or any linear combination of these functions, and l an integer, then

$$\mathcal{B}_l(|\vec{x} - \vec{y}|) e^{\pm i l \theta_{x-y}} = \sum_{p=-\infty}^{\infty} \mathcal{B}_{l+p}(|\vec{x}|) e^{\pm i(l+p)\theta_x} J_p(|\vec{y}|) e^{\mp i p \theta_y}, \quad |\vec{y}| < |\vec{x}|. \quad (2.47)$$

When $\mathcal{B} = J$, the restriction $|\vec{y}| < |\vec{x}|$ is unnecessary.

2.3 The Dirac Delta in $\mathbb{R}^n$

The Dirac delta function in n -dimensional Cartesian space, $\delta(\vec{r})$, $\vec{r} \in \mathbb{R}^n$ is defined by the following two properties

$$\delta(\vec{r}) = 0 \text{ for } \vec{r} \neq 0 \quad (2.48)$$

$$\int_{\mathbb{R}^n} d\vec{r} \delta(\vec{r}) = 1. \quad (2.49)$$

Let $\vec{r} \in \mathbb{R}^n$ be an n -dimensional vector, if when it's represented in Cartesian coordinates, the vector is written as

$$\vec{r} = \sum_{i=1}^n x_i \hat{x}_i \quad (2.50)$$

then, it's Dirac delta is defined

$$\delta(\vec{r}) = \prod_{i=1}^n \delta(x_i) \quad (2.51)$$

and thus, the integration is carried out independently in each dimension. In the same form for $\vec{r}_0 \in \mathbb{R}^n$

$$\delta(\vec{r} - \vec{r}_0) = 0 \text{ for } \vec{r} \neq \vec{r}_0 \quad (2.52)$$

$$\int_{\mathbb{R}^n} d\vec{r} \delta(\vec{r} - \vec{r}_0) = 1. \quad (2.53)$$

$$\delta(\vec{r} - \vec{r}_0) = \prod_{i=1}^n \delta(x_i - x_{0i}), \quad (2.54)$$

also

$$\int_{\mathbb{R}^n} d\vec{r} f(\vec{r}) \delta(\vec{r} - \vec{r}_0) = f(\vec{r}_0). \quad (2.55)$$

$$\delta(\alpha \vec{r}) = \frac{1}{|\alpha|^n} \delta(\vec{r}). \quad (2.56)$$

2.3.1 Dirac delta in curvilinear coordinates

Now, let's consider a coordinate system different from the Cartesian defined by the direct and inverse relations

$$u_i = u_i(x_1, \dots, x_n) \quad \forall i = 1, \dots, n \quad (2.57)$$

$$x_i = x_i(u_1, \dots, u_n) \quad \forall i = 1, \dots, n \quad (2.58)$$

with

$$d\vec{r} = dx_1 \cdot dx_2 \cdots dx_n = |J| du_1 du_2 \cdots du_n \quad (2.59)$$

where J is the Jacobian or Jacobian determinant of the transformation,

$$J = J(u_1, \dots, u_n) = \frac{\partial(x_1, \dots, x_n)}{\partial(u_1, \dots, u_n)} = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{vmatrix} \quad (2.60)$$

then for Eq.(2.49) it must be met

$$\int_{\mathbb{R}^n} \delta(\vec{r}) |J| du_1 \cdots du_n = 1 \quad (2.61)$$

and by Eq.(2.51)

$$\Rightarrow \delta(\vec{r}) = \frac{\prod_{i=1}^n \delta(u_i)}{|J|} \quad (2.62)$$

and in the same form by Eq.(2.53)

$$\delta(\vec{r} - \vec{r}_0) = \frac{\prod_{i=1}^n \delta(u_i - u_{0i})}{|J|}. \quad (2.63)$$

For example, in polar coordinates

$$x = r \cos \theta \quad y = r \sin \theta \quad (2.64)$$

$$x_0 = r_0 \cos \theta_0 \quad y_0 = r_0 \sin \theta_0 \quad (2.65)$$

$$J(r, \theta) = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r, \quad (2.66)$$

then, the representation for the Dirac delta function at the origin is

$$\delta(x, y) = \frac{1}{r} \delta(r, \theta) = \frac{1}{r} \delta(r) \delta(\theta) \quad (2.67)$$

and for the shifted Dirac delta function

$$\delta(x - x_0, y - y_0) = \frac{1}{r} \delta(r - r_0) \delta(\theta - \theta_0). \quad (2.68)$$

$$\delta(x + x_0, y + y_0) = \frac{1}{r} \delta(r - r_0) \delta(\theta - (\theta_0 + \pi)). \quad (2.69)$$

A graphic representation of both cases can be seen in Fig. 2.4

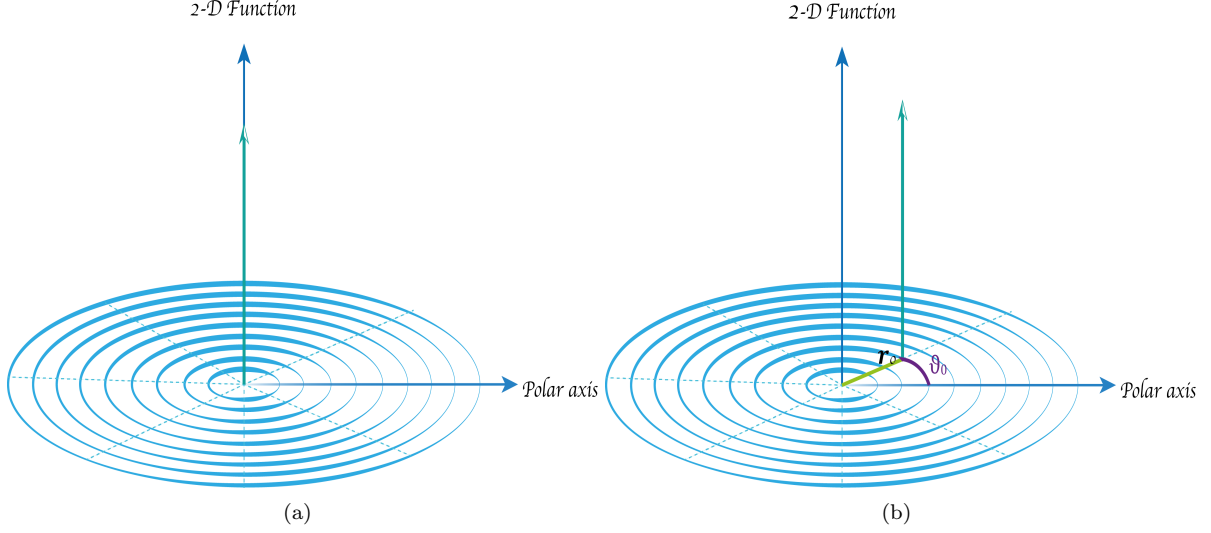


Figure 2.4: (a) Dirac delta function at the origin and (b) shifted Dirac delta function.

For the theoretical development of the following sections, the previous definitions correspond to the space of the object, in the Fourier space they are expressed as

$$\delta(\mu, \nu) = \frac{1}{\rho} \delta(\rho) \delta(\varphi). \quad (2.70)$$

$$\delta(\mu - \mu_0, \nu - \nu_0) = \frac{1}{\rho} \delta(\rho - \rho_0) \delta(\varphi - \varphi_0). \quad (2.71)$$

$$\delta(\mu + \mu_0, \nu + \nu_0) = \frac{1}{\rho} \delta(\rho - \rho_0) \delta(\varphi - (\varphi_0 + \pi)). \quad (2.72)$$

Unnecessary coordinates

If $x_{01}, \dots, x_{0n}$ and $u_{01}, \dots, u_{0n}$ are coordinates of a point P and if $J(u_{01}, \dots, u_{0n}) \neq 0$ then

$$\delta(x_1 - x_{01}) \cdots \delta(x_n - x_{0n}) = \frac{1}{|J|} \delta(u_1 - u_{01}) \cdots \delta(u_n - u_{0n}). \quad (2.73)$$

However, if $J(u_{01}, \dots, u_{0n}) = 0$ and the point P is specified by k coordinates $u_{01}, \dots, u_{0k}$ (that means that $n - k$ coordinates $u_{0(k+1)}, \dots, u_{0n}$ are unnecessary for the specification of

the point P), the Jacobian determinant is denoted by

$$J_k(u_1, \dots, u_k) = \int \cdots \int J(u_1, \dots, u_n) du_{(k+1)}, \dots, du_n \quad (2.74)$$

and

$$\delta(x_1 - x_{01}) \cdots \delta(x_n - x_{0n}) = \frac{1}{|J_k(u_1, \dots, u_k)|} \delta(u_1 - u_{01}) \cdots \delta(u_k - u_{0k}). \quad (2.75)$$

For example for polar coordinates

$$J(r, \theta) = r, \quad u_1 = r, u_{01} = r_0, u_2 = \theta, u_{02} = \theta_0, \quad k = 1 \quad (2.76)$$

- At points $P(r_0, \theta_0)$, if $r_0 \neq 0$, then

$$\delta(x - x_0) \delta(y - y_0) = \frac{1}{r} \delta(r - r_0) \delta(\theta - \theta_0) \quad (2.77)$$

- At the point $P(r_0, \theta_0)$, if $r_0 = 0$, then the coordinate θ_0 is unnecessary and

$$J_{k=1}(r) = \int_0^{2\pi} J(r, \theta) d\theta = \int_0^{2\pi} r d\theta = 2\pi r \quad (2.78)$$

$$\Rightarrow \delta(x, y) = \frac{1}{2\pi r} \delta(r). \quad (2.79)$$

2.3.2 The Comb function

The Comb function or also called two-dimensional Dirac pulse train is defined as a set of point sources that are symmetrically distributed respect to the origin with the same separation x_p with respect to the x axis and y_p with respect the y axis can be represented as follows

$$\zeta(x, y) = \delta_{x_p y_p}(x, y) = \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta(x - jx_p, y - ly_p) \quad (2.80)$$

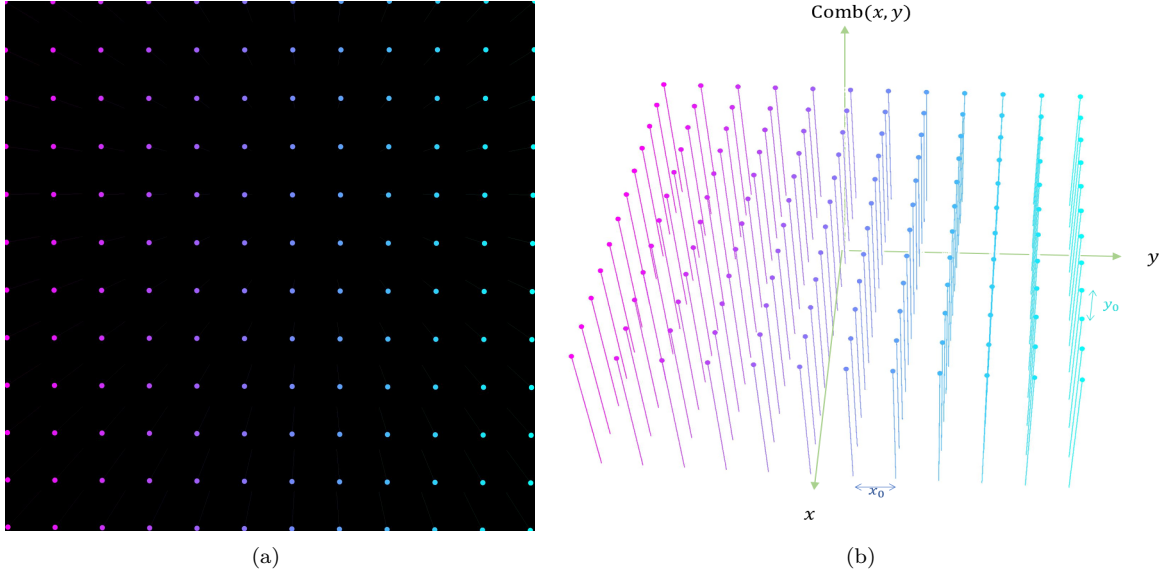


Figure 2.5: $\text{Comb}(x, y)$ function in (a) 2D and (b) 3D.

because the function is separable, it can be seen in the way

$$\zeta(x, y) = \zeta_x(x)\zeta_y(y) = \sum_{j=-\infty}^{\infty} \delta(x - jx_p) \sum_{l=-\infty}^{\infty} \delta(y - ly_p) \quad (2.81)$$

note that each factor has Fourier series of the form

$$\zeta_x(x) = \sum_{j=-\infty}^{\infty} C_j e^{ik_j x} \quad (2.82)$$

$$\zeta_y(y) = \sum_{l=-\infty}^{\infty} C_l e^{ik_l y} \quad (2.83)$$

with $k_j = \frac{2\pi j}{x_p}$, $k_l = \frac{2\pi l}{y_p}$ where x_p , y_p represents the periods and the coefficients C_j , C_l are given by

$$C_j = \frac{1}{x_p} \int_{-\frac{x_p}{2}}^{\frac{x_p}{2}} dx \zeta(x) e^{-ik_j x}. \quad (2.84)$$

$$C_l = \frac{1}{y_p} \int_{-\frac{y_p}{2}}^{\frac{y_p}{2}} dy \zeta(y) e^{-ik_l y}. \quad (2.85)$$

calculating C_j for Eq.(2.82) we obtain

$$\begin{aligned} C_j &= \frac{1}{x_p} \int_{-\frac{x_p}{2}}^{\frac{x_p}{2}} dx \delta_{x_p}(x) e^{-ik_j x} \\ &= \frac{1}{x_p} \end{aligned} \quad (2.86)$$

and in the same way $C_l = \frac{1}{y_p}$ is obtained, then

$$\zeta_x(x) = \frac{1}{x_p} \sum_{j=-\infty}^{\infty} e^{i2\pi \frac{j}{x_p} x} \quad (2.87)$$

and

$$\zeta_y(y) = \frac{1}{y_p} \sum_{l=-\infty}^{\infty} e^{i2\pi \frac{l}{y_p} y} \quad (2.88)$$

thus

$$\zeta(x, y) = \left(\frac{1}{x_p} \sum_{j=-\infty}^{\infty} e^{i2\pi \frac{j}{x_p} x} \right) \left(\frac{1}{y_p} \sum_{l=-\infty}^{\infty} e^{i2\pi \frac{l}{y_p} y} \right) \quad (2.89)$$

and the Fourier transform is

$$\begin{aligned} \mathcal{F} \{ \zeta(x, y) \} &= \mathcal{F} \left\{ \frac{1}{x_p} \sum_{j=-\infty}^{\infty} e^{i2\pi \frac{j}{x_p} x} \right\} \mathcal{F} \left\{ \frac{1}{y_p} \sum_{l=-\infty}^{\infty} e^{i2\pi \frac{l}{y_p} y} \right\} \\ &= \frac{1}{x_p} \sum_{j=-\infty}^{\infty} \delta \left(\mu - \frac{j}{x_p} \right) \frac{1}{y_p} \sum_{l=-\infty}^{\infty} \delta \left(\nu - \frac{l}{y_p} \right) \\ &= \frac{1}{x_p} \frac{1}{y_p} \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta \left(\mu - \frac{j}{x_p}, \nu - \frac{l}{y_p} \right) \end{aligned} \quad (2.90)$$

therefore the Fourier transform for a pulse train of period x_p in x and y_p in y is another pulse train of period $\frac{1}{x_p}$ in μ and $\frac{1}{y_p}$ in ν .

2.4 The Hankel transform

Two-dimensional system may often show circular symmetry, in these cases, a Fourier transform applied to a radial variable will be sufficient instead of applying it to two independent variables x and y . The appropriate expression of such problems is in terms of the Hankel transform, a one-dimensional transform with Bessel function as kernel.

When circular symmetry exists, that is

$$f(x, y) = f(r) \text{ with } r^2 = x^2 + y^2 \quad (2.91)$$

then $\tilde{f}(\mu, \nu) = \mathcal{F}\{f(x, y)\}$ proves also to be circularly symmetrical, i.e.,

$$\tilde{f}(\mu, \nu) = \tilde{f}(\rho) \text{ with } \rho^2 = \mu^2 + \nu^2. \quad (2.92)$$

Applying a coordinate change to Fourier transform definition and integrating over the angular variable, we can find the expression for the Hankel transform, this will be discussed in chapter 3. Let $f(r)$ be a function defined for $r \geq 0$. The m th order Hankel transform of $f(r)$ is defined as

$$F_m(\rho) = \mathcal{H}_m\{f(r)\} = 2\pi \int_0^\infty r dr f(r) J_m(2\pi\rho r). \quad (2.93)$$

If $m > 1/2$, Hankel's repeated integral immediately gives the inversion formula

$$f(r) = \mathcal{H}_m^{-1}\{F_m(\rho)\} = 2\pi \int_0^\infty \rho d\rho F_m(\rho) J_m(2\pi\rho r). \quad (2.94)$$

The factors 2π in the previous formulas can be canceled by the appropriate redefinition of the variables, but their retention is logically derived from the form adopted for the Fourier transforms. The most important special cases of the Hankel transform correspond to $m = 0$ and $m = 1$.

Table 2.2: Some Hankel Transforms of Order Zero.

$f(r)$	$F_0(\rho) = \mathcal{H}_0\{f(r)\}$
1	$\frac{1}{2\pi r}\delta(r)$
$\frac{1}{r}$	$\frac{1}{\rho}$
$\delta(r-a)$	$2\pi a J_0(2\pi a\rho)$
$e^{-\pi r^2}$	$e^{-\pi \rho^2}$
e^{-ar}	$\frac{2\pi a}{(4\pi^2 \rho^2 + a^2)^{\frac{3}{2}}}$
$\frac{e^{-ar}}{r}$	$\frac{2\pi}{(4\pi^2 \rho^2 + a^2)^{\frac{1}{2}}}$
e^{ir^2}	$i\pi e^{-i\pi^2 \rho^2}$
$r^2 e^{-\pi r^2}$	$\left(\frac{1}{\pi} - \rho^2\right) e^{\pi \rho^2}$
$\frac{1}{(a^2 + r^2)^{\frac{1}{2}}}$	$\frac{e^{-2\pi a\rho}}{\rho}$
$\frac{1}{(a^2 + r^2)^{\frac{3}{2}}}$	$\frac{2\pi e^{-2\pi a\rho}}{a}$
$\text{sinc}^2(ar)$	$\frac{1}{\pi a^2} \frac{1}{\cosh(\frac{a}{\rho})}$

Some of these relations will be used in the next chapter.

Chapter 3

The Fourier transform in polar coordinates

Using the theoretical concepts addressed in the previous chapter, the basic properties of the Fourier transform in polar coordinates are developed in this chapter in order to obtain the concepts that will be applied to develop the stated objectives.

3.1 Forward Fourier transform

In cartesian coordinates, the two-dimensional Fourier transform of an arbitrary function $\zeta(x, y)$ is given by

$$\begin{aligned}\tilde{\zeta}(\mu, \nu) = \mathcal{F}\{\zeta(x, y)\} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \zeta(x, y) e^{-i(\omega_x x + \omega_y y)} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \zeta(x, y) e^{-i2\pi(\mu x + \nu y)},\end{aligned}\tag{3.1}$$

where

$$\omega_x = 2\pi\mu, \quad \omega_y = 2\pi\nu\tag{3.2}$$

changing to polar coordinates

$$\begin{aligned}x &= r \cos \theta & y &= r \sin \theta \\ \mu &= \rho \cos \varphi & \nu &= \rho \sin \varphi\end{aligned}\tag{3.3}$$

we obtain

$$\begin{aligned}\tilde{\zeta}(\rho, \varphi) = \mathcal{F}\{\zeta(r, \theta)\} &= \int_0^\infty \int_0^{2\pi} r dr d\theta \zeta(r, \theta) e^{-i2\pi(\rho r \cos \theta \cos \varphi + \rho r \sin \theta \sin \varphi)} \\ &= \int_0^\infty \int_0^{2\pi} r dr d\theta \zeta(r, \theta) e^{-i2\pi \rho r \cos(\theta - \varphi)}.\end{aligned}\quad (3.4)$$

based on this, the equivalence in polar coordinates of the transforms of some functions already known in cartesian coordinates can be established.

3.1.1 Fourier transform of a constant

If $\zeta(r, \theta) = 1$

$$\begin{aligned}\Rightarrow \mathcal{F}\{1\} &= \int_0^\infty \int_0^{2\pi} r dr d\theta e^{-i2\pi \rho r \cos(\theta - \varphi)} \\ &= \int_0^\infty r dr \left[\int_0^{2\pi} d\theta e^{-i2\pi \rho r \cos(\theta - \varphi)} \right],\end{aligned}\quad (3.5)$$

let

$$\alpha = \theta - \varphi, \quad d\alpha = d\theta, \quad (3.6)$$

then (3.5) takes the form

$$\begin{aligned}&\int_0^\infty r dr \left[\int_{-\varphi}^{2\pi - \varphi} d\alpha e^{-i2\pi \rho r \cos \alpha} \right] \\ &\int_0^\infty r dr \left[\int_0^{2\pi} d\alpha e^{-i2\pi \rho r \cos \alpha} \right] \\ &= \int_0^\infty r dr [2\pi J_0(-2\pi \rho r)] \\ &= 2\pi \int_0^\infty r dr J_0(2\pi \rho r) \\ &= \mathcal{H}_0\{1\},\end{aligned}\quad (3.7)$$

making use of the Table 2.2, we can find

$$\text{i.e., } \mathcal{F}\{1\} = \mathcal{H}_0\{1\} = \frac{\delta(\rho)}{2\pi\rho} \quad (3.8)$$

which in effect is the 2D-Dirac delta at the origin as would be expected from the results in cartesian coordinates. Here we have used Eq.(2.44) with $m = 0$, i.e.,

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha e^{ix \sin \alpha} = \frac{1}{2\pi} \int_0^{2\pi} d\alpha e^{ix \cos \alpha}, \quad (3.9)$$

the relation derived from Eq.(2.38)

$$J_0(x) = J_0(-x) \quad (3.10)$$

and the definition of Hankel transform of order m (2.93) for $m = 0$.

An alternative way is to apply the Fourier transform in cartesian coordinates and change the variable to the final result as shown below.

$$\begin{aligned} \mathcal{F}\{1\} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy e^{-i2\pi(\mu x + \nu y)} \\ &= \int_{-\infty}^{\infty} dx e^{-i2\pi\mu x} \int_{-\infty}^{\infty} dy e^{-i2\pi\nu y} \\ &= \int_{-\infty}^{\infty} dx e^{-i2\pi\mu x} \int_{-\infty}^{\infty} dy e^{-i2\pi\nu y} \\ &= \delta(\mu)\delta(\nu) \\ &= \delta(\mu, \nu). \end{aligned} \quad (3.11)$$

Making the change to polar coordinates based on the relations (3.3) we have

$$\mathcal{F}\{1\} = \delta(\mu, \nu) = \frac{1}{\rho} \delta(\rho, \varphi) = \frac{1}{2\pi\rho} \delta(\rho), \quad (3.12)$$

thereby the result in (3.8) is verified.

3.1.2 Fourier transform of the Dirac delta function at the origin

Since the Fourier transform is reciprocal for the previous case, transform of the Dirac delta Function at the Origin will result in unity, however, development is shown. Given the Dirac delta in polar coordinates we have

$$\begin{aligned}
 \mathcal{F} \left\{ \frac{1}{r} \delta(r, \theta) \right\} &= \int_0^\infty \int_0^{2\pi} r dr d\theta \frac{1}{r} \delta(r, \theta) e^{-i2\pi \rho r \cos(\theta - \varphi)} \\
 &= \int_0^\infty dr \delta(r) \left[\int_0^{2\pi} d\theta \delta(\theta) e^{-i2\pi \rho r \cos(\theta - \varphi)} \right] \\
 &= \int_0^\infty dr \delta(r) \left[e^{-i2\pi \rho r \cos(\varphi)} \right] \\
 &= e^0 = 1.
 \end{aligned}$$

Here, Eq.(2.55) has been used.

$$\therefore \mathcal{F} \left\{ \frac{1}{r} \delta(r, \theta) \right\} = 1 \tag{3.13}$$

Performing the same procedure in the form that only depends on r (we saw in section 2.3.1 that there are unnecessary coordinates), with the help of the integral representation for the bessel function (Eq.(2.44)) something analogous is obtained

$$\begin{aligned}
 \mathcal{F} \left\{ \frac{1}{2\pi r} \delta(r) \right\} &= \int_0^\infty \int_0^{2\pi} r dr d\theta \frac{1}{2\pi r} \delta(r) e^{-i2\pi \rho r \cos(\theta - \varphi)} \\
 &= \frac{1}{2\pi} \int_0^\infty dr \delta(r) \left[\int_0^{2\pi} d\theta e^{-i2\pi \rho r \cos(\theta - \varphi)} \right] \\
 &= \frac{1}{2\pi} \int_0^\infty dr \delta(r) [2\pi J_0(-2\pi \rho r)] \\
 &= \int_0^\infty dr \delta(r) J_0(2\pi \rho r) \\
 &= J_0(0) = 1.
 \end{aligned} \tag{3.14}$$

In cartesian coordinates

$$\begin{aligned}
 \mathcal{F} \{ \delta(x, y) \} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \delta(x, y) e^{-i2\pi(\mu x + \nu y)} \\
 &= \int_{-\infty}^{\infty} dx \delta(x) e^{-i2\pi\mu x} \int_{-\infty}^{\infty} dy \delta(y) e^{-i2\pi\nu y} \\
 &= 1
 \end{aligned} \tag{3.15}$$

3.1.3 Fourier transform of the shifted Dirac delta function

Considering the relation in Eq.(2.68), then

$$\begin{aligned}
 \mathcal{F} \left\{ \frac{1}{r} \delta(r - r_0, \theta - \theta_0) \right\} &= \int_0^{\infty} \int_0^{2\pi} r dr d\theta \frac{1}{r} \delta(r - r_0, \theta - \theta_0) e^{-i2\pi\rho r \cos(\theta - \varphi)} \\
 &= \int_0^{\infty} dr \delta(r - r_0) \left[\int_0^{2\pi} d\theta \delta(\theta - \theta_0) e^{-i2\pi\rho r \cos(\theta - \varphi)} \right] \\
 &= \int_0^{\infty} dr \delta(r - r_0) \left[e^{-i2\pi\rho r \cos(\theta_0 - \varphi)} \right] \\
 &= e^{-i2\pi\rho r_0 \cos(\theta_0 - \varphi)}
 \end{aligned}$$

$$\text{i.e., } \mathcal{F} \left\{ \frac{1}{r} \delta(r - r_0, \theta - \theta_0) \right\} = e^{-i2\pi\rho r_0 \cos(\theta_0 - \varphi)} \tag{3.16}$$

and in cartesian coordinates

$$\begin{aligned}
 \mathcal{F} \{ \delta(x - x_0, y - y_0) \} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \delta(x - x_0, y - y_0) e^{-i2\pi(\mu x + \nu y)} \\
 &= \int_{-\infty}^{\infty} dx \delta(x - x_0) e^{-i2\pi\mu x} \int_{-\infty}^{\infty} dy \delta(y - y_0) e^{-i2\pi\nu y} \\
 &= e^{-i2\pi(\mu x_0 + \nu y_0)} \\
 &= e^{-i2\pi\rho r_0 \cos(\theta_0 - \varphi)}
 \end{aligned} \tag{3.17}$$

where the conversion was made based on Eq.(3.3) and Eq.(2.65), we can see that the complex exponential is given in the Fourier space but the displacements (angular and radial) remain in the object space. In the same form, using the relation in Eq.(2.69) we have

$$\begin{aligned}
 \mathcal{F} \left\{ \frac{1}{r} \delta(r - r_0, \theta - (\theta_0 + \pi)) \right\} &= \int_0^\infty \int_0^{2\pi} r dr d\theta \frac{1}{r} \delta(r - r_0, \theta - (\theta_0 + \pi)) e^{-i2\pi \rho r \cos(\theta - \varphi)} \\
 &= \int_0^\infty dr \delta(r - r_0) \left[\int_0^{2\pi} d\theta \delta(\theta - (\theta_0 + \pi)) e^{-i2\pi \rho r \cos(\theta - \varphi)} \right] \\
 &= \int_0^\infty dr \delta(r - r_0) \left[e^{-i2\pi \rho r \cos(\theta_0 - \varphi + \pi)} \right] \\
 &= e^{i2\pi \rho r_0 \cos(\theta_0 - \varphi)}
 \end{aligned}$$

$$i.e., \mathcal{F} \left\{ \frac{1}{r} \delta(r - r_0, \theta - (\theta_0 + \pi)) \right\} = e^{i2\pi \rho r_0 \cos(\theta_0 - \varphi)} \quad (3.18)$$

and in cartesian coordinates

$$\begin{aligned}
 \mathcal{F} \{ \delta(x + x_0, y + y_0) \} &= \int_{-\infty}^\infty \int_{-\infty}^\infty dx dy \delta(x, y) e^{-i2\pi(\mu x + \nu y)} \\
 &= \int_{-\infty}^\infty dx \delta(x + x_0) e^{-i2\pi \mu x} \int_{-\infty}^\infty dy \delta(y + y_0) e^{-i2\pi \nu y} \\
 &= e^{i2\pi(\mu x_0 + \nu y_0)} \\
 &= e^{i2\pi \rho r_0 \cos(\theta_0 - \varphi)}
 \end{aligned} \quad (3.19)$$

3.1.4 Fourier transform of a complex exponential

Considering the positive complex exponential $e^{i2\pi r \rho_0 \cos(\varphi_0 - \theta)}$ (given in the object space with the displacements angular and radial in the Fourier space), then

$$\begin{aligned}
 \mathcal{F} \{ e^{i2\pi r \rho_0 \cos(\varphi_0 - \theta)} \} &= \int_0^\infty \int_0^{2\pi} r dr d\theta e^{i2\pi r \rho_0 \cos(\varphi_0 - \theta)} e^{-i2\pi \rho r \cos(\theta - \varphi)} \\
 &= \int_0^\infty \int_0^{2\pi} r dr d\theta e^{i2\pi r \rho_0 (\cos \varphi_0 \cos \theta + \sin \varphi_0 \sin \theta)} e^{-i2\pi \rho r (\cos \theta \cos \varphi + \sin \theta \sin \varphi)} \\
 &= \int_0^\infty \int_0^{2\pi} r dr d\theta e^{i2\pi r [(\rho_0 \cos \varphi_0 - \rho \cos \varphi) \cos \theta + (\rho_0 \sin \varphi_0 - \rho \sin \varphi) \sin \theta]},
 \end{aligned}$$

let

$$a \cos \beta = \rho_0 \cos \varphi_0 - \rho \cos \varphi \quad (3.20)$$

$$a \sin \beta = \rho_0 \sin \varphi_0 - \rho \sin \varphi \quad (3.21)$$

$$\Rightarrow a = \sqrt{\rho_0^2 + \rho^2 - 2\rho_0\rho \cos(\varphi - \varphi_0)} \quad (3.22)$$

$$\begin{aligned} \Rightarrow \mathcal{F} \left\{ e^{i2\pi r \rho_0 \cos(\varphi_0 - \theta)} \right\} &= \int_0^\infty \int_0^{2\pi} r dr d\theta e^{i2\pi r [a \cos \beta \cos \theta + a \sin \beta \sin \theta]} \\ &= \int_0^\infty \int_0^{2\pi} r dr d\theta e^{i2\pi a r \cos(\theta - \beta)}. \end{aligned} \quad (3.23)$$

Using the change of variable

$$\alpha = \theta - \beta \Rightarrow d\alpha = d\theta \text{ and } \theta = \alpha + \beta \quad (3.24)$$

$$\begin{aligned} \Rightarrow \mathcal{F} \left\{ e^{i2\pi r \rho_0 \cos(\varphi_0 - \theta)} \right\} &= \int_0^\infty r dr \int_0^{2\pi} d\alpha e^{i2\pi a r \cos \alpha} \\ &= 2\pi \int_0^\infty r dr J_0 \left(2\pi r \sqrt{\rho_0^2 + \rho^2 - 2\rho_0\rho \cos(\varphi - \varphi_0)} \right). \end{aligned}$$

Following the Graf's addition theorem (section 2.2.5) with $\mathcal{B} = J$, i.e.,

$$J_l(|\vec{x} - \vec{y}|) e^{\pm i l \theta_{x-y}} = \sum_{m=-\infty}^{\infty} J_{l+m}(|\vec{x}|) e^{\pm i(l+m)\theta_x} J_m(|\vec{y}|) e^{\mp i m \theta_y} \quad (3.25)$$

and if

$$\vec{x} = 2\pi r (\rho \cos \varphi, \rho \sin \varphi) \quad (3.26)$$

$$\vec{y} = 2\pi r (\rho_0 \cos \varphi_0, \rho_0 \sin \varphi_0) \quad (3.27)$$

$$\Rightarrow |\vec{x}| = 2\pi r \rho \quad (3.28)$$

$$|\vec{y}| = 2\pi r \rho_0 \quad (3.29)$$

$$|\vec{x} - \vec{y}| = 2\pi r \sqrt{\rho_0^2 + \rho^2 - 2\rho_0\rho \cos(\varphi - \varphi_0)} \quad (3.30)$$

$$\theta_x = \varphi \quad (3.31)$$

$$\theta_y = \varphi_0, \quad (3.32)$$

here the angle θ_{x-y} between $\vec{x} - \vec{y}$ and the x axis can be calculated but it's unnecessary, then Eq.(3.25) takes the form

$$\begin{aligned} J_l(|\vec{x} - \vec{y}|)e^{\pm il\theta_{x-y}} &= J_l\left(2\pi r\sqrt{\rho_0^2 + \rho^2 - 2\rho_0\rho\cos(\varphi - \varphi_0)}\right)e^{\pm il\theta_{x-y}} \\ &= \sum_{m=-\infty}^{\infty} J_{l+m}(2\pi r\rho)e^{\pm i(l+m)\varphi} J_m(2\pi r\rho_0)e^{\mp im\varphi_0}, \end{aligned}$$

if also $l = 0$,

$$J_0\left(2\pi r\sqrt{\rho_0^2 + \rho^2 - 2\rho_0\rho\cos(\varphi - \varphi_0)}\right) = \sum_{m=-\infty}^{\infty} J_m(2\pi r\rho)e^{\pm im\varphi} J_m(2\pi r\rho_0)e^{\mp im\varphi_0} \quad (3.33)$$

$$\begin{aligned} \Rightarrow \mathcal{F}\left\{e^{i2\pi r\rho_0\cos(\theta-\varphi_0)}\right\} &= 2\pi \int_0^\infty r dr \sum_{m=-\infty}^{\infty} J_m(2\pi r\rho)e^{\pm im\varphi} J_m(2\pi r\rho_0)e^{\mp im\varphi_0} \\ &= 2\pi \sum_{m=-\infty}^{\infty} e^{\pm im\varphi} e^{\mp im\varphi_0} \int_0^\infty r dr J_m(2\pi r\rho) J_m(2\pi r\rho_0). \end{aligned}$$

Let $\omega = 2\pi r \Rightarrow r = \frac{\omega}{2\pi}, dr = \frac{d\omega}{2\pi}$, then by Eq.(2.46)

$$\begin{aligned} \int_0^\infty r dr J_m(2\pi r\rho) J_m(2\pi r\rho_0) &= \frac{1}{(2\pi)^2} \int_0^\infty \omega d\omega J_m(\omega\rho) J_m(\omega\rho_0) \\ &= \frac{1}{(2\pi)^2} \frac{1}{\rho} \delta(\rho - \rho_0) \end{aligned} \quad (3.34)$$

$$\Rightarrow \mathcal{F}\left\{e^{i2\pi r\rho_0\cos(\theta-\varphi_0)}\right\} = \frac{1}{\rho} \delta(\rho - \rho_0) \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{\pm im\varphi} e^{\mp im\varphi_0}, \quad (3.35)$$

also the sum can be interpreted as a Fourier series in the form

$$\delta(\varphi - \varphi_0) = \sum_{m=-\infty}^{\infty} \chi_m e^{im\varphi} \quad (3.36)$$

$$\begin{aligned}
 \chi_m &= \frac{1}{2\pi} \int_0^{2\pi} d\varphi \delta(\varphi - \varphi_0) e^{-im\varphi} \\
 &= \frac{1}{2\pi} e^{-im\varphi_0}
 \end{aligned} \tag{3.37}$$

$$\Rightarrow \delta(\varphi - \varphi_0) = \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} e^{-im\varphi_0} e^{im\varphi} \tag{3.38}$$

$$\text{i.e., } \mathcal{F} \left\{ e^{i2\pi r \rho_0 \cos(\varphi_0 - \theta)} \right\} = \frac{1}{\rho} \delta(\rho - \rho_0) \delta(\varphi - \varphi_0). \tag{3.39}$$

In cartesian coordinates

$$\begin{aligned}
 \mathcal{F} \left\{ e^{i2\pi(x\mu_0 + y\nu_0)} \right\} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy e^{i2\pi(x\mu_0 + y\nu_0)} e^{-i2\pi(\mu x + \nu y)} \\
 &= \int_{-\infty}^{\infty} dx e^{i2\pi x \mu_0} e^{-i2\pi \mu x} \int_{-\infty}^{\infty} dy e^{i2\pi y \nu_0} e^{-i2\pi \nu y} \\
 &= \int_{-\infty}^{\infty} dx e^{-i2\pi x(\mu - \mu_0)} \int_{-\infty}^{\infty} dy e^{-i2\pi y(\nu - \nu_0)} \\
 &= \delta(\mu - \mu_0, \nu - \nu_0) \\
 &= \frac{1}{\rho} \delta(\rho - \rho_0) \delta(\varphi - \varphi_0).
 \end{aligned} \tag{3.40}$$

In the same form, considering the negative complex exponential $e^{-i2\pi r \rho_0 \cos(\theta - \varphi_0)}$, a change can be made to the previous result, replacing φ_0 with $\varphi_0 + \pi$, i.e.,

$$\mathcal{F} \left\{ e^{i2\pi r \rho_0 \cos(\theta - \varphi_0)} \right\} = \mathcal{F} \left\{ e^{-i2\pi r \rho_0 \cos(\theta - (\varphi_0 + \pi))} \right\}$$

$$\text{i.e., } \mathcal{F} \left\{ e^{-i2\pi r \rho_0 \cos(\varphi_0 - \theta)} \right\} = \frac{1}{\rho} \delta(\rho - \rho_0) \delta(\varphi - (\varphi_0 + \pi)) \tag{3.41}$$

and in cartesian coordinates

$$\begin{aligned}
 \mathcal{F} \left\{ e^{-i2\pi(x\mu_0+y\nu_0)} \right\} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy e^{-i2\pi(x\mu_0+y\nu_0)} e^{-i2\pi(\mu x+\nu y)} \\
 &= \int_{-\infty}^{\infty} dx e^{-i2\pi x\mu_0} e^{-i2\pi\mu x} \int_{-\infty}^{\infty} dy e^{-i2\pi y\nu_0} e^{-i2\pi\nu y} \\
 &= \int_{-\infty}^{\infty} dx e^{-i2\pi x(\mu+\mu_0)} \int_{-\infty}^{\infty} dy e^{-i2\pi y(\nu+\nu_0)} \\
 &= \delta(\mu + \mu_0) \delta(\nu + \nu_0) \\
 &= \delta(\mu + \mu_0, \nu + \nu_0) \\
 &= \frac{1}{\rho} \delta(\rho - \rho_0) \delta(\varphi - (\varphi_0 + \pi)).
 \end{aligned} \tag{3.42}$$

This result is consistent with Eq.(2.72).

3.2 Fourier transform of the Cylinder (circle) function

The Cylinder (circle) function describes the transmission function of a circular aperture and it's defined as

$$\text{circ} \left(\frac{r}{R} \right) = \begin{cases} 1 & \text{if } r < R, \\ 0 & \text{if } r > R. \end{cases} \tag{3.43}$$

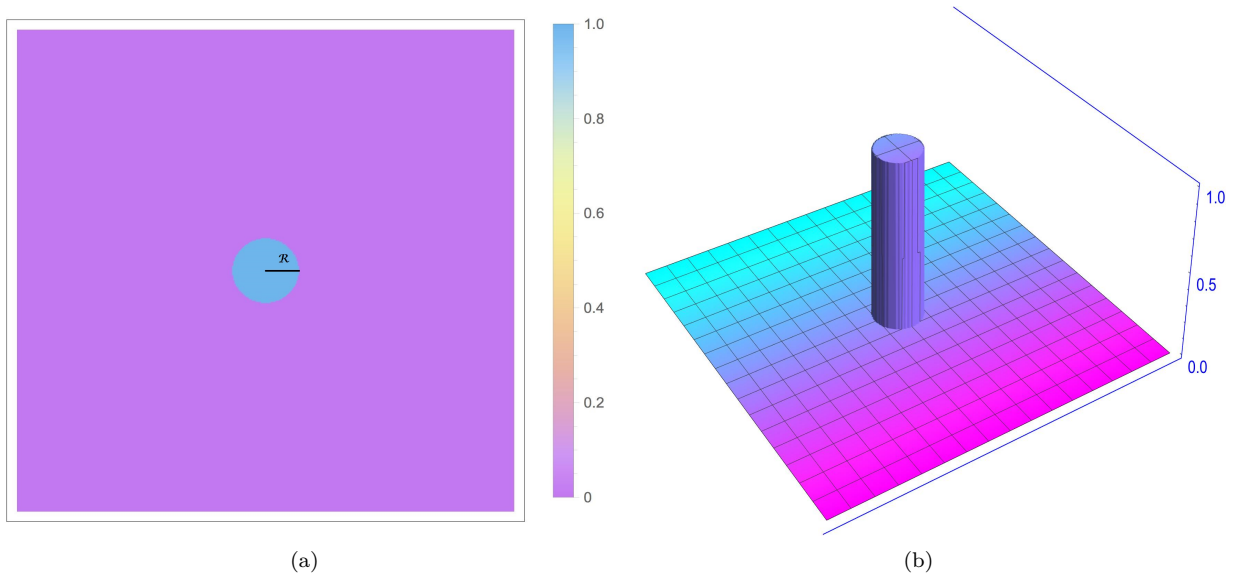


Figure 3.1: $\text{Circ} \left(\frac{r}{R} \right)$ in (a) 2D and (b) 3D.

It has the following properties

- Unit amplitude inside radius R
- Null amplitude outside
- Circularly symmetric version of 2-D rectangle
- Area of enclosed circle of unit diameter is $\frac{\pi}{4} \simeq 0.7854 < \text{unit area of } \text{rect}(x, y)$.

its Fourier transform is

$$\begin{aligned}
 \tilde{\zeta}(\rho) = \mathcal{F} \left\{ \text{circ} \left(\frac{r}{R} \right) \right\} &= \int_0^\infty \int_0^{2\pi} r dr d\theta \text{circ} \left(\frac{r}{R} \right) e^{-i2\pi\rho r \cos(\theta-\varphi)} \\
 &= \int_0^R \int_0^{2\pi} r dr d\theta e^{-i2\pi\rho r \cos(\theta-\varphi)} \\
 &= \int_0^R r dr 2\pi J_0(-2\pi\rho r) \\
 &= 2\pi \int_0^R r dr J_0(2\pi\rho r) \\
 &= 2\pi \frac{R}{2\pi\rho} J_1(2\pi\rho R) \\
 &= 2\pi R^2 \frac{J_1(2\pi\rho R)}{2\pi\rho R} \tag{3.44}
 \end{aligned}$$

$$= \pi R^2 \text{Somb}(2\rho R) \tag{3.45}$$

$$= \pi R^2 \text{Jinc}(2\rho R) \tag{3.46}$$

$$= \pi R^2 \text{Bessinc}(2\rho R) \tag{3.47}$$

where Eq.(2.45) has been considered and the follow definition has been used

$$\text{Somb}(a\rho) = \text{Jinc}(a\rho) = \text{Bessinc}(a\rho) = 2 \frac{J_1(\pi a\rho)}{\pi a\rho}. \tag{3.48}$$

A sombrero function (sometimes called bessinc function or jinc function) is the 2-dimensional polar coordinate analog of the sinc function, it can be defined through the Bessel function of the first kind, in some definitions constant 2 may or may not be considered, but with the constant, the maximum amplitude that occurs at the origin is 1.

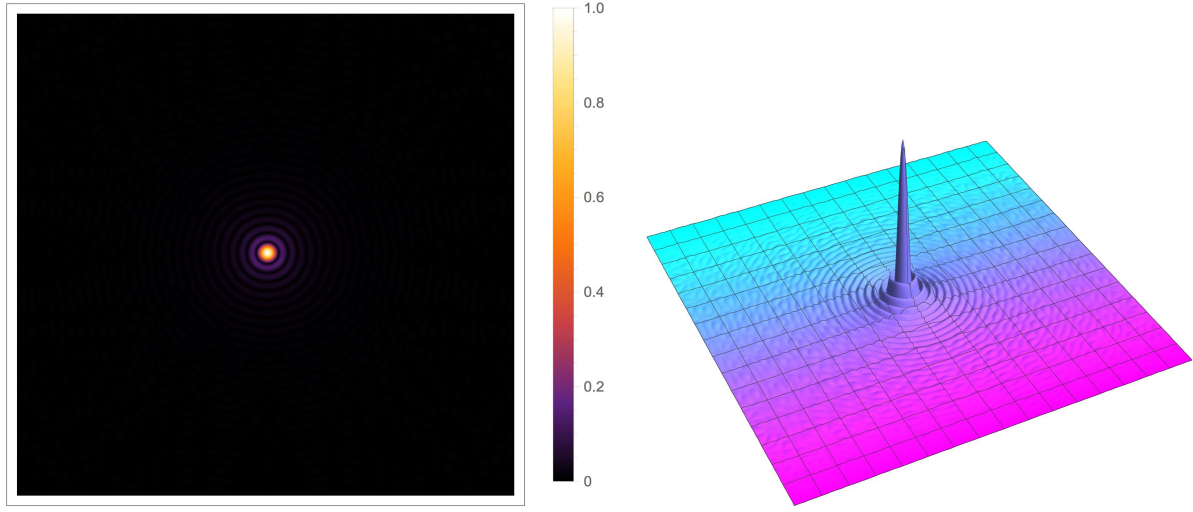


Figure 3.2: Diffraction pattern of a circular aperture in (a) 2D and (b) 3D. On the color scale 0 represents the minimum value of intensity and 1 represents the maximum value.

3.3 Inverse Fourier transform

In cartesian coordinates, the two-dimensional inverse Fourier transform of a function $\tilde{\zeta}(\mu, \nu)$ is given by

$$\begin{aligned}\zeta(r, \theta) = \mathcal{F}^{-1}\{\tilde{\zeta}(\mu, \nu)\} &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega_x d\omega_y \tilde{\zeta}(\omega_x, \omega_y) e^{-i(\omega_x x + \omega_y y)} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\mu d\nu \tilde{\zeta}(\omega_x, \omega_y) e^{i2\pi(\mu x + \nu y)},\end{aligned}\quad (3.49)$$

where

$$\omega_x = 2\pi\mu, \quad \omega_y = 2\pi\nu \quad (3.50)$$

making a change to polar coordinates with the relations (3.3), we obtain

$$\zeta(r, \theta) = \mathcal{F}^{-1}\{\tilde{\zeta}(\rho, \varphi)\} = \int_0^{\infty} \int_0^{2\pi} \rho d\rho d\varphi \tilde{\zeta}(\rho, \varphi) e^{i2\pi\rho r \cos(\theta - \varphi)}. \quad (3.51)$$

Just like in the forward transform, we can find the equivalence in polar coordinates of the inverse transforms of some functions known in cartesian coordinates. However, only one example will be shown that will be useful in the next section.

3.3.1 Inverse Fourier transform of the shifted Dirac delta function

Considering the previous definition and $\tilde{\zeta}(\rho, \varphi) = \frac{1}{\rho}\delta(\rho - \rho_0, \varphi - \varphi_0)$ we have

$$\begin{aligned}
 \mathcal{F}^{-1} \left\{ \frac{1}{\rho} \delta(\rho - \rho_0, \varphi - \varphi_0) \right\} &= \int_0^\infty \int_0^{2\pi} \rho d\rho d\varphi \frac{1}{\rho} \delta(\rho - \rho_0, \varphi - \varphi_0) e^{i2\pi \rho r \cos(\theta - \varphi)} \\
 &= \int_0^\infty d\rho \delta(\rho - \rho_0) \int_0^{2\pi} d\varphi \delta(\varphi - \varphi_0) e^{i2\pi \rho r \cos(\theta - \varphi)} \\
 &= \int_0^\infty d\rho \delta(\rho - \rho_0) e^{i2\pi \rho r \cos(\theta - \varphi_0)} \\
 &= e^{i2\pi \rho_0 r \cos(\theta - \varphi_0)}
 \end{aligned}$$

$$i.e., \mathcal{F}^{-1} \left\{ \frac{1}{\rho} \delta(\rho - \rho_0, \varphi - \varphi_0) \right\} = e^{i2\pi \rho_0 r \cos(\theta - \varphi_0)} \quad (3.52)$$

Similarly

$$\begin{aligned}
 \mathcal{F}^{-1} \left\{ \frac{1}{\rho} \delta(\rho - \rho_0, \varphi - (\varphi_0 + \pi)) \right\} &= \int_0^\infty \int_0^{2\pi} \rho d\rho d\varphi \frac{1}{\rho} \delta(\rho - \rho_0, \varphi - (\varphi_0 + \pi)) e^{i2\pi \rho r \cos(\theta - \varphi)} \\
 &= \int_0^\infty d\rho \delta(\rho - \rho_0) \int_0^{2\pi} d\varphi \delta(\varphi - (\varphi_0 + \pi)) e^{i2\pi \rho r \cos(\theta - \varphi)} \\
 &= \int_0^\infty d\rho \delta(\rho - \rho_0) e^{i2\pi \rho r \cos(\theta - (\varphi_0 + \pi))} \\
 &= e^{-i2\pi \rho_0 r \cos(\theta - \varphi_0)}
 \end{aligned}$$

$$i.e., \mathcal{F}^{-1} \left\{ \frac{1}{\rho} \delta(\rho - \rho_0, \varphi - (\varphi_0 + \pi)) \right\} = e^{-i2\pi \rho_0 r \cos(\theta - \varphi_0)} \quad (3.53)$$

3.3.2 Inverse Fourier transform of a complex exponential

Following the same procedure as in the section 3.1.4, the inverse Fourier transform of a negative and positive complex exponential can be found, however the development is too long, but with a little patience can be seen that

$$i.e., \mathcal{F}^{-1} \left\{ e^{-i2\pi r \rho_0 \cos(\varphi_0 - \theta)} \right\} = \frac{1}{r} \delta(r - r_0, \theta - \theta_0) \quad (3.54)$$

$$i.e., \mathcal{F}^{-1} \left\{ e^{i2\pi r \rho_0 \cos(\varphi - \theta_0)} \right\} = \frac{1}{r} \delta(r - r_0, \theta - (\theta_0 + \pi)). \quad (3.55)$$

3.4 Functions with radial symmetry

If a function has radial symmetry then it can be written as a function of r only, then its Fourier transform is given by

$$\begin{aligned} \tilde{\zeta}(\rho) = \mathcal{F}\{\zeta(r)\} &= \int_0^\infty \int_0^{2\pi} r dr d\theta \zeta(r) e^{-i2\pi \rho r \cos(\theta - \varphi)} \\ &= \int_0^\infty r \zeta(r) dr \int_0^{2\pi} d\theta e^{-i2\pi \rho r \cos(\theta - \varphi)} \\ &= \int_0^\infty r \zeta(r) dr 2\pi J_0(-2\pi \rho r) \\ &= 2\pi \int_0^\infty r \zeta(r) dr J_0(2\pi \rho r) \\ &= \mathcal{H}_0\{\zeta(r)\}. \end{aligned} \quad (3.56)$$

Thus, the special case of the 2D Fourier transform of a function with radial symmetry is the same as the zeroth-order Hankel transform of that function.

In the same way, the inverse Fourier transform is given by

$$\begin{aligned}
 \zeta(r) = \mathcal{F}^{-1}\{\tilde{\zeta}(\rho)\} &= \int_0^\infty \int_0^{2\pi} \rho d\rho d\varphi \tilde{\zeta}(\rho) e^{i2\pi\rho r \cos(\theta-\varphi)} \\
 &= \int_0^\infty \rho d\rho \tilde{\zeta}(\rho) \int_0^{2\pi} d\varphi e^{-i2\pi\rho r \cos(\theta-\varphi)} \\
 &= \int_0^\infty \rho d\rho \tilde{\zeta}(\rho) 2\pi J_0(2\pi\rho r) \\
 &= \mathcal{H}_0\{\tilde{\zeta}(\rho)\}.
 \end{aligned} \tag{3.57}$$

3.4.1 Ring delta function

The ring delta function describes the transmission function of a annular aperture, is defined as

$$\zeta(r) = \frac{1}{2\pi r} \delta(r - r_0) \tag{3.58}$$

which, is nonzero only on the ring of radius r_0 (Fig. 3.3).

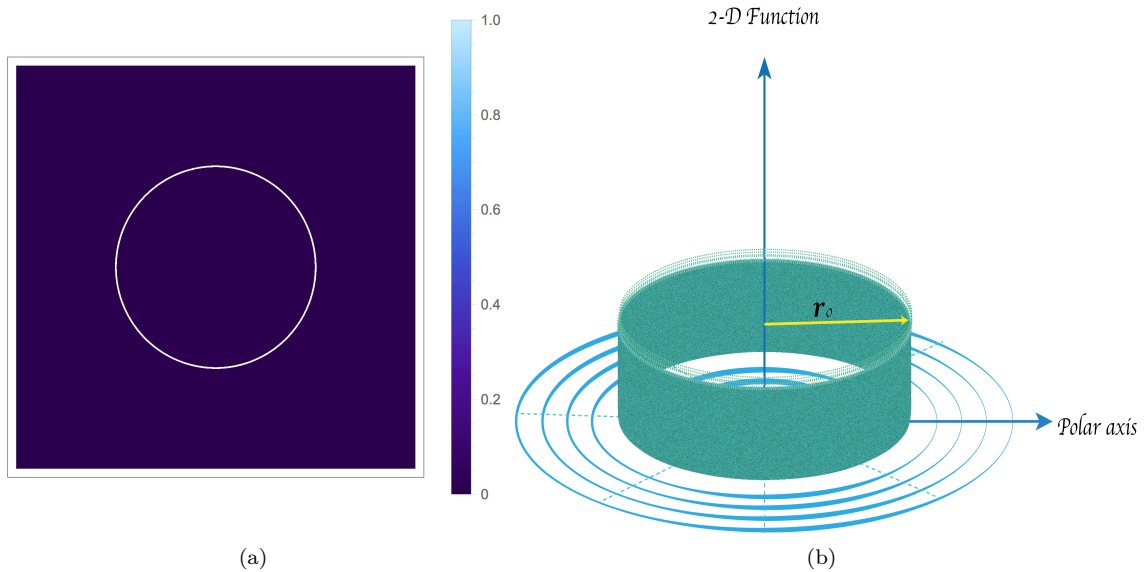


Figure 3.3: Ring delta function in (a) 2D and (b) 3D.

Its Fourier transform is

$$\begin{aligned}
 \tilde{\zeta}(\rho) = \mathcal{F} \left\{ \frac{1}{2\pi r} \delta(r - r_0) \right\} &= \mathcal{H}_0 \left\{ \frac{1}{2\pi r} \delta(r - r_0) \right\} \\
 &= 2\pi \int_0^\infty r dr \frac{1}{2\pi r} \delta(r - r_0) J_0(2\pi \rho r) \\
 &= J_0(2\pi \rho r_0)
 \end{aligned} \tag{3.59}$$

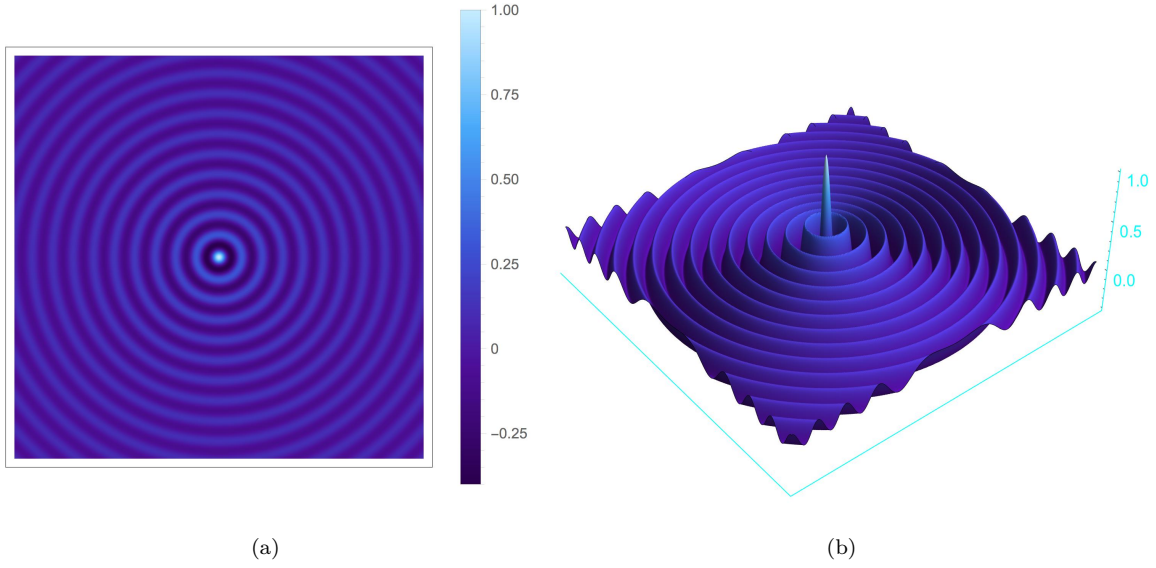


Figure 3.4: Diffraction pattern of an annular aperture in (a) 2D and (b) 3D.

3.5 Functions without radial symmetry: Fourier series for a function in polar coordinates

A two-dimensional arbitrary function $\zeta(r, \theta)$, expressed in polar coordinates, is periodic in the variable θ , with period 2π . As a result, it is possible to express $\zeta(r, \theta)$ in a Fourier series in the angular variable

$$\zeta(r, \theta) = \sum_{m=-\infty}^{\infty} \zeta_m(r) e^{im\theta} \tag{3.60}$$

where

$$\zeta_m(r) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \zeta(r, \theta) e^{-im\theta}. \tag{3.61}$$

It's clear that the Fourier coefficients $(\zeta_m(r))$ are only functions of radius, each term in Eq.(3.60) is known as a circular harmonic of the function ζ .

3.5.1 Series representation for the forward Fourier transform

Considering the representation shown in Eq.(3.60), the Fourier transform can be expressed as

$$\begin{aligned}\tilde{\zeta}(\rho, \varphi) = \mathcal{F}\{\zeta(r, \theta)\} &= \int_0^\infty \int_0^{2\pi} r dr d\theta \sum_{m=-\infty}^{\infty} \zeta_m(r) e^{im\theta} e^{-i2\pi\rho r \cos(\theta-\varphi)} \\ &= \sum_{m=-\infty}^{\infty} \int_0^\infty r dr \zeta_m(r) \int_0^{2\pi} d\theta e^{i[-2\pi\rho r \cos(\theta-\varphi)+m\theta]}.\end{aligned}\quad (3.62)$$

Using the change of variable

$$\alpha = \theta - \varphi \Rightarrow d\alpha = d\theta \text{ and } \theta = \alpha + \varphi \quad (3.63)$$

$$\begin{aligned}\Rightarrow \tilde{\zeta}(\rho, \varphi) &= \sum_{m=-\infty}^{\infty} \int_0^\infty r dr \zeta_m(r) \int_0^{2\pi} d\alpha e^{i[-2\pi\rho r \cos(\alpha)+m(\alpha+\varphi)]} \\ &= \sum_{m=-\infty}^{\infty} e^{im\varphi} \int_0^\infty r dr \zeta_m(r) \int_0^{2\pi} d\alpha e^{i[-2\pi\rho r \cos(\alpha)+m\alpha]} \\ &= \sum_{m=-\infty}^{\infty} e^{im\varphi} \int_0^\infty r dr \zeta_m(r) \frac{2\pi}{i-m} J_m(-2\pi\rho r) \\ &= \sum_{m=-\infty}^{\infty} e^{im\varphi} \int_0^\infty r dr \zeta_m(r) \frac{2\pi}{i-m} (-1)^{-m} J_m(2\pi\rho r) \\ &= \sum_{m=-\infty}^{\infty} 2\pi(-i)^m e^{im\varphi} \int_0^\infty r dr \zeta_m(r) J_m(2\pi\rho r) \\ &= \sum_{m=-\infty}^{\infty} (-i)^m e^{im\varphi} \mathcal{H}_m\{\zeta_m(r)\}\end{aligned}\quad (3.64)$$

where Eq.(2.44) has been used. Then, the Fourier transform of a general function in polar coordinates can be expressed as an infinite sum of weighted Hankel transforms of order m . It's clear that the result of Hankel transform will be in terms of ρ ; in addition, it can be seen that Eq.(3.65) represents a Fourier series in the frequency space.

$$\text{i.e., } \tilde{\zeta}(\rho, \varphi) = \sum_{m=-\infty}^{\infty} (-i)^m e^{im\varphi} \mathcal{H}_m\{\zeta_m(r)\} = \sum_{m=-\infty}^{\infty} \chi_m(\rho) e^{im\varphi} \quad (3.65)$$

$$\chi_m(\rho) = (-i)^m \mathcal{H}_m\{\zeta_m(r)\} = 2\pi(-i)^m \int_0^{\infty} r dr \zeta_m(r) J_m(2\pi\rho r) \quad (3.66)$$

Following this fact, some of the results found in the previous section can be written in an alternative form.

However, the result found in Eq.(3.16) allows to establish the following relation

$$e^{-i2\pi\rho r_0 \cos(\theta_0 - \varphi)} = \sum_{m=-\infty}^{\infty} (-i)^m e^{im\varphi} e^{-im\theta_0} J_m(2\pi\rho r_0) \quad (3.67)$$

3.5.2 Series representation for the inverse Fourier transform

Similarly the inverse Fourier transform can be expressed in series, using the fact that

$$\tilde{\zeta}(\rho, \varphi) = \sum_{m=-\infty}^{\infty} \chi_m(\rho) e^{im\varphi} \quad (3.68)$$

where

$$\chi_m(\rho) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \tilde{\zeta}(\rho, \varphi) e^{-im\varphi}. \quad (3.69)$$

but also meets the definition in Eq.(3.66)

$$\chi_m(\rho) = 2\pi(-i)^m \int_0^{\infty} r dr \zeta_m(r) J_m(2\pi\rho r),$$

i.e., the coefficient in the frequency space in terms of the coefficient in the object space.

$$\begin{aligned}\Rightarrow \mathcal{F}^{-1}\{\tilde{\zeta}(\rho, \varphi)\} = \zeta(r, \theta) &= \int_0^\infty \int_0^{2\pi} \rho d\rho d\varphi \sum_{m=-\infty}^{\infty} \chi_m(\rho) e^{im\varphi} e^{i2\pi\rho r \cos(\theta-\varphi)} \\ &= \sum_{m=-\infty}^{\infty} \int_0^\infty \rho d\rho \chi_m(\rho) \int_0^{2\pi} d\varphi e^{i[2\pi\rho r \cos(\varphi-\theta)+m\varphi]}. \quad (3.70)\end{aligned}$$

Using the change of variable $\alpha = \varphi - \theta$

$$\begin{aligned}\Rightarrow \mathcal{F}^{-1}\{\tilde{\zeta}(\rho, \varphi)\} &= \sum_{m=-\infty}^{\infty} e^{im\theta} \int_0^\infty \rho d\rho \chi_m(\rho) \int_0^{2\pi} d\alpha e^{i[2\pi\rho r \cos(\alpha)+m\alpha]} \\ &= \sum_{m=-\infty}^{\infty} e^{im\theta} \int_0^\infty \rho d\rho \chi_m(\rho) \frac{2\pi}{i-m} J_m(2\pi\rho r) \\ &= \sum_{m=-\infty}^{\infty} 2\pi(i)^m e^{im\theta} \int_0^\infty \rho d\rho \chi_m(\rho) J_m(2\pi\rho r) \\ &= \sum_{m=-\infty}^{\infty} (i)^m e^{im\theta} \mathcal{H}_m\{\chi_m(\rho)\}\end{aligned}$$

$$i.e., \mathcal{F}^{-1}\{\tilde{\zeta}(\rho, \varphi)\} = \sum_{m=-\infty}^{\infty} (i)^m e^{im\theta} \mathcal{H}_m\{\chi_m(\rho)\} \quad (3.71)$$

$$\zeta_m(r) = (i)^m \mathcal{H}_m\{\chi_m(\rho)\} = 2\pi(i)^m \int_0^\infty \rho d\rho \chi_m(\rho) J_m(2\pi\rho r) \quad (3.72)$$

Eq.(3.72) is the representation of coefficient in the object space in terms of the coefficient in the frequency space. Similary it can be obtained the inverse Fourier transform for a shifted Dirac delta function given as $\tilde{\zeta}(\rho, \varphi) = \frac{1}{\rho} \delta(\rho - \rho_0, \varphi - \varphi_0)$, where

$$\begin{aligned}\chi_m(\rho) &= \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{1}{\rho} \delta(\rho - \rho_0, \varphi - \varphi_0) e^{-im\varphi} \\ &= \frac{1}{2\pi\rho} \delta(\rho - \rho_0) \int_0^{2\pi} d\varphi \delta(\varphi - \varphi_0) e^{-im\varphi} \\ &= \frac{1}{2\pi\rho} \delta(\rho - \rho_0) e^{-im\varphi_0}\end{aligned} \quad (3.73)$$

$$\begin{aligned}
\Rightarrow \mathcal{F}^{-1}\{\tilde{\zeta}(\rho, \varphi)\} &= \sum_{m=-\infty}^{\infty} 2\pi(i)^m e^{im\theta} \int_0^{\infty} \rho d\rho \chi_m(\rho) J_m(2\pi\rho r) \\
&= \sum_{m=-\infty}^{\infty} 2\pi(i)^m e^{im\theta} \int_0^{\infty} \rho d\rho \left[\frac{1}{2\pi\rho} \delta(\rho - \rho_0) e^{-im\varphi_0} \right] J_m(2\pi\rho r) \\
&= \sum_{m=-\infty}^{\infty} (i)^m e^{im\theta} e^{-im\varphi_0} \int_0^{\infty} d\rho \delta(\rho - \rho_0) J_m(2\pi\rho r) \\
&= \sum_{m=-\infty}^{\infty} (i)^m e^{im\theta} e^{-im\varphi_0} J_m(2\pi\rho_0 r). \tag{3.74}
\end{aligned}$$

This and Eq.(3.52) allows to establish another relation

$$e^{i2\pi\rho_0 r \cos(\theta-\varphi_0)} = \sum_{m=-\infty}^{\infty} (i)^m e^{im\theta} e^{-im\varphi_0} J_m(2\pi\rho_0 r). \tag{3.75}$$

Eq.(3.67) and Eq.(3.75) can be written more generally as

$$e^{-i2\pi\rho r \cos(\theta-\varphi)} = \sum_{m=-\infty}^{\infty} (-i)^m e^{im\varphi} e^{-im\theta} J_m(2\pi\rho r). \tag{3.76}$$

$$e^{i2\pi\rho r \cos(\theta-\varphi)} = \sum_{m=-\infty}^{\infty} (i)^m e^{im\theta} e^{-im\varphi} J_m(2\pi\rho r). \tag{3.77}$$

These relations will be used in the following sections. It's important to mention that the forward and inverse transform for a function with radial symmetry mentioned in the section 3.4 are a particular case of the development presented in this section since these functions have only the $m = 0$ term in their Fourier series, thus if we consider $m = 0$ in Eq.(3.65)

and Eq.(3.72), we get something similar except for the terms $\zeta_0(r)$ and $\chi_0(\rho)$. However, the definition of the coefficients allows to see the following

$$\zeta_0(r) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \zeta(r) e^{-i0\theta} = \zeta(r) \quad (3.78)$$

and in the same way for $\chi_0(\rho)$, so Eq.(3.56) and Eq.(3.57) are recovered.

3.6 Fourier transform and series representation for a shifted function

In cartesian coordinates we know that

$$\mathcal{F}\{\zeta(x - x_0, y - y_0)\} = e^{-i2\pi(\mu x_0 + \nu y_0)} \mathcal{F}\{\zeta(x, y)\}, \quad (3.79)$$

to express this in polar coordinates, we can see that $e^{-i2\pi(\mu x_0 + \nu y_0)}$ is given by Eq.(3.17) also, we know that the Fourier transform of a general function in polar coordinates can be expressed as Eq.(3.65). Then, for $\vec{r}$ and $\vec{r}_0$, vectors in polar coordinates we found

$$\mathcal{F}\{\zeta(\vec{r} - \vec{r}_0)\} = e^{-i2\pi\rho r_0 \cos(\theta_0 - \varphi)} \mathcal{F}\{\zeta(r, \theta)\} \quad (3.80)$$

$$\begin{aligned} &= \left[\sum_{m=-\infty}^{\infty} (-i)^m e^{im\varphi} e^{-im\theta_0} J_m(2\pi\rho r_0) \right] \left[\sum_{n=-\infty}^{\infty} \chi_n(\rho) e^{in\varphi} \right] \\ &= \sum_{m=-\infty}^{\infty} (-i)^m e^{im\varphi} e^{-im\theta_0} J_m(2\pi\rho r_0) \sum_{n=-\infty}^{\infty} (-i)^n e^{in\varphi} \mathcal{H}_n\{\zeta_n(r)\} \end{aligned} \quad (3.81)$$

$$i.e., \mathcal{F}\{\zeta(\vec{r} - \vec{r}_0)\} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-i)^{m+n} e^{i(m+n)\varphi} e^{-im\theta_0} J_m(2\pi\rho r_0) \mathcal{H}_n\{\zeta_n(r)\}. \quad (3.82)$$

For simplicity, applying the inverse Fourier transform to Eq.(3.81) it is possible to find an expression for a shifted function

$$\begin{aligned}
 \zeta(\vec{r} - \vec{r}_0) &= \mathcal{F}^{-1} \left\{ \sum_{m=-\infty}^{\infty} (-i)^m e^{im\varphi} e^{-im\theta_0} J_m(2\pi\rho r_0) \sum_{n=-\infty}^{\infty} \chi_n(\rho) e^{in\varphi} \right\} \\
 &= \int_0^\infty \int_0^{2\pi} \rho d\rho d\varphi \left[\sum_{m=-\infty}^{\infty} (-i)^m e^{im\varphi} e^{-im\theta_0} J_m(2\pi\rho r_0) \sum_{n=-\infty}^{\infty} \chi_n(\rho) e^{in\varphi} \right] e^{i2\pi\rho r \cos(\theta-\varphi)} \\
 &= \int_0^\infty \int_0^{2\pi} \rho d\rho d\varphi \sum_{m=-\infty}^{\infty} (-i)^m e^{im\varphi} e^{-im\theta_0} J_m(2\pi\rho r_0) \sum_{n=-\infty}^{\infty} \chi_n(\rho) e^{in\varphi} \\
 &\quad \times \sum_{q=-\infty}^{\infty} (i)^q e^{iq\theta} e^{-iq\varphi} J_q(2\pi\rho r) \\
 &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} (i)^{-m} (i)^q e^{-im\theta_0} e^{iq\theta} \int_0^\infty \rho d\rho \chi_n(\rho) J_m(2\pi\rho r_0) J_q(2\pi\rho r) \\
 &\quad \times \int_0^{2\pi} d\varphi e^{i(m+n-q)\varphi}. \tag{3.83}
 \end{aligned}$$

The integral over φ is nonzero only if $m + n - q = 0 \rightarrow m = q - n$ and $(-i)^m = i^{-m}$

$$\begin{aligned}
 \Rightarrow \zeta(\vec{r} - \vec{r}_0) &= 2\pi \sum_{n=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} (i)^n e^{-i(q-n)\theta_0} e^{iq\theta} \int_0^\infty \rho d\rho \chi_n(\rho) J_{q-n}(2\pi\rho r_0) J_q(2\pi\rho r) \\
 &= \sum_{q=-\infty}^{\infty} [\zeta(\vec{r} - \vec{r}_0)]_q(r) e^{iq\theta}.
 \end{aligned}$$

$$\text{i.e., } \zeta(\vec{r} - \vec{r}_0) = \sum_{q=-\infty}^{\infty} [\zeta(\vec{r} - \vec{r}_0)]_q(r) e^{iq\theta} \tag{3.84}$$

$$[\zeta(\vec{r} - \vec{r}_0)]_q(r) = 2\pi \sum_{n=-\infty}^{\infty} (i)^n e^{-i(q-n)\theta_0} \int_0^\infty \rho d\rho \chi_n(\rho) J_{q-n}(2\pi\rho r_0) J_q(2\pi\rho r), \tag{3.85}$$

The last expression is the Fourier series representation, where the coefficients of the shifted function in r are represented in terms of the unshifted frequency-space coefficients $\chi_n(\rho)$.

However if the values of the shifted coefficients are desired in terms of the original unshifted coefficients $\zeta_n(r) \rightarrow \zeta_n(u)$, then Eq.(3.66) gives us an expression for $\chi_n(\rho)$.

$$\begin{aligned}
 \zeta(\vec{r} - \vec{r}_0) &= 2\pi \sum_{n=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} (i)^n e^{-i(q-n)\theta_0} e^{iq\theta} \int_0^{\infty} \rho d\rho [(-i)^n \mathcal{H}_n\{\zeta_n(u)\}] J_{q-n}(2\pi\rho r_0) J_q(2\pi\rho r) \\
 &= (2\pi)^2 \sum_{n=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} e^{-i(q-n)\theta_0} e^{iq\theta} \int_0^{\infty} \rho d\rho \int_0^{\infty} u du \zeta_n(u) J_n(2\pi\rho u) J_{q-n}(2\pi\rho r_0) J_q(2\pi\rho r) \\
 &= (2\pi)^2 \sum_{n=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} e^{-i(q-n)\theta_0} e^{iq\theta} \int_0^{\infty} u du \zeta_n(u) S_n^q(u, r_0, r) \tag{3.86}
 \end{aligned}$$

where $S_n^q(u, r_0, r)$ is defined as a shift-type operator given by the integral of the triple-Bessel product

$$S_n^q(u, r_0, r) = \int_0^{\infty} \rho d\rho J_n(2\pi\rho u) J_{q-n}(2\pi\rho r_0) J_q(2\pi\rho r) \tag{3.87}$$

3.6.1 Fourier transform and series representation for a shifted function with radial symmetry

Although the function has radial symmetry, the shift to imply the linear translation in r and angular rotation in θ . Hence $\zeta(\vec{r} - \vec{r}_0)$ indicates that the radially symmetric $\zeta(r)$ is now centered at $\vec{r}_0$. Then the expression (3.80) is valid and takes the form

$$\mathcal{F}\{\zeta(\vec{r} - \vec{r}_0)\} = e^{-i2\pi\rho r_0 \cos(\theta_0 - \varphi)} \mathcal{F}\{\zeta(r)\} = e^{-i2\pi\rho r_0 \cos(\theta_0 - \varphi)} \tilde{\zeta}(\rho), \tag{3.88}$$

substituting directly or using Eq.(3.82) with $n = 0$ the following expression is reached

$$\mathcal{F}\{\zeta(\vec{r} - \vec{r}_0)\} = \sum_{m=-\infty}^{\infty} (-i)^m e^{i(m)\varphi} e^{-im\theta_0} J_m(2\pi\rho r_0) \mathcal{H}_0\{\zeta(r)\}. \tag{3.89}$$

To find an expression for a shifted function with radial symmetry it's necessary to apply the inverse Fourier transform to the previous result

$$\begin{aligned}
\zeta(\vec{r} - \vec{r}_0) &= \mathcal{F}^{-1} \left\{ \sum_{m=-\infty}^{\infty} (-i)^m e^{i(m)\varphi} e^{-im\theta_0} J_m(2\pi\rho r_0) \mathcal{H}_0\{\zeta(r)\} \right\} \\
&= \int_0^\infty \int_0^{2\pi} \rho d\rho d\varphi \sum_{m=-\infty}^{\infty} (-i)^m e^{i(m)\varphi} e^{-im\theta_0} J_m(2\pi\rho r_0) \mathcal{H}_0\{\zeta(r)\} \\
&\quad \times \sum_{q=-\infty}^{\infty} (i)^q e^{iq\theta} e^{-iq\varphi} J_q(2\pi\rho r) \\
&= \sum_{m=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} (-i)^{m-q} e^{-im\theta_0} e^{iq\theta} \int_0^\infty \rho d\rho J_m(2\pi\rho r_0) J_q(2\pi\rho r) \mathcal{H}_0\{\zeta(r)\} \\
&\quad \times \int_0^{2\pi} d\varphi e^{i(m-q)\varphi}.
\end{aligned} \tag{3.90}$$

The integral over φ is nonzero only if $m - q = 0 \rightarrow m = q$, also making the change $\zeta_n(r) \rightarrow \zeta_n(u)$ in the Hankel transform, as it was done in the previous treatment to Eq.(3.86) in order to avoid confusion, we have

$$\begin{aligned}
\zeta(\vec{r} - \vec{r}_0) &= 2\pi \sum_{q=-\infty}^{\infty} e^{-iq\theta_0} e^{iq\theta} \int_0^\infty \rho d\rho J_q(2\pi\rho r_0) J_q(2\pi\rho r) \mathcal{H}_0\{\zeta(u)\} \\
&= (2\pi)^2 \sum_{q=-\infty}^{\infty} e^{-iq\theta_0} e^{iq\theta} \int_0^\infty \rho d\rho J_q(2\pi\rho r_0) J_q(2\pi\rho r) \int_0^\infty u du \zeta(u) J_0(2\pi\rho u) \\
&= (2\pi)^2 \sum_{q=-\infty}^{\infty} e^{-iq\theta_0} e^{iq\theta} \int_0^\infty \rho d\rho \int_0^\infty u du \zeta(u) J_0(2\pi\rho u) J_q(2\pi\rho r_0) J_q(2\pi\rho r) \\
&= (2\pi)^2 \sum_{q=-\infty}^{\infty} e^{-iq\theta_0} e^{iq\theta} \int_0^\infty u du \zeta(u) S_0^q(u, r_0, r),
\end{aligned} \tag{3.91}$$

with

$$S_0^q(u, r_0, r) = \int_0^\infty \rho d\rho J_0(2\pi\rho u) J_q(2\pi\rho r_0) J_q(2\pi\rho r). \tag{3.92}$$

It's clear that the same result can be obtained making $n = 0$ in Eq.(3.86). In the same way, the inverse Fourier transform and series representation for a shifted function for the

two previous cases can be found, we now that

$$\begin{aligned}\mathcal{F}^{-1}\{\tilde{\zeta}(\mu - \mu_0, \nu - \nu_0)\} &= e^{-i2\pi(\mu_0 x + \nu_0 y)} \mathcal{F}^{-1}\{\tilde{\zeta}(\mu, \nu)\} \\ &= e^{-i2\pi(\mu_0 x + \nu_0 y)} \zeta(x, y)\end{aligned}\tag{3.93}$$

then for a function with radial symmetry we have

$$\mathcal{F}^{-1}\{\tilde{\zeta}(\rho - \vec{\rho}_0)\} = e^{-i2\pi\rho_0 r \cos(\theta - \varphi_0)} \zeta(r)$$

$$i.e., \mathcal{F}^{-1}\{\tilde{\zeta}(\rho - \vec{\rho}_0)\} = \sum_{m=-\infty}^{\infty} (i)^m e^{im\theta} e^{-im\varphi_0} J_m(2\pi\rho_0 r) \zeta(r)\tag{3.94}$$

$$\begin{aligned}\tilde{\zeta}(\rho - \vec{\rho}_0) &= \mathcal{F}\left\{\sum_{m=-\infty}^{\infty} (i)^m e^{im\theta} e^{-im\varphi_0} J_m(2\pi\rho_0 r) \zeta(r)\right\} \\ &= \int_0^{\infty} \int_0^{2\pi} r dr d\theta \left[\sum_{m=-\infty}^{\infty} (i)^m e^{im\theta} e^{-im\varphi_0} J_m(2\pi\rho_0 r) \zeta(r) \right] \\ &\quad \times \sum_{q=-\infty}^{\infty} (-i)^q e^{iq\varphi} e^{-iq\theta} J_q(2\pi\rho r) \\ &= \sum_{m=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} (i)^m (-i)^q e^{-im\varphi_0} e^{iq\varphi} \int_0^{\infty} r dr J_m(2\pi\rho_0 r) \zeta(r) J_q(2\pi\rho r) \\ &\quad \times \int_0^{2\pi} d\theta e^{i(m-q)\theta} \\ &= 2\pi \sum_{q=-\infty}^{\infty} e^{-iq\varphi_0} e^{iq\varphi} \int_0^{\infty} r dr J_q(2\pi\rho_0 r) \zeta(r) J_q(2\pi\rho r)\end{aligned}\tag{3.95}$$

it can be expressed as

$$\tilde{\zeta}(\rho - \vec{\rho}_0) = 2\pi \int_0^\infty r dr \zeta(r) J_0 \left(2\pi r \sqrt{\rho_0^2 + \rho^2 - 2\rho\rho_0 \cos(\varphi - \varphi_0)} \right) \quad (3.96)$$

3.7 Fourier transform of a product

We consider the product of two functions $\zeta(r, \theta) = f(r, \theta)g(r, \theta)$, where $f(r, \theta)$ and $g(r, \theta)$ can be expressed in Fourier Series (Eq.3.60), then for expressed $\zeta(r, \theta)$ in Fourier series we need to find a relation between the coefficients by using the Fourier transform. Let

$$\begin{aligned} \zeta(r, \theta) &= \sum_{m=-\infty}^{\infty} \zeta_m(r) e^{im\theta} \\ f(r, \theta) &= \sum_{n=-\infty}^{\infty} f_n(r) e^{in\theta} \\ g(r, \theta) &= \sum_{q=-\infty}^{\infty} g_q(r) e^{iq\theta} \end{aligned}$$

where the all coefficients are given by Eq.(3.61), then

$$\begin{aligned} \tilde{\zeta}(\rho, \varphi) &= \mathcal{F}\{f(r, \theta)g(r, \theta)\} \\ &= \int_0^\infty \int_0^{2\pi} r dr d\theta \left[\sum_{n=-\infty}^{\infty} f_n(r) e^{in\theta} \sum_{q=-\infty}^{\infty} g_q(r) e^{iq\theta} \right] e^{-i2\pi\rho r \cos(\theta - \varphi)} \\ &= \int_0^\infty \int_0^{2\pi} r dr d\theta \left[\sum_{n=-\infty}^{\infty} f_n(r) e^{in\theta} \sum_{q=-\infty}^{\infty} g_q(r) e^{iq\theta} \right] \sum_{m=-\infty}^{\infty} (-i)^m e^{im\varphi} e^{-im\theta} J_m(2\pi\rho r) \\ &= \sum_{n=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (-i)^m e^{im\varphi} \int_0^\infty r dr f_n(r) g_q(r) J_m(2\pi\rho r) \int_0^{2\pi} d\theta e^{in\theta} e^{iq\theta} e^{-im\theta} \end{aligned}$$

where as in the section 3.6, the last integral is nonzero only if $n + q - m = 0$ which leads to $n = m - q$ and so

$$\begin{aligned}\tilde{\zeta}(\rho, \varphi) &= \sum_{q=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} 2\pi(-i)^m e^{im\varphi} \int_0^{\infty} r dr f_{m-q}(r) g_q(r) J_m(2\pi\rho r) \\ &= \sum_{m=-\infty}^{\infty} \chi_m(\rho) e^{im\varphi}\end{aligned}\quad (3.97)$$

$$\Leftrightarrow \chi_m(\rho) = \sum_{q=-\infty}^{\infty} 2\pi(-i)^m \int_0^{\infty} r dr f_{m-q}(r) g_q(r) J_m(2\pi\rho r). \quad (3.98)$$

However, we found (Eq.3.66) that for a general function, the Fourier coefficients in the frequency space $\chi_m(\rho)$ are given by

$$\begin{aligned}\chi_m(\rho) &= (-i)^m \mathcal{H}_m\{\zeta_m(r)\} \\ &= 2\pi(-i)^m \int_0^{\infty} r dr \zeta_m(r) J_m(2\pi\rho r),\end{aligned}$$

comparing this expression with Eq.(3.98) leads to

$$\zeta_m(r) = \sum_{q=-\infty}^{\infty} f_{m-q}(r) g_q(r) \quad (3.99)$$

which is the convolution of the Fourier series of $f(r, \theta)$ and $g(r, \theta)$. In other words, the coefficients of the two series product is the convolution of the coefficients, so

$$\zeta_m(r) = (fg)_m(r) = (f_m * g_m)(r) = \sum_{q=-\infty}^{\infty} f_{m-q}(r) g_q(r). \quad (3.100)$$

Note that $(fg)_m(r)$ doesn't represent the product of coefficients, it symbolizes the coefficients of the product.

3.8 Fourier transform of a product of a radially symmetric function with a function without radial symmetry

We consider the product of two functions $\zeta(r, \theta) = f(r, \theta)g(r)$, where only $f(r, \theta)$ can be expressed in Fourier Series (Eq.3.60), then Let

$$f(r, \theta) = \sum_{n=-\infty}^{\infty} f_n(r)e^{in\theta}$$

and

$$\begin{aligned}\tilde{\zeta}(\rho, \varphi) &= \mathcal{F}\{f(r, \theta)g(r)\} \\ &= \int_0^{\infty} \int_0^{2\pi} r dr d\theta \left[\sum_{n=-\infty}^{\infty} f_n(r)e^{in\theta} g(r) \right] e^{-i2\pi\rho r \cos(\theta-\varphi)} \\ &= \int_0^{\infty} \int_0^{2\pi} r dr d\theta \left[\sum_{n=-\infty}^{\infty} f_n(r)e^{in\theta} g(r) \right] \sum_{m=-\infty}^{\infty} (-i)^m e^{im\varphi} e^{-im\theta} J_m(2\pi\rho r) \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (-i)^m e^{im\varphi} \int_0^{\infty} r dr f_n(r) g(r) J_m(2\pi\rho r) \int_0^{2\pi} d\theta e^{in\theta} e^{-im\theta}\end{aligned}$$

where as in the section 3.6, the last integral is nonzero only if $n - m = 0$ which leads to $n = m$ and so

$$\begin{aligned}\tilde{\zeta}(\rho, \varphi) &= \sum_{m=-\infty}^{\infty} 2\pi(-i)^m e^{im\varphi} \int_0^{\infty} r dr f_m(r) g(r) J_m(2\pi\rho r) \\ &= \sum_{m=-\infty}^{\infty} \chi_m(\rho) e^{im\varphi}\end{aligned}\tag{3.101}$$

$$\Leftrightarrow \chi_m(\rho) = 2\pi(-i)^m \int_0^{\infty} r dr f_m(r) g(r) J_m(2\pi\rho r).\tag{3.102}$$

However, we found (Eq.3.66) that for a general function, the Fourier coefficients in the frequency space $\chi_m(\rho)$ are given by

$$\begin{aligned}\chi_m(\rho) &= (-i)^m \mathcal{H}_m\{\zeta_m(r)\} \\ &= 2\pi(-i)^m \int_0^{\infty} r dr \zeta_m(r) J_m(2\pi\rho r),\end{aligned}$$

comparing this expression with Eq.(3.98) leads to

$$\zeta_m(r) = f_m(r)g(r). \quad (3.103)$$

3.8.1 In terms of the Fourier transforms

Eqs.(3.57) and (3.72) allows to express

$$g(r) = \int_0^\infty u du G(u) 2\pi J_0(2\pi ur)$$

$$f_m(r) = 2\pi(i)^m \int_0^\infty v dv F_m(v) J_m(2\pi vr)$$

then

$$\begin{aligned} \tilde{\zeta}(\rho, \varphi) &= \sum_{m=-\infty}^{\infty} 2\pi(-i)^m e^{im\varphi} \int_0^\infty r dr 2\pi(i)^m \\ &\quad \times \int_0^\infty v dv F_m(v) J_m(2\pi vr) \int_0^\infty u du G(u) 2\pi J_0(2\pi ur) J_m(2\pi \rho r) \\ &= \sum_{m=-\infty}^{\infty} (2\pi)^3 e^{im\varphi} \int_0^\infty r dr \int_0^\infty v dv F_m(v) J_m(2\pi vr) \\ &\quad \times \int_0^\infty u du G(u) J_0(2\pi ur) J_m(2\pi \rho r) \\ &= \sum_{m=-\infty}^{\infty} (2\pi)^3 e^{im\varphi} \int_0^\infty v dv F_m(v) \int_0^\infty u du G(u) \\ &\quad \times \int_0^\infty r dr J_0(2\pi ur) J_m(2\pi vr) J_m(2\pi \rho r) \\ &= \sum_{m=-\infty}^{\infty} (2\pi)^3 e^{im\varphi} \int_0^\infty v dv F_m(v) \int_0^\infty u du G(u) S_0^m(u, v, \rho) \\ &= \sum_{m=-\infty}^{\infty} (2\pi)^3 e^{im\varphi} \int_0^\infty v dv F_m(v) S_0^m(u, v, \rho) \int_0^\infty u du G(u) \end{aligned} \quad (3.104)$$

3.9 Fourier transform of a product of two radially symmetric functions

We consider the product of two functions $\zeta(r) = f(r)g(r)$, then

$$\begin{aligned}
 \tilde{\zeta}(\rho) &= \mathcal{F}\{f(r)g(r)\} \\
 &= 2\pi \int_0^\infty r dr f(r)g(r) J_0(2\pi\rho r)
 \end{aligned} \tag{3.105}$$

3.9.1 In terms of the Fourier transforms

Eq.(3.57) allow to express

$$g(r) = \int_0^\infty u du G(u) 2\pi J_0(2\pi ur)$$

$$f(r) = \int_0^\infty v dv F(v) 2\pi J_0(2\pi vr)$$

then

$$\begin{aligned}
 \tilde{\zeta}(\rho) &= 2\pi \int_0^\infty r dr \int_0^\infty v dv F(v) 2\pi J_0(2\pi vr) \int_0^\infty u du G(u) 2\pi J_0(2\pi ur) J_0(2\pi\rho r) \\
 &= (2\pi)^3 \int_0^\infty v dv F(v) \int_0^\infty u du G(u) \int_0^\infty r dr J_0(2\pi vr) J_0(2\pi ur) J_0(2\pi\rho r) \\
 &= (2\pi)^3 \int_0^\infty v dv F(v) \int_0^\infty u du G(u) S_0^0(u, v, \rho)
 \end{aligned} \tag{3.106}$$

3.10 Convolution

In two dimensions, the convolution of two functions is defined as

$$\zeta(r, \theta) = \zeta(\vec{r}) = f(\vec{r}) * g(\vec{r}) = \iint_{\vec{r}_0} d\vec{r}_0 f(\vec{r}_0) g(\vec{r} - \vec{r}_0), \tag{3.107}$$

so we can find an expression for the Fourier transform of a convolution with the help of the Fourier expansions for f and the shifted version of g given by Eq.(3.86), so

$$\begin{aligned}
 \zeta(\vec{r}) &= \int_0^\infty \int_0^{2\pi} r_0 dr_0 d\theta_0 \left[\sum_{m=-\infty}^\infty f_m(r_0) e^{im\theta_0} \right] \\
 &\times \left[(2\pi)^2 \sum_{n=-\infty}^\infty \sum_{q=-\infty}^\infty e^{-i(q-n)\theta_0} e^{iq\theta} \int_0^\infty u du g_n(u) S_n^q(u, r_0, r) \right]
 \end{aligned}$$

$$\begin{aligned}
 &= (2\pi)^2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \int_0^{\infty} r_0 dr_0 f_m(r_0) \int_0^{2\pi} d\theta_0 e^{im\theta_0} e^{-i(q-n)\theta_0} \\
 &\quad \times e^{iq\theta} \int_0^{\infty} u du g_n(u) S_n^q(u, r_0, r),
 \end{aligned}$$

it can be seen that the second integral is nonzero only if $m + n - q = 0$; i.e., $m = q - n$

$$\begin{aligned}
 \Rightarrow \zeta(\vec{r}) &= (2\pi)^3 \sum_{n=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \int_0^{\infty} r_0 dr_0 f_{q-n}(r_0) e^{iq\theta} \int_0^{\infty} u du g_n(u) S_n^q(u, r_0, r) \\
 &= (2\pi)^3 \sum_{n=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \int_0^{\infty} r_0 dr_0 f_{q-n}(r_0) e^{iq\theta} \int_0^{\infty} u du g_n(u) \\
 &\quad \times \int_0^{\infty} \rho d\rho J_n(2\pi\rho u) J_{q-n}(2\pi\rho r_0) J_q(2\pi\rho r) \\
 &= 2\pi \sum_{q=-\infty}^{\infty} e^{iq\theta} \int_0^{\infty} \rho d\rho J_q(2\pi\rho r) \\
 &\quad \times \sum_{n=-\infty}^{\infty} 2\pi \int_0^{\infty} r_0 dr_0 f_{q-n}(r_0) J_{q-n}(2\pi\rho r_0) 2\pi \int_0^{\infty} u du g_n(u) J_n(2\pi\rho u),
 \end{aligned}$$

the second and third integral have a similar form as in Eq.(3.66); i.e, for

$$\tilde{F}(\rho, \varphi) = \sum_{q-n=-\infty}^{\infty} F_{q-n}(\rho) e^{i(q-n)\varphi} \quad \wedge \quad \tilde{G}(\rho, \varphi) = \sum_{n=-\infty}^{\infty} G_n(\rho) e^{in\varphi} \quad (3.108)$$

we have

$$F_{q-n}(\rho) = 2\pi(-i)^{q-n} \int_0^{\infty} r_0 dr_0 f_{q-n}(r_0) J_{q-n}(2\pi\rho r_0) \quad (3.109)$$

$$G_n(\rho) = 2\pi(-i)^n \int_0^{\infty} u du g_n(u) J_n(2\pi\rho u) \quad (3.110)$$

and

$$\begin{aligned}
 \zeta(\vec{r}) &= 2\pi \sum_{q=-\infty}^{\infty} e^{iq\theta} \int_0^{\infty} \rho d\rho J_q(2\pi\rho r) \sum_{n=-\infty}^{\infty} i^{q-n} F_{q-n}(\rho) i^n G_n(\rho) \\
 &= 2\pi \sum_{q=-\infty}^{\infty} i^q e^{iq\theta} \int_0^{\infty} \rho d\rho \left[\sum_{n=-\infty}^{\infty} F_{q-n}(\rho) G_n(\rho) \right] J_q(2\pi\rho r) \\
 &= \sum_{q=-\infty}^{\infty} \zeta_q(r) e^{iq\theta}
 \end{aligned} \quad (3.111)$$

$$\Leftrightarrow \zeta_q(r) = 2\pi i^q \int_0^\infty \rho d\rho \left[\sum_{n=-\infty}^{\infty} F_{q-n}(\rho) G_n(\rho) \right] J_q(2\pi\rho r), \quad (3.112)$$

Similarly in (Eq.3.72) we find that for a general function, the Fourier coefficients in the object space $\zeta_q(r)$ are given by

$$\zeta_q(r) = 2\pi(i)^q \int_0^\infty \rho d\rho \chi_q(\rho) J_q(2\pi\rho r)$$

comparing this expression with Eq.(3.112) leads to

$$\chi_q(\rho) = \sum_{n=-\infty}^{\infty} F_{q-n}(\rho) G_n(\rho). \quad (3.113)$$

Also, in Eq.(3.100) we found that the convolution of the coefficients is equal to the coefficients of the two series product; i.e.,

$$\chi_q(\rho) = \sum_{n=-\infty}^{\infty} F_{q-n}(\rho) G_n(\rho) = (F_q * G_q)(\rho) = (FG)_q(\rho). \quad (3.114)$$

The comparison made with $\chi_q(\rho)$, means that by obtaining the Fourier transform of a convolution and express it in series, the coefficient of the convolution of two functions is equal to the multiplication of coefficients, this provides

$$\tilde{\zeta}(\rho, \varphi) = \mathcal{F}\{\zeta(\vec{r})\} = \mathcal{F}\{f(\vec{r}) * g(\vec{r})\} = F(\rho, \varphi) G(\rho, \varphi). \quad (3.115)$$

Which is consistent with the result in cartesian coordinates i.e., the Fourier transform of the two functions convolution is the multiplication of their transforms. This equation defines the convolution operation; therefore, to find the convolution of two functions given in Fourier-series form, one must first find the qth coefficient of the Fourier transform of each function $F_q(\rho), G_q(\rho)$ and then convolve the resulting series follow Eq.(3.114) to get $\chi_q(\rho)$. Finally inverse Hankel transform must be applied to obtain $\zeta_q(r)$.

3.11 Convolution of a radially symmetric function with a function without radial symmetry

For the development of this particular case, Eq.(3.107) is considered with $g(\vec{r}) = g(r)$, the radially symmetric function, then

$$\zeta(r, \theta) = \zeta(\vec{r}) = f(\vec{r}) * g(\vec{r}) = f(\vec{r}) * g(r) = \iint_{\vec{r}_0} d\vec{r}_0 f(\vec{r}_0) g(\vec{r} - \vec{r}_0), \quad (3.116)$$

due to the convolution definition, the Fourier expansion for $f(\vec{r}_0)$ and the shifted version of g for a function with radial symmetry given by Eq.(3.91) are necessary

$$\begin{aligned} \zeta(\vec{r}) &= \int_0^\infty \int_0^{2\pi} r_0 dr_0 d\theta_0 \left[\sum_{m=-\infty}^\infty f_m(r_0) e^{im\theta_0} \right] \\ &\quad \times \left[(2\pi)^2 \sum_{q=-\infty}^\infty e^{-iq\theta_0} e^{iq\theta} \int_0^\infty u du g(u) S_0^q(u, r_0, r) \right] \\ &= \int_0^\infty \int_0^{2\pi} r_0 dr_0 d\theta_0 \sum_{m=-\infty}^\infty f_m(r_0) e^{im\theta_0} \\ &\quad \times (2\pi)^2 \sum_{q=-\infty}^\infty e^{-iq\theta_0} e^{iq\theta} \int_0^\infty \rho d\rho \int_0^\infty u du g(u) J_0(2\pi\rho u) J_q(2\pi\rho r_0) J_q(2\pi\rho r) \\ &= (2\pi)^2 \sum_{m=-\infty}^\infty \sum_{q=-\infty}^\infty e^{iq\theta} \int_0^\infty r_0 dr_0 f_m(r_0) \int_0^{2\pi} d\theta_0 e^{-i(q-m)\theta_0} \\ &\quad \times \int_0^\infty \rho d\rho \int_0^\infty u du g(u) J_0(2\pi\rho u) J_q(2\pi\rho r_0) J_q(2\pi\rho r), \end{aligned}$$

the second integral is nonzero only if $q - m = 0$; i.e., $q = m$, then the convolution is given by

$$\zeta(\vec{r}) = (2\pi)^3 \sum_{q=-\infty}^\infty e^{iq\theta} \int_0^\infty r_0 dr_0 f_q(r_0) \int_0^\infty \rho d\rho \int_0^\infty u du g(u) J_0(2\pi\rho u) J_q(2\pi\rho r_0) J_q(2\pi\rho r). \quad (3.117)$$

By rearranging the terms, a relationship can be found to express the Fourier transform of the convolution

$$\begin{aligned} \Rightarrow \zeta(\vec{r}) &= 2\pi \sum_{q=-\infty}^{\infty} e^{iq\theta} \int_0^{\infty} \rho d\rho J_q(2\pi\rho r) \\ &\quad \times 2\pi \int_0^{\infty} r_0 dr_0 f_q(r_0) J_q(2\pi\rho r_0) 2\pi \int_0^{\infty} u du g(u) J_0(2\pi\rho u). \end{aligned} \quad (3.118)$$

However, for

$$\tilde{F}(\rho, \varphi) = \sum_{q=-\infty}^{\infty} F_q(\rho) e^{iq\varphi} \quad (3.119)$$

we have

$$F_q(\rho) = 2\pi(-i)^q \int_0^{\infty} r_0 dr_0 f_q(r_0) J_q(2\pi\rho r_0), \quad (3.120)$$

also Eq.(3.56) indicates

$$\tilde{G}(\rho) = 2\pi \int_0^{\infty} u g(u) du J_0(2\pi\rho u) \quad (3.121)$$

then

$$\zeta(\vec{r}) = 2\pi \sum_{q=-\infty}^{\infty} i^q e^{iq\theta} \int_0^{\infty} \rho d\rho F_q(\rho) \tilde{G}(\rho) J_q(2\pi\rho r) \quad (3.122)$$

but by definition

$$\begin{aligned} \zeta(\vec{r}) &= \sum_{q=-\infty}^{\infty} 2\pi(i)^q e^{iq\theta} \int_0^{\infty} \rho d\rho \chi_q(\rho) J_q(2\pi\rho r) \\ &\Leftrightarrow \chi_q(\rho) = F_q(\rho) \tilde{G}(\rho). \end{aligned} \quad (3.123)$$

Then the convolution Fourier transform of two series when one of them has radial symmetry (and thus consists of only the first term) becomes a simple matter of multiplication. Also this result could be found considering $n = 0$ in Eq.(3.111) since $G(\rho)$ is not expressed in

series. It's clear that by considering only the $m, n \vee q = 0$ term in the general equations, the particular case of radial symmetry can be reached; however it was necessary to make a more detailed demonstration.

3.12 Convolution of two radially symmetric functions

Let $g(\vec{r}) = g(r)$ and $f(\vec{r}) = f(r)$ be radially symmetric functions, then

$$\begin{aligned}\zeta(\vec{r}) &= f(r) * * g(r) = \iint_{\vec{r}_0} d\vec{r}_0 f(\vec{r}_0) g(\vec{r} - \vec{r}_0) = \iint_{\vec{r}_0} d\vec{r}_0 f(r_0) g(r - r_0), \quad (3.124) \\ \Rightarrow \zeta(\vec{r}) &= \int_0^\infty \int_0^{2\pi} r_0 dr_0 d\theta_0 f(r_0) \times \left[(2\pi)^2 \sum_{q=-\infty}^\infty e^{-iq\theta_0} e^{iq\theta} \int_0^\infty u du g(u) S_0^q(u, r_0, r) \right] \\ &= \int_0^\infty \int_0^{2\pi} r_0 dr_0 d\theta_0 f(r_0) \\ &\quad \times (2\pi)^2 \sum_{q=-\infty}^\infty e^{-iq\theta_0} e^{iq\theta} \int_0^\infty \rho d\rho \int_0^\infty u du g(u) J_0(2\pi\rho u) J_q(2\pi\rho r_0) J_q(2\pi\rho r) \\ &= (2\pi)^2 \sum_{q=-\infty}^\infty e^{iq\theta} \int_0^\infty r_0 dr_0 f(r_0) \int_0^{2\pi} d\theta_0 e^{-iq\theta_0} \\ &\quad \times \int_0^\infty \rho d\rho \int_0^\infty u du g(u) J_0(2\pi\rho u) J_q(2\pi\rho r_0) J_q(2\pi\rho r),\end{aligned}$$

the second integral is nonzero only if $q = 0$, then the convolution for two radially symmetric functions is given by

$$\zeta(\vec{r}) = (2\pi)^3 \int_0^\infty r_0 dr_0 f(r_0) \int_0^\infty \rho d\rho \int_0^\infty u du g(u) J_0(2\pi\rho u) J_0(2\pi\rho r_0) J_0(2\pi\rho r). \quad (3.125)$$

Also

$$\tilde{G}(\rho) = 2\pi \int_0^\infty u g(u) du J_0(2\pi\rho u) \quad (3.126)$$

$$\tilde{F}(\rho) = 2\pi \int_0^\infty r_0 dr_0 f(r_0) J_0(2\pi \rho r_0) \quad (3.127)$$

$$\Rightarrow \zeta(\vec{r}) = 2\pi \int_0^\infty \tilde{F}(\rho) \tilde{G}(\rho) J_0(2\pi \rho r)$$

and comparing with Eq.(3.57)

$$\zeta(\vec{r}) = \int_0^\infty \rho d\rho \tilde{\zeta}(\rho) 2\pi J_0(2\pi \rho r)$$

the following result can be set

$$\tilde{\zeta}(\rho) = \tilde{F}(\rho) \tilde{G}(\rho) \quad (3.128)$$

Chapter 4

Problem Statement

By using the Young interferometer of n -slits distributed on a contour, when n is large, the resulting interference pattern is similar to the diffraction pattern provided by a continuous aperture (that follows the contour shape) [2]. Following this qualitative fact, it is desired to find the numerical value for n in order to form zero-order and first-order Bessel beams. Below are two proposals to find these desired values.

The first proposal consists of circular slits distributed in a way that they follow the desired contour shape. Initially circular slits are established because in real life point slits can't be made; however, when the radius tends to zero, the ideal case must be reached.

Given the possible complexity of the first method, the second proposal is to use the sampling theorem, where the object to be sampled would be the desired contour, this method has proven to be effective in understanding the result more simply at least in cartesian coordinates as it is presented below.

4.1 Sampling theorem in cartesian coordinates

In cartesian coordinates, by acquiring discretized information of an object $\zeta(x, y)$ (sampling), a set of values $\zeta_s(x, y)$ is obtained, if this sampling is done in equal intervals for x and y then

$$\begin{aligned}\zeta_s(x, y) &= \zeta(x, y) \cdot \text{Comb}(x, y) \\ &= \zeta(x, y) \cdot \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(x - jx_p, y - ky_p).\end{aligned}\tag{4.1}$$

using the fact that the transform of a multiplication is the convolution of the transforms, the Fourier transform of the sampling is given by

$$\begin{aligned}
 \mathcal{F}\{\zeta_s(x, y)\} &= \tilde{\zeta}(\mu, \nu) ** \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{1}{x_p} \frac{1}{y_p} \delta\left(\mu - \frac{j}{x_p}, \nu - \frac{k}{y_p}\right) \\
 &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{1}{x_p} \frac{1}{y_p} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\xi d\eta \tilde{\zeta}(\xi, \eta) \delta\left(\mu - \frac{j}{x_p} - \xi, \nu - \frac{k}{y_p} - \eta\right) \\
 &= \frac{1}{x_p} \frac{1}{y_p} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \tilde{\zeta}\left(\mu - \frac{j}{x_p}, \nu - \frac{k}{y_p}\right)
 \end{aligned} \tag{4.2}$$

which indicates that the $\tilde{\zeta}(\mu, \nu)$ spectrum of the unsampled object is repeated periodically in $\mu = \frac{j}{x_p}$ and $\nu = \frac{k}{y_p}$. Then the values of x_p and y_p to get the spectrum sufficiently separated can be found.

For example let's consider a sampling for the contour of a square shown in Fig. 4.1 using two different Comb functions, one with periods $x_p = y_p$ (Fig. 4.2 (a)) and another with smaller periods $x_p = y_p$ (Fig. 4.3 a)).

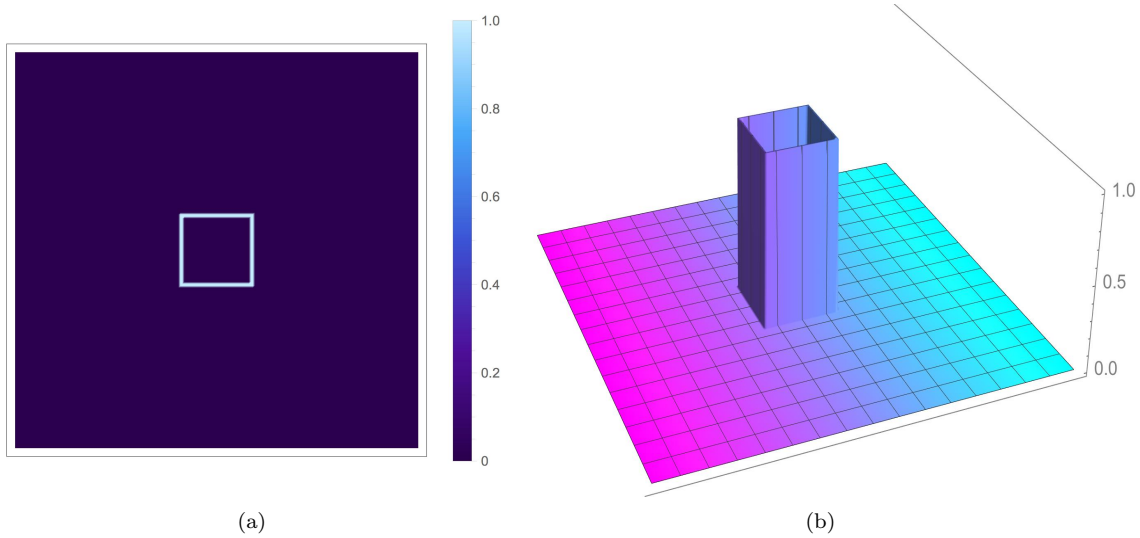


Figure 4.1: Contour of a square in (a) 2D and (b) 3D.

Since $x_p = y_p$ are large, it can be seen that for the first sampling, the spectrum of the square contour is repeated with a smaller period and therefore they are mixed.

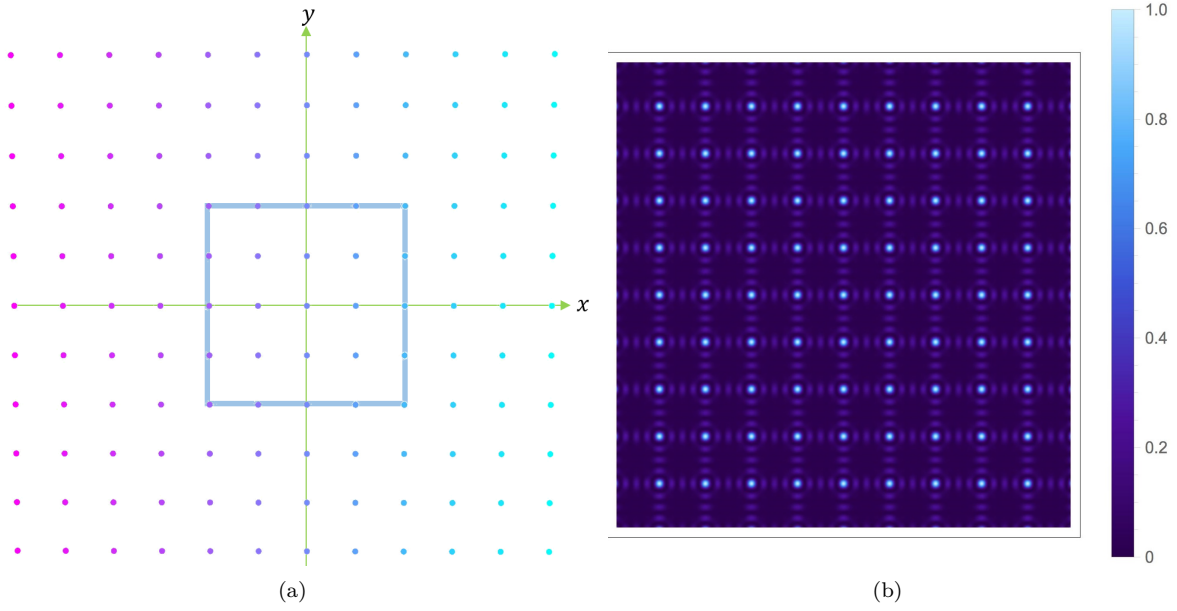


Figure 4.2: (a) Comb function with periods $x_p = y_p$, (b) spectrum of the square contour replicated with periods $\frac{1}{x_p} = \frac{1}{y_p}$.

Instead, for the second sampling, as $x_p = y_p$ are smaller, the spectrum replicas are farther apart.

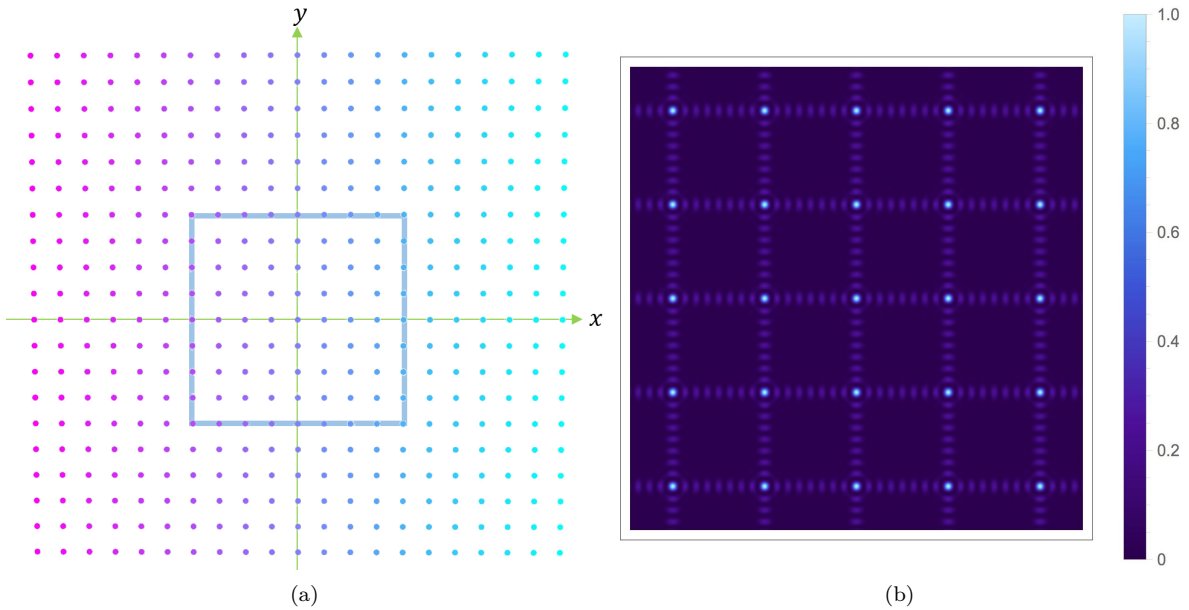


Figure 4.3: (a) Comb function with periods $x_p = y_p$, (b) spectrum of the square contour replicated with periods $\frac{1}{x_p} = \frac{1}{y_p}$.

Then using very small periods, the replicated spectra can be obtained far enough to distin-

guish only one.

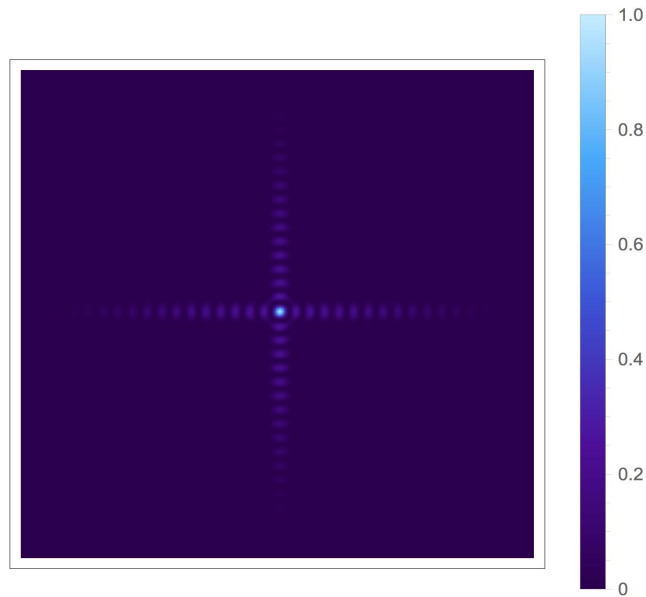


Figure 4.4: Spectrum of the square contour. On the color scale 0 represents the minimum value of intensity and 1 represents the maximum value.

This treatment is versatile because the transformation of the pulse train is another train, this allows the convolution with any function gives as a result such replicated function. Following this idea, a process analogous for polar coordinates can be done. However the Comb function can't be used, we need a train that can sample circumferences.

4.2 Sampling in polar coordinates

A sampling in polar coordinates can be defined as

$$\zeta_s(r, \theta) = \zeta(r) \cdot t_p(r, \theta). \quad (4.3)$$

where $\zeta(r)$ is a function with radial symmetry and $t_p(r, \theta)$ is an impulse train in polar coordinates.

4.2.1 Dirac impulse train in polar coordinates

To sample circumferences, a delta train that follows the form shown in Fig. 4.5 is needed

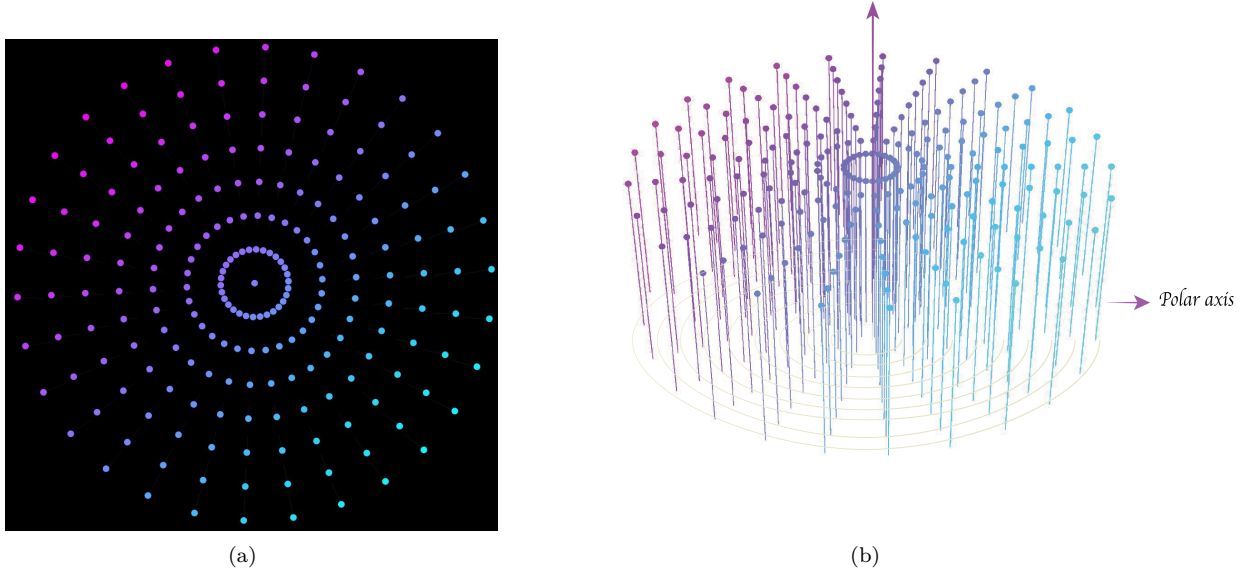


Figure 4.5: Dirac impulse train in polar coordinates in (a) 2D and (b) in 3D.

An impulse train that follows the form showed in Fig. 4.5 can be written as

$$t(x, y) = \delta(x, y) + \sum_{j=1}^{\infty} \sum_{k=0}^{n-1} \delta(x - jr_p \cos(k\theta_p), y - jr_p \sin(k\theta_p)), \quad (4.4)$$

where θ_p is the angular period with n the desired number of slits and r_p the radial period. However, if we want to find the Fourier transform of this train, the treatment that is done to the $\text{Comb}(x, y)$ function can't be applied due to that the function is not separable. To avoid this, a notation in polar coordinates can be expressed as

$$t_p(r, \theta) = \frac{1}{2\pi r} \delta(r) + \sum_{j=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{r} \delta(r - jr_p, \theta - k\theta_p). \quad (4.5)$$

Eq.(4.5) will be called polar train (t_p). One way to find the Fourier transform of the impulse train is to directly apply the definition

$$\begin{aligned}
\tilde{t}_p(\rho, \varphi) &= \mathcal{F} \left\{ \frac{1}{2\pi r} \delta(r) + \sum_{j=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{r} \delta(r - jr_p) \delta(\theta - k\theta_p) \right\} \\
&= 1 + \sum_{j=1}^{\infty} \sum_{k=0}^{n-1} \int_0^{\infty} \int_0^{2\pi} r dr d\theta \frac{1}{r} \delta(r - jr_p) \delta(\theta - k\theta_p) e^{-i2\pi\rho r \cos(\theta - \varphi)} \\
&= 1 + \sum_{j=1}^{\infty} \sum_{k=0}^{n-1} e^{-i2\pi\rho jr_p \cos(k\theta_p - \varphi)}.
\end{aligned} \tag{4.6}$$

If the polar train (Eq 4.5) is expressed in series but only for the angular part, this is

$$t_p(r, \theta) = \frac{1}{2\pi r} \delta(r) + \sum_{m=-\infty}^{\infty} t_{pm}(r) e^{im\theta}$$

with

$$\begin{aligned}
t_{pm}(r) &= \frac{1}{2\pi} \int_0^{2\pi} d\theta \left[\sum_{j=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{r} \delta(r - jr_p) \delta(\theta - k\theta_p) \right] e^{-im\theta} \\
&= \frac{1}{2\pi} \sum_{j=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{r} \delta(r - jr_p) \int_0^{2\pi} d\theta \delta(\theta - k\theta_p) e^{-im\theta} \\
&= \frac{1}{2\pi} \sum_{j=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{r} \delta(r - jr_p) e^{-imk\theta_p}
\end{aligned} \tag{4.7}$$

$$\Rightarrow t_p(r, \theta) = \frac{1}{2\pi r} \delta(r) + \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \sum_{j=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{r} \delta(r - jr_p) e^{-imk\theta_p} e^{im\theta} \tag{4.8}$$

and

$$\Rightarrow \tilde{t}_p(\rho, \varphi) = 1 + \sum_{m=-\infty}^{\infty} \sum_{j=1}^{\infty} \sum_{k=0}^{n-1} (-i)^m e^{im\varphi} e^{-imk\theta_p} J_m(2\pi\rho jr_p). \tag{4.9}$$

In cartesian coordinates the transform of an impulse train results in another train, making use of its series representation. Following the same procedure for the polar train, both the angular and the radial part need to be developed in series, so the following definition is used.

A function $\zeta(r, \theta)$ can be expressed as a Fourier-Bessel expansion

$$\zeta(r, \theta) = \sum_{m=-\infty}^{\infty} \sum_{s=1}^{\infty} C_{\nu s} J_{\nu} \left(\lambda_{\nu s} \frac{r}{\alpha} \right) e^{im\theta}$$

with

$$C_{\nu s} = \frac{2}{\alpha^2 J_{\nu+1}^2(\lambda_{\nu s})} \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\alpha} r dr d\theta \zeta(r, \theta) J_{\nu} \left(\lambda_{\nu s} \frac{r}{\alpha} \right) e^{-im\theta}, \quad (4.10)$$

where $\lambda_{\nu s}$ are the zeros of the function J_{ν} and α is the upper limit of the interval $(0, \alpha]$ where the function can be approximated with the Fourier-Bessel series. Then, for the polar train

$$t_p(r, \theta) = \frac{1}{2\pi r} \delta(r) + \sum_{j=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{r} \delta(r - jr_p) \delta(\theta - k\theta_p) \quad (4.11)$$

we have

$$\begin{aligned} C_{\nu s} &= \frac{2}{\alpha^2 J_{\nu+1}^2(\lambda_{\nu s})} \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\alpha} r dr d\theta \frac{1}{2\pi r} \delta(r) J_{\nu} \left(\lambda_{\nu s} \frac{r}{\alpha} \right) e^{-im\theta} \\ &\quad + \frac{2}{\alpha^2 J_{\nu+1}^2(\lambda_{\nu s})} \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\alpha} r dr d\theta \sum_{k=0}^{n-1} \sum_{j=1}^{\infty} \frac{1}{r} \delta(r - jr_p) \delta(\theta - k\theta_p) J_{\nu} \left(\lambda_{\nu s} \frac{r}{\alpha} \right) e^{-im\theta} \\ &= \frac{J_{\nu}(0)}{\pi \alpha^2 J_{\nu+1}^2(\lambda_{\nu s})} + \sum_{k=0}^{n-1} e^{-imk\theta_p} \frac{2}{\alpha^2 J_{\nu+1}^2(\lambda_{\nu s})} \frac{1}{2\pi} \int_0^{\alpha} r dr \sum_{j=1}^{\infty} \frac{1}{r} \delta(r - jr_p) J_{\nu} \left(\lambda_{\nu s} \frac{r}{\alpha} \right) \\ &= \frac{J_{\nu}(0)}{\pi \alpha^2 J_{\nu+1}^2(\lambda_{\nu s})} + \sum_{k=0}^{n-1} e^{-imk\theta_p} \frac{2}{\alpha^2 J_{\nu+1}^2(\lambda_{\nu s})} \frac{1}{2\pi} \sum_{j=1}^{\infty} J_{\nu} \left(\lambda_{\nu s} \frac{jr_p}{\alpha} \right) \\ &= \frac{J_{\nu}(0)}{\pi \alpha^2 J_{\nu+1}^2(\lambda_{\nu s})} + \sum_{k=0}^{n-1} \sum_{j=1}^{\infty} e^{-imk\theta_p} \frac{1}{\pi \alpha^2 J_{\nu+1}^2(\lambda_{\nu s})} J_{\nu} \left(\lambda_{\nu s} \frac{jr_p}{\alpha} \right). \end{aligned} \quad (4.12)$$

The first part implies $m = 0$, so its series would be only for the zeros, thus,

$$\begin{aligned} t_p(r, \theta) &= \sum_{s=1}^{\infty} \frac{J_{\nu}(0)}{\pi \alpha^2 J_{\nu+1}^2(\lambda_{\nu s})} J_{\nu} \left(\lambda_{\nu s} \frac{r}{\alpha} \right) \\ &\quad + \sum_{m=-\infty}^{\infty} \sum_{s=1}^{\infty} \sum_{k=0}^{n-1} \sum_{j=1}^{\infty} e^{-imk\theta_p} \frac{1}{\pi \alpha^2 J_{\nu+1}^2(\lambda_{\nu s})} J_{\nu} \left(\lambda_{\nu s} \frac{jr_p}{\alpha} \right) J_{\nu} \left(\lambda_{\nu s} \frac{r}{\alpha} \right) e^{im\theta}. \end{aligned} \quad (4.13)$$

Here $jr_p \leq \alpha$ for some j and the first term is canceled if $\nu \neq 0$. If $\nu = 0$, the polar train is represented by

$$t_p(r, \theta) = \sum_{s=1}^{\infty} \frac{1}{\pi \alpha^2 J_1^2(\lambda_{0s})} J_0\left(\lambda_{0s} \frac{r}{\alpha}\right) + \sum_{m=-\infty}^{\infty} \sum_{s=1}^{\infty} \sum_{k=0}^{n-1} \sum_{j=1}^{\alpha} e^{-imk\theta_p} \frac{1}{\pi \alpha^2 J_1^2(\lambda_{0s})} J_0\left(\lambda_{0s} \frac{jr_p}{\alpha}\right) J_0\left(\lambda_{0s} \frac{r}{\alpha}\right) e^{im\theta}. \quad (4.14)$$

The simulation of Eq.(4.14) is presented bellow

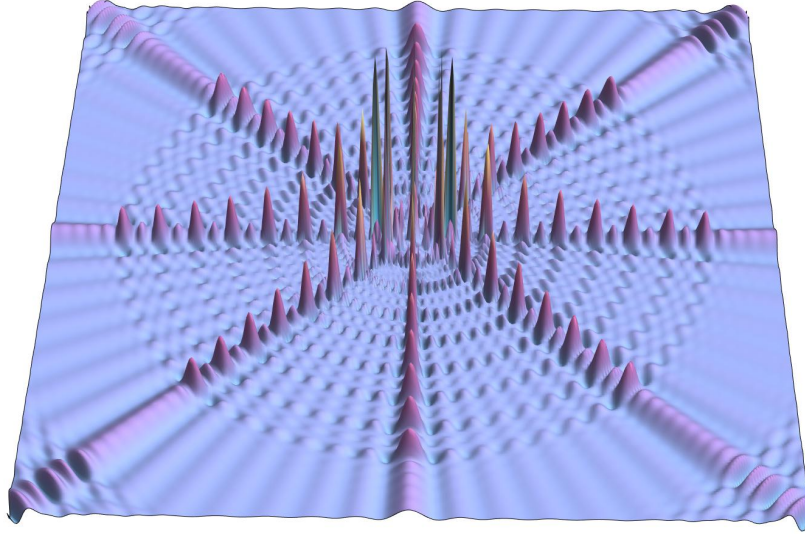


Figure 4.6: Polar train represented by Fourier-Bessel series with unitary radial period, $\alpha = 10$ (arbitrary units), angular period $\frac{2\pi}{8}$, $j = 8$, $m \in [-60, 60]$ and $s \in [0, 50]$.

To obtain the Fourier transform of the entire polar train, for the first part the definition for a function with radial symmetry must be applied, the second part must apply the definition in series. For that reason, the second part of Eq.(4.14) has to be seen as a Fourier series in the form

$$\sum_{m=-\infty}^{\infty} t_{pm}(r) e^{im\theta}$$

the coefficient $t_{pm}(r)$ is

$$t_{pm}(r) = \sum_{s=1}^{\infty} \sum_{k=0}^{n-1} \sum_{j=1}^{\alpha} e^{-imk\theta_p} \frac{1}{\pi\alpha^2 J_1^2(\lambda_{0s})} J_0\left(\lambda_{0s} \frac{jr_p}{\alpha}\right) J_0\left(\lambda_{0s} \frac{r}{\alpha}\right), \quad (4.15)$$

i.e.,

$$\begin{aligned} t_p(r, \theta) &= \sum_{s=1}^{\infty} \frac{1}{\pi\alpha^2 J_1^2(\lambda_{0s})} J_0\left(\lambda_{0s} \frac{r}{\alpha}\right) \\ &\quad + \sum_{m=-\infty}^{\infty} \sum_{s=1}^{\infty} \sum_{k=0}^{n-1} \sum_{j=1}^{\alpha} e^{-imk\theta_p} \frac{1}{\pi\alpha^2 J_1^2(\lambda_{0s})} J_0\left(\lambda_{0s} \frac{jr_p}{\alpha}\right) J_0\left(\lambda_{0s} \frac{r}{\alpha}\right) e^{im\theta} \\ &= \sum_{s=1}^{\infty} \frac{1}{\pi\alpha^2 J_1^2(\lambda_{0s})} J_0\left(\lambda_{0s} \frac{r}{\alpha}\right) + \sum_{m=-\infty}^{\infty} t_{pm}(r) e^{im\theta}. \end{aligned} \quad (4.16)$$

Then the Fourier transform of the polar train is

$$\begin{aligned} \tilde{t}_p(\rho, \varphi) &= 2\pi \int_0^{\infty} r dr \sum_{s=1}^{\infty} \frac{1}{\pi\alpha^2 J_1^2(\lambda_{0s})} J_0\left(\lambda_{0s} \frac{r}{\alpha}\right) J_0(2\pi\rho r) \\ &\quad + \sum_{m=-\infty}^{\infty} 2\pi(-i)^m e^{im\varphi} \int_0^{\infty} r dr t_{pm}(r) J_m(2\pi\rho r) \\ &= \sum_{s=1}^{\infty} \frac{1}{\pi\alpha^2 J_1^2(\lambda_{0s})} \frac{1}{(2\pi)^2 \rho} \delta\left(\rho - \frac{\lambda_{\nu s}}{2\pi\alpha}\right) \\ &\quad + \sum_{m=-\infty}^{\infty} 2\pi(-i)^m e^{im\varphi} \sum_{s=1}^{\infty} \sum_{k=0}^{n-1} \sum_{j=1}^{\alpha} e^{-imk\theta_p} \frac{J_0\left(\lambda_{0s} \frac{jr_p}{\alpha}\right)}{\pi\alpha^2 J_1^2(\lambda_{0s})} \int_0^{\infty} r dr J_0\left(\lambda_{0s} \frac{r}{\alpha}\right) J_m(2\pi\rho r) \end{aligned} \quad (4.17)$$

In contrast to the treatment in cartesian coordinates, in polar coordinates, the transformation of a pulse train can't be clearly interpreted.

Chapter 5

Method to form zero-order Bessel beams

5.1 Overlap of Circ functions

The Fourier transform of an annular aperture when the thickness tends to zero gives, as a result zero-order Bessel function. Let's consider n circular slits for the interferometer, which are distributed over an imaginary circumference with radius r_0 as shown in Fig. 5.1

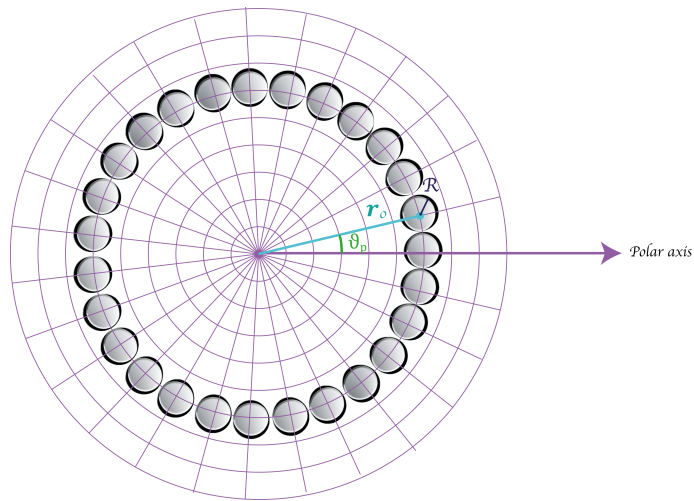


Figure 5.1: Overlap of circ functions.

Every slit can be described as a shifted circle function; also, in the section 3.6.1 we found that for a shifted function with radial symmetry

$$\mathcal{F}\{\zeta(\vec{r} - \vec{r}_0)\} = e^{-i2\pi\rho r_0 \cos(k\theta_p - \varphi)} \mathcal{F}\{\zeta(r)\}, \quad (5.1)$$

it's obvious that for Fig. 5.1 the radius r_0 is the same for each slit, but the angle changes according to

$$k\theta_p = k\frac{2\pi}{n}; \quad k = 0, \dots, n-1. \quad (5.2)$$

Then, for each slit shifted according to $\vec{r}_k = \vec{r}_k(r_0, k\theta_p)$ we have

$$\begin{aligned} \mathcal{F}\left\{\text{circ}\left(\frac{r - \vec{r}_k}{R}\right)\right\} &= e^{-i2\pi\rho r_0 \cos(k\theta_p - \varphi)} \mathcal{F}\left\{\text{circ}\left(\frac{r}{R}\right)\right\} \\ &= e^{-i2\pi\rho r_0 \cos(k\theta_p - \varphi)} \pi R^2 \text{Bessinc}(2\rho R) \\ &= e^{-i2\pi\rho r_0 \cos(k\theta_p - \varphi)} 2\pi R^2 \frac{J_1(2\pi\rho R)}{2\pi\rho R} \end{aligned} \quad (5.3)$$

The previous result can be divided by a factor πR^2 , then the Fourier transform of the complete configuration is given by

$$\begin{aligned} \mathcal{F}\left\{\sum_{k=0}^{n-1} \text{circ}\left(\frac{r - \vec{r}_k}{R}\right)\right\} &= 2 \sum_{k=0}^{n-1} e^{-i2\pi\rho r_0 \cos(k\theta_p - \varphi)} \frac{J_1(2\pi\rho R)}{2\pi\rho R} \\ &= 2 \sum_{m=-\infty}^{\infty} \sum_{k=0}^{n-1} (-i)^m e^{im\varphi} e^{-imk\theta_p} J_m(2\pi\rho r_0) \frac{J_1(2\pi\rho R)}{2\pi\rho R}, \end{aligned} \quad (5.4)$$

where the exponential term has been developed in series. It can be seen that the different values of the exponential term are accompanied by the term $\text{Bessinc}(2\rho R)$, this means that the spectrum formed by the sum of exponentials will be contained in the envelope $\text{Bessinc}(2\rho R)$. Then if the radius of the circular slit tends to zero, the case of point slits can be reached, then for $\text{Bessinc}(2\rho R)$ we have

$$\begin{aligned}
\lim_{R \rightarrow 0} 2 \frac{J_1(2\pi\rho R)}{2\pi\rho R} &= \lim_{R \rightarrow 0} \frac{2}{2\pi\rho R} \sum_{l=0}^{\infty} \frac{(-1)^l}{\Gamma(l+1)\Gamma(l+1+1)} \left(\frac{2\pi\rho R}{2}\right)^{2l+1} \\
&= \lim_{R \rightarrow 0} \sum_{l=0}^{\infty} \frac{(-1)^l}{\Gamma(l+1)\Gamma(l+2)} \frac{(2\pi\rho R)^{2l}}{(2)^{2l}} \\
&= \lim_{R \rightarrow 0} \left[1 + \sum_{l=1}^{\infty} \frac{(-1)^l}{\Gamma(l+1)\Gamma(l+2)} \frac{(2\pi\rho R)^{2l}}{(2)^{2l}} \right] \\
&= 1.
\end{aligned}$$

Therefore for the case of point slits its Fourier transform is given by

$$\sum_{k=0}^{n-1} e^{-i2\pi\rho r_0 \cos(k\theta_p - \varphi)} = \sum_{m=-\infty}^{\infty} \sum_{k=0}^{n-1} (-i)^m e^{im\varphi} e^{-imk\theta_p} J_m(2\pi\rho r_0), \quad (5.5)$$

which is in effect the Fourier transform of a superposition of Dirac deltas following the circumference shape with radius r_0 , we can verify it by comparing with Eq.(4.9) for $jr_p = r_0$. In this case the envelope is a constant. Using the squared module of Eq.(5.4) and Eq.(5.5) in exponential form, the diffraction pattern for $n = 5$ for the two types of slits is compared below

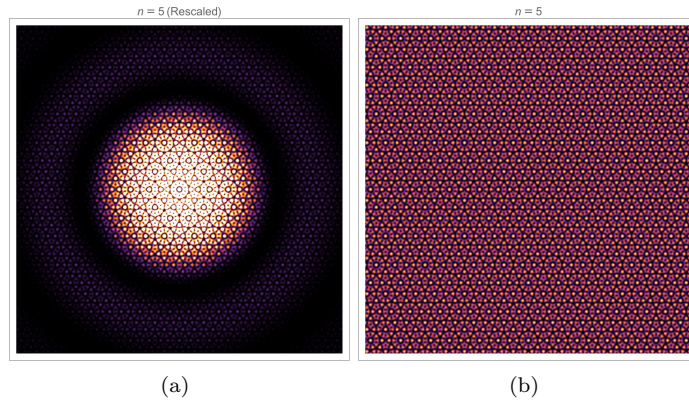


Figure 5.2: Diffraction pattern for $r_0 = 3$ for (a) circular slits ($R = 0.125$), here the color scale was modified to appreciate the envelope $\text{Bessinc}(2\rho R)$ and (b) for point slits, here it's clear that the envelope is a constant.

The diffraction pattern for different values of n were obtained only for circular slits with the squared module of Eq.(5.4), without a change in the color scale, it's difficult to see the envelope clearly.

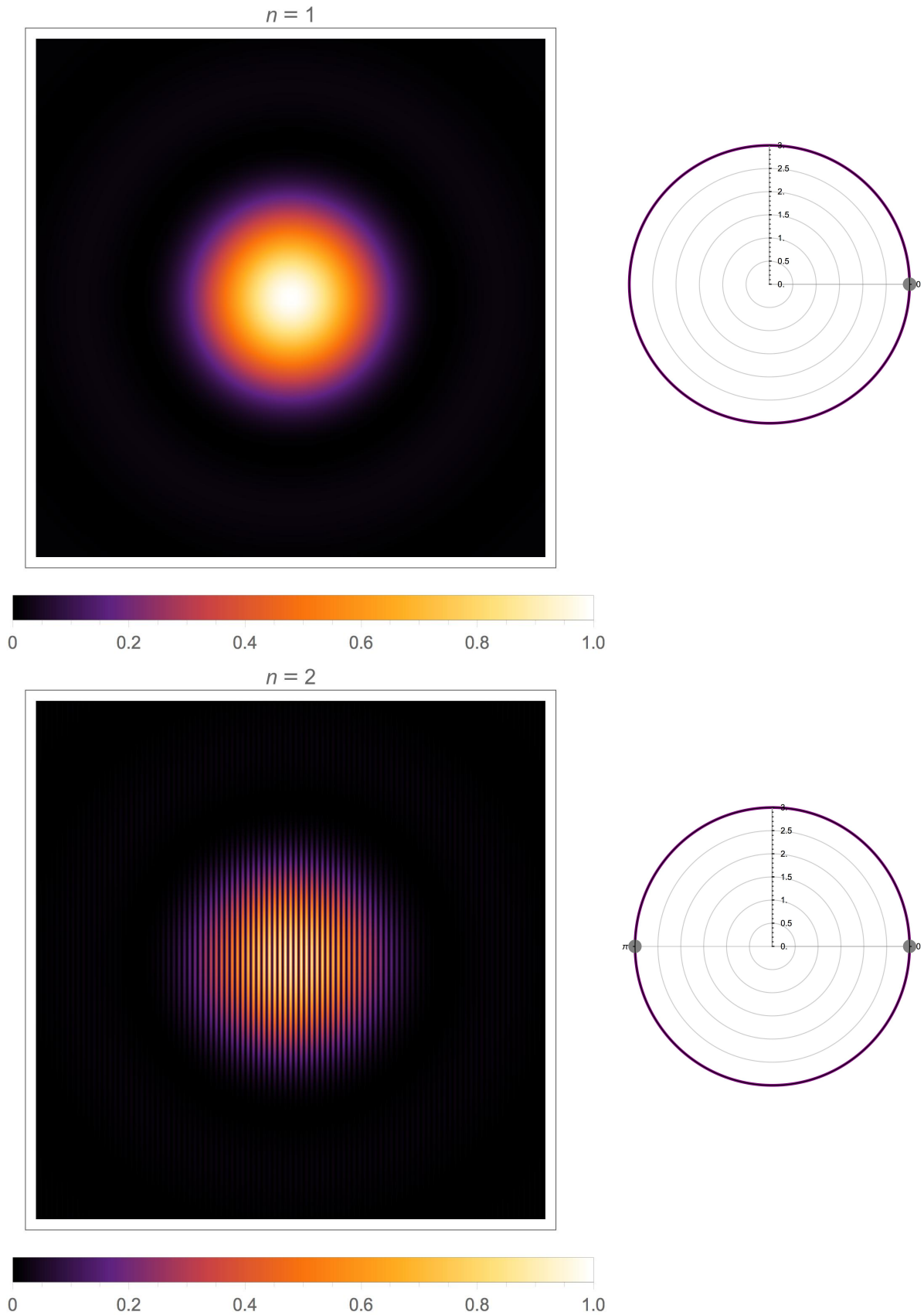


Figure 5.3: The figures on the left show the diffraction pattern for $R = 0.125$, $r_0 = 3$ (arbitrary units). On the color scale 0 represents the minimum value of intensity and 1 represents the maximum value. The figures on the right show the distribution of the circular slits on an imaginary ring (in purple).

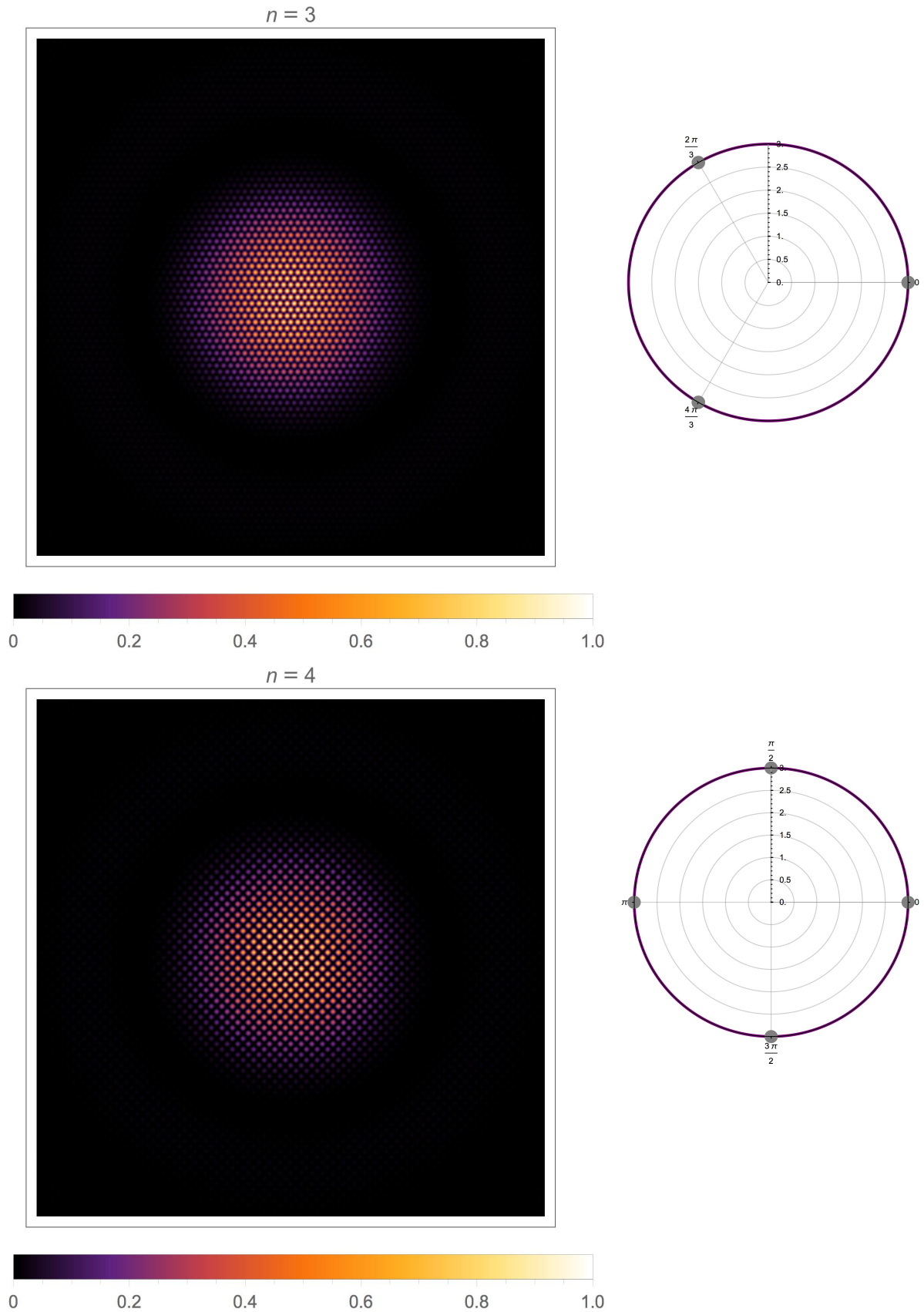


Figure 5.4: The figures on the left show the diffraction patterns for $R = 0.125$, $r_0 = 3$ (arbitrary units). On the color scale 0 represents the minimum value of intensity and 1 represents the maximum value. The figures on the right show the distribution of the circular slits on an imaginary ring (in purple).

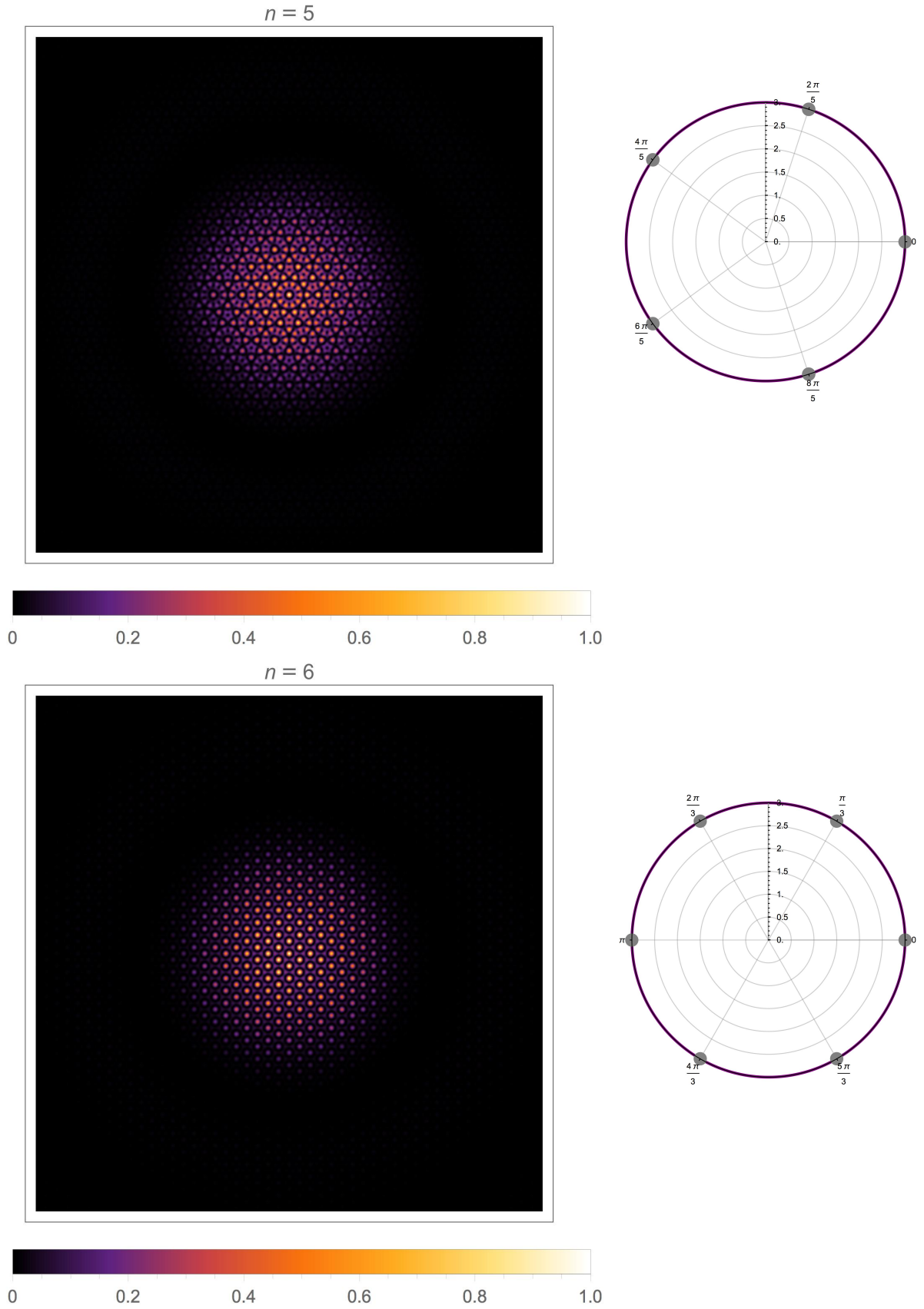


Figure 5.5: The figures on the left show the diffraction patterns for $R = 0.125$, $r_0 = 3$ (arbitrary units). On the color scale 0 represents the minimum value of intensity and 1 represents the maximum value. The figures on the right show the distribution of the circular slits on an imaginary ring (in purple).

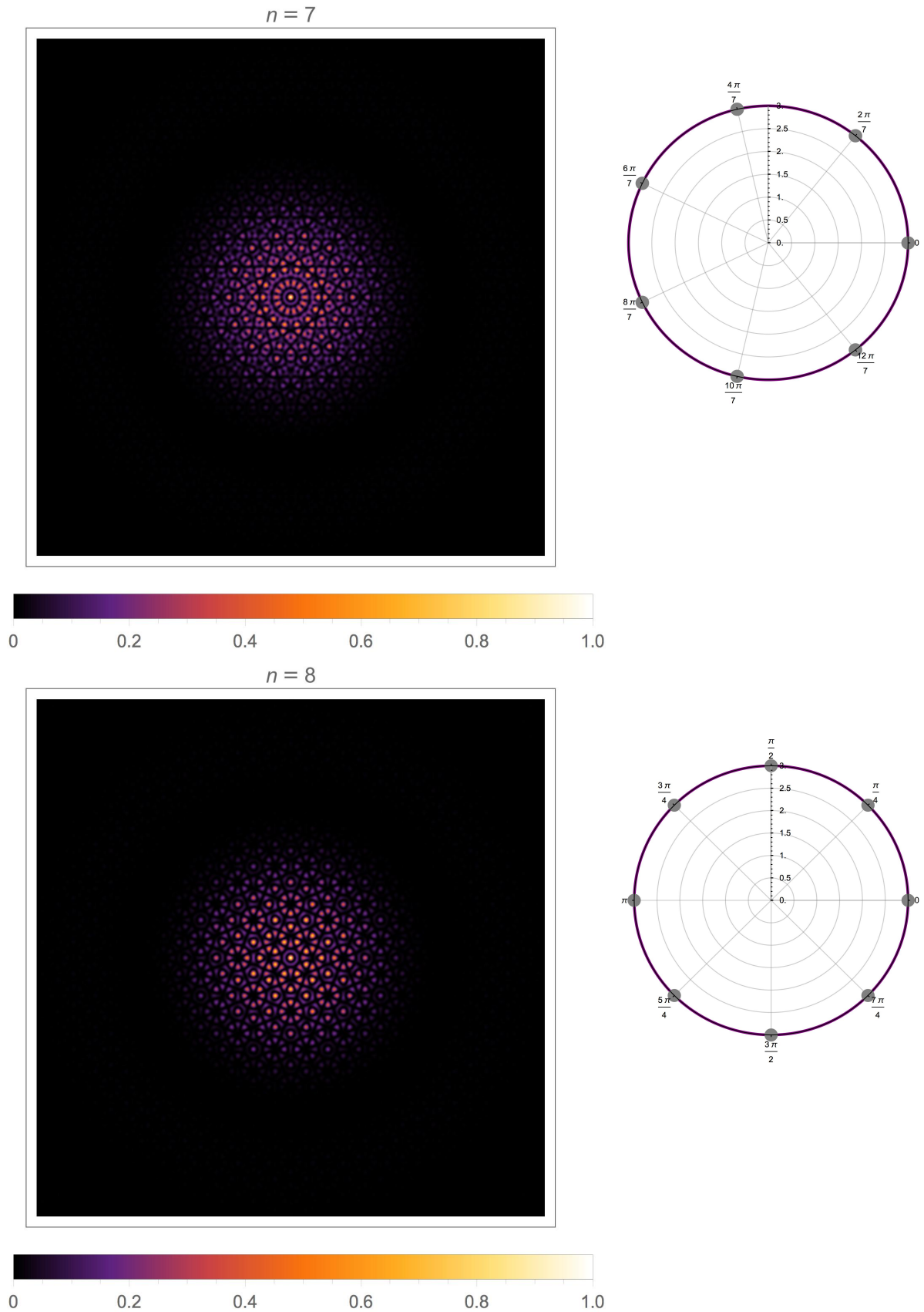


Figure 5.6: The figures on the left show the diffraction patterns for $R = 0.125$, $r_0 = 3$ (arbitrary units). On the color scale 0 represents the minimum value of intensity and 1 represents the maximum value. The figures on the right show the distribution of the circular slits on an imaginary ring (in purple).

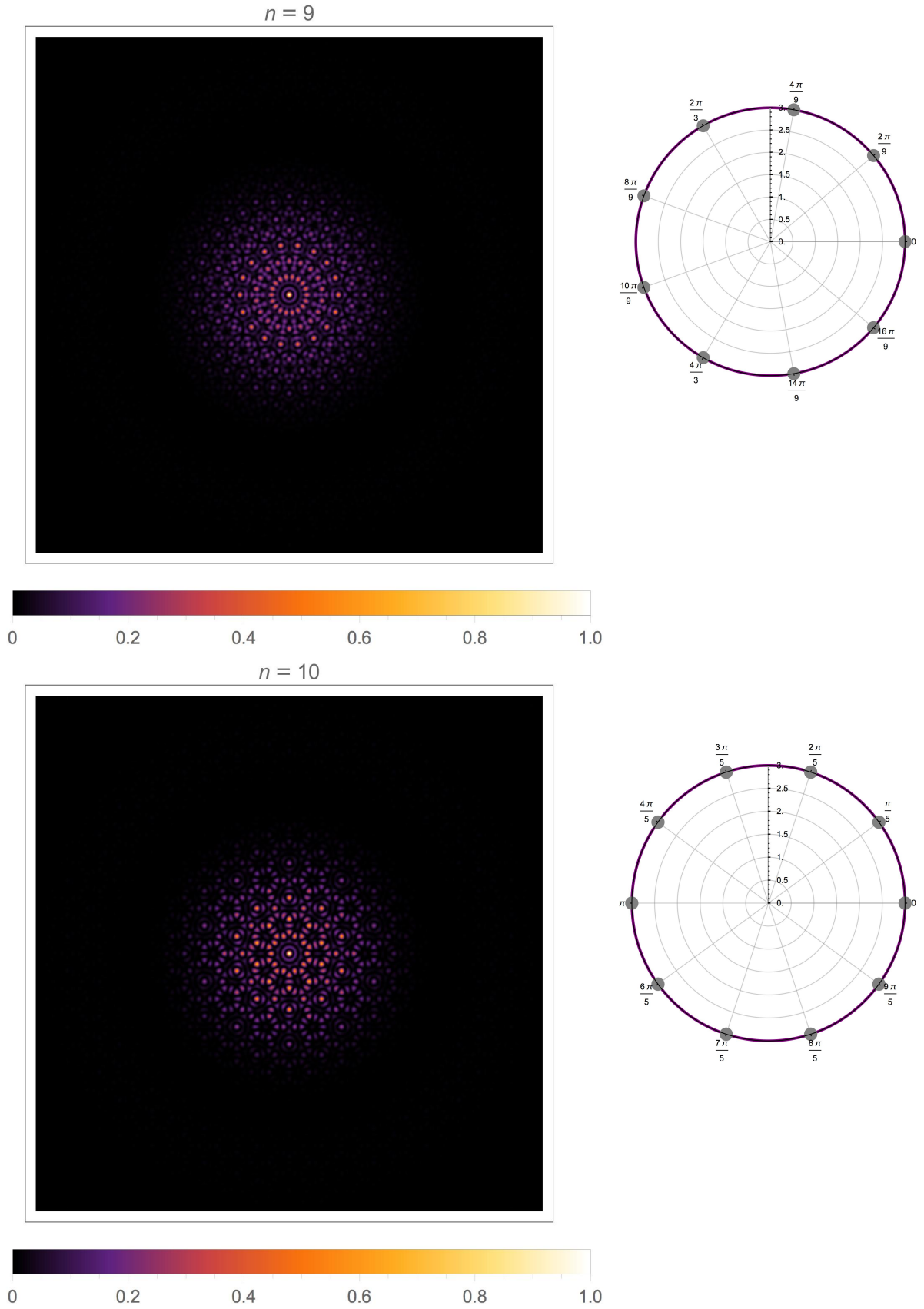


Figure 5.7: The figures on the left show the diffraction patterns for $R = 0.125$, $r_0 = 3$ (arbitrary units). On the color scale 0 represents the minimum value of intensity and 1 represents the maximum value. The figures on the right show the distribution of the circular slits on an imaginary ring (in purple).

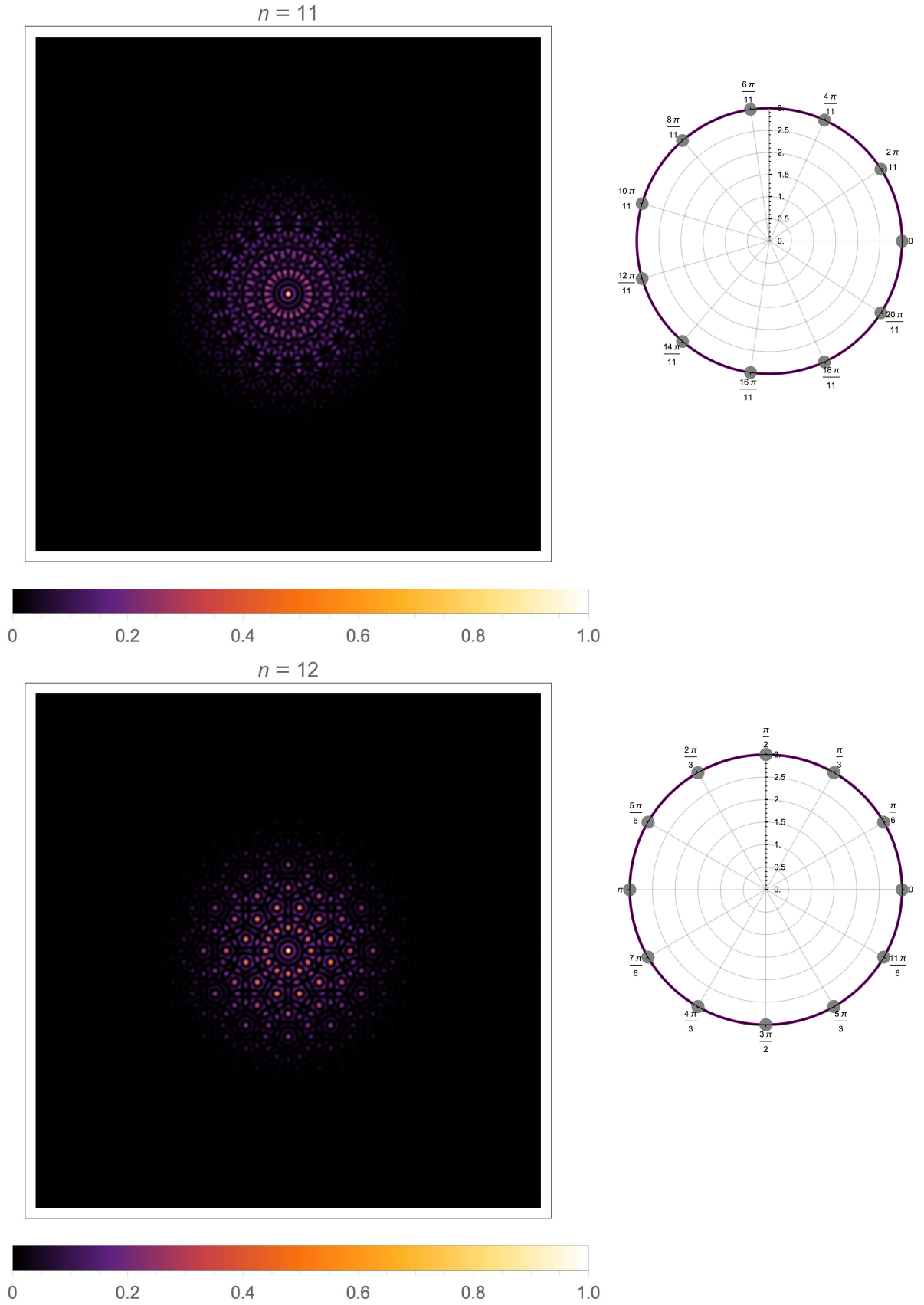


Figure 5.8: The figures on the left show the diffraction patterns for $R = 0.125$, $r_0 = 3$ (arbitrary units). On the color scale 0 represents the minimum value of intensity and 1 represents the maximum value. The figures on the right show the distribution of the circular slits on an imaginary ring (in purple).

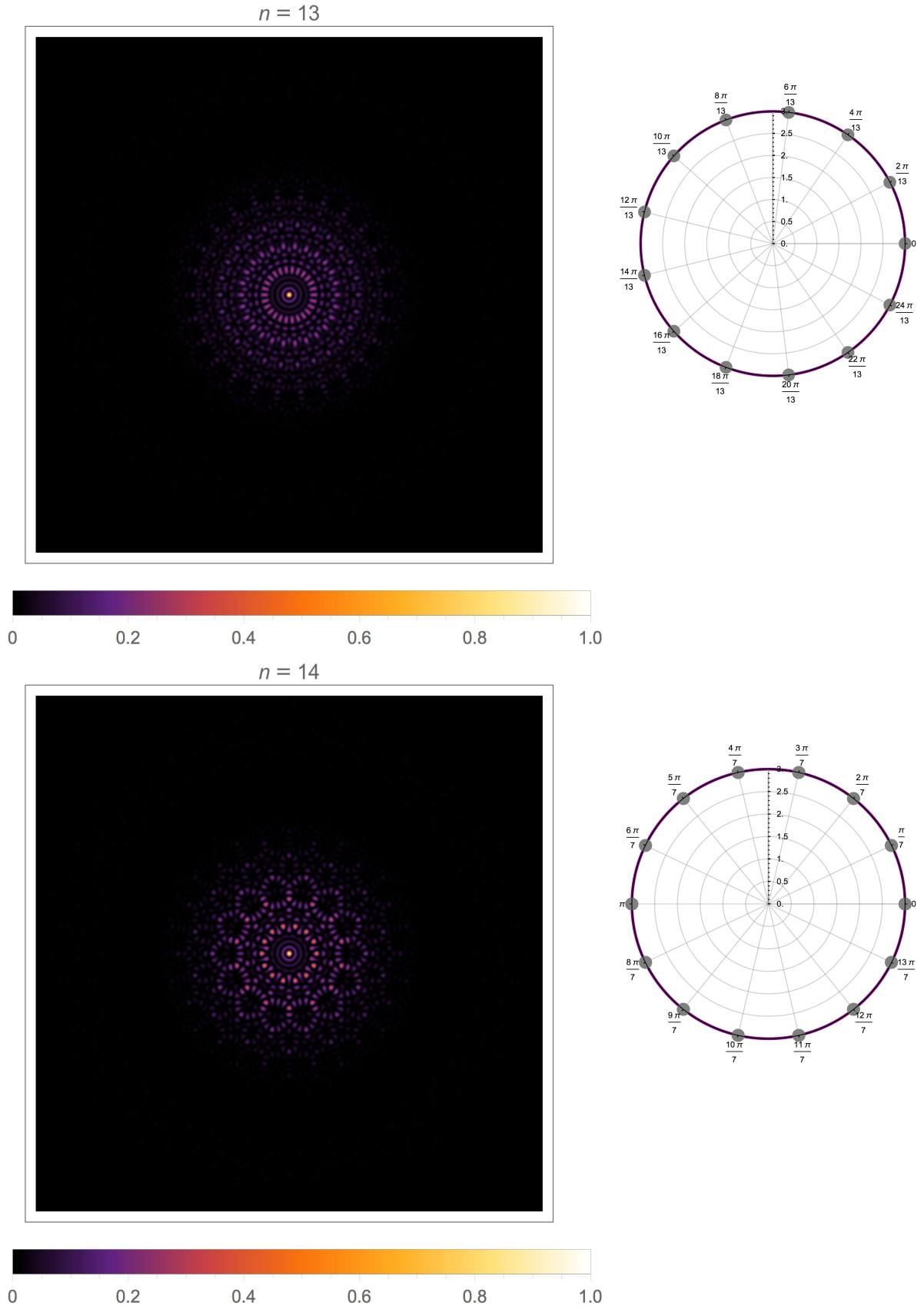


Figure 5.9: The figures on the left show the diffraction patterns for $R = 0.125$, $r_0 = 3$ (arbitrary units). On the color scale 0 represents the minimum value of intensity and 1 represents the maximum value. The figures on the right show the distribution of the circular slits on an imaginary ring (in purple).

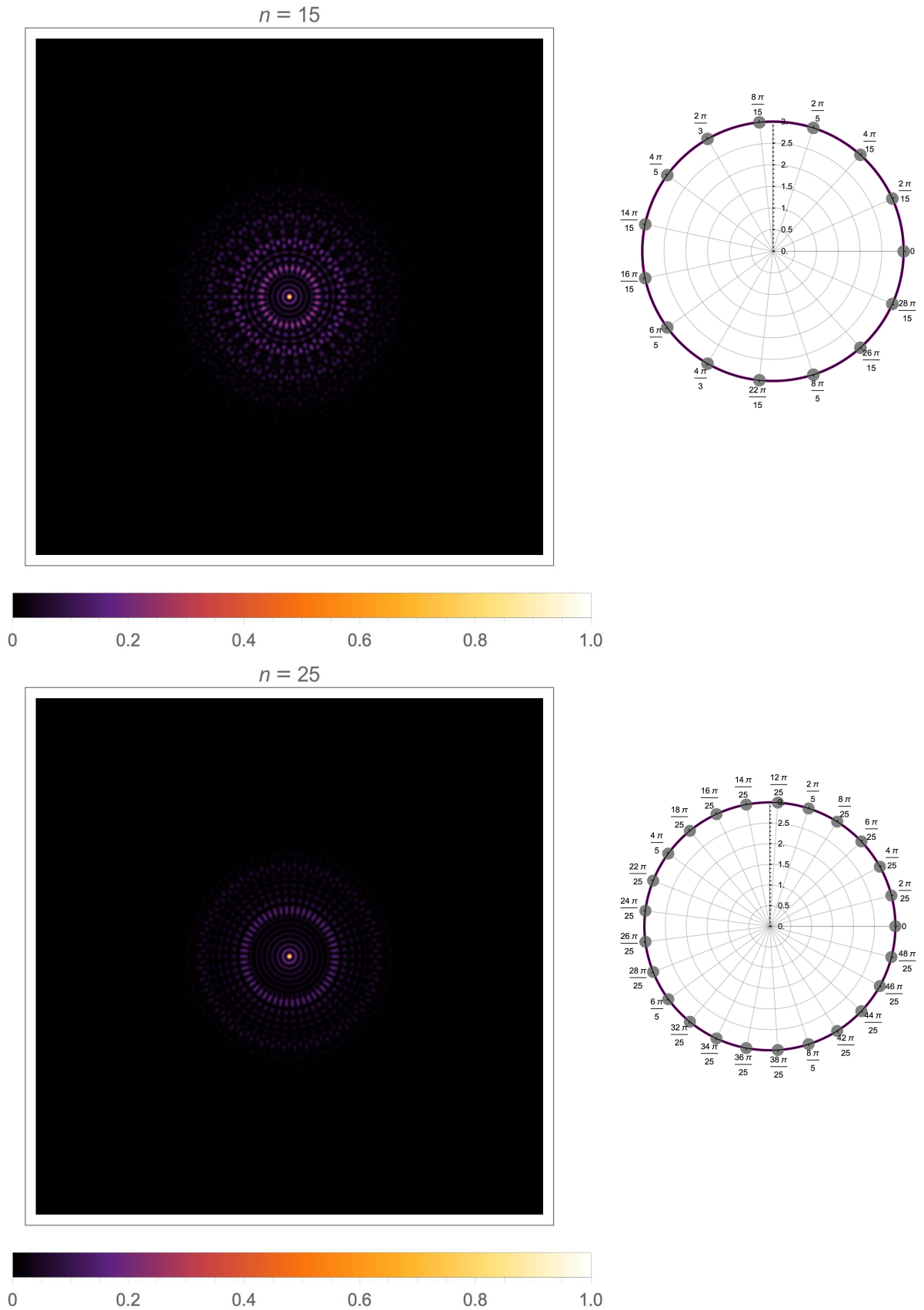


Figure 5.10: The figures on the left show the diffraction patterns for $R = 0.125$, $r_0 = 3$ (arbitrary units). On the color scale 0 represents the minimum value of intensity and 1 represents the maximum value. The figures on the right show the distribution of the circular slits on an imaginary ring (in purple).

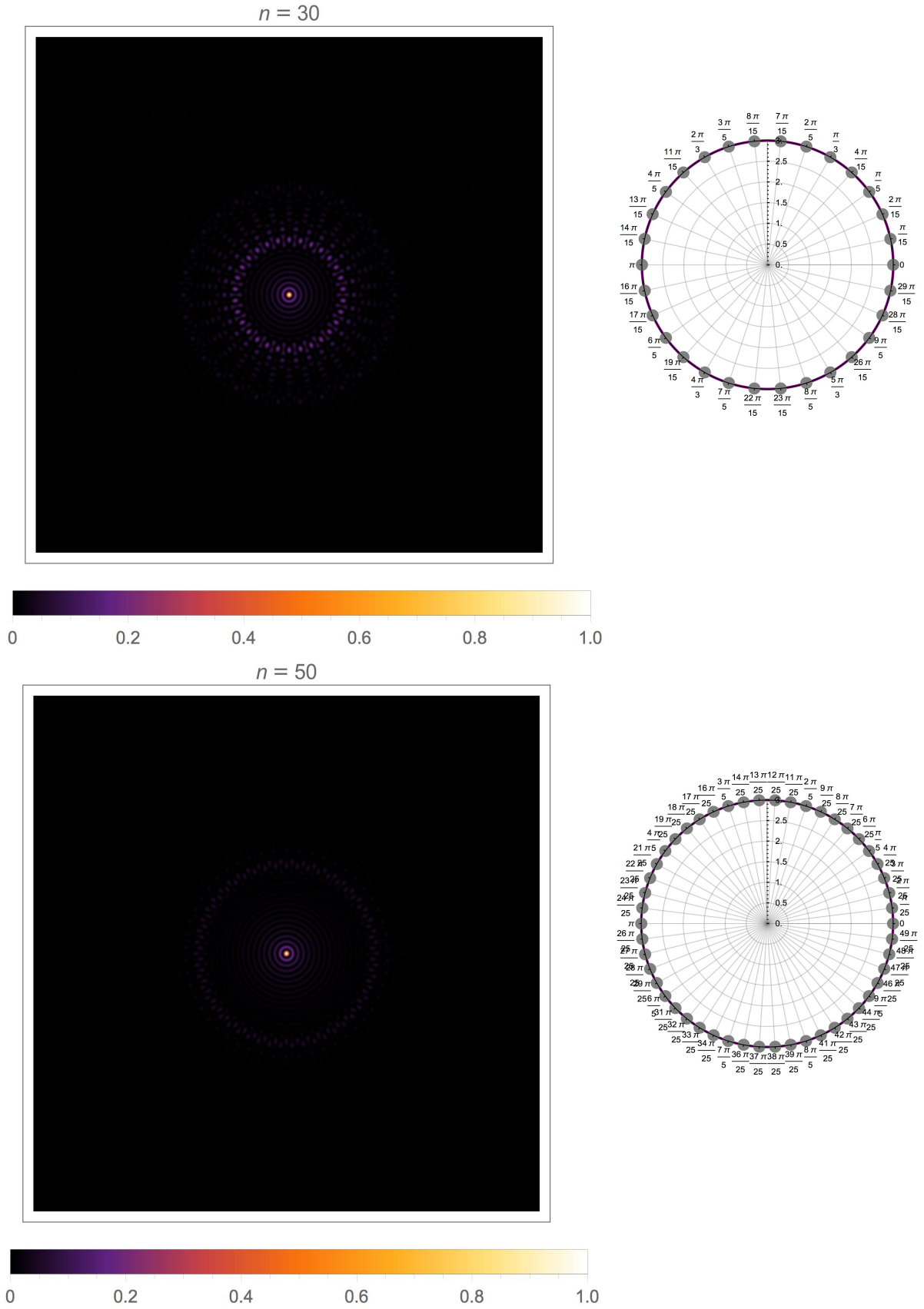


Figure 5.11: The figures on the left show the diffraction patterns for $R = 0.125$, $r_0 = 3$ (arbitrary units). On the color scale 0 represents the minimum value of intensity and 1 represents the maximum value. The figures on the right show the distribution of the circular slits on an imaginary ring (in purple).

Eq.(5.4) shows that it's difficult to obtain values to form a Bessel beam due to the number of terms, so the second method is presented in the next section.

5.2 Sampling of a ring delta function

To find the Fourier transform of the sampling ($\tilde{\zeta}_s(\rho, \varphi)$) of a ring delta function (transform of a product), it's necessary to apply the theory of the section 3.8; i.e., considering the definition for a sampling in polar coordinates (Eq.(4.3))

$$\zeta_s(r, \theta) = \zeta(r) \cdot t_p(r, \theta),$$

we have

$$\begin{aligned} \zeta_s(r, \theta) = & \frac{1}{2\pi r} \delta(r - r_0) \cdot \sum_{s=1}^{\infty} \frac{1}{\pi \alpha^2 J_1^2(\lambda_{0s})} J_0\left(\lambda_{0s} \frac{r}{\alpha}\right) \\ & + \frac{1}{2\pi r} \delta(r - r_0) \cdot \sum_{m=-\infty}^{\infty} \sum_{s=1}^{\infty} \sum_{k=0}^{n-1} \sum_{j=1}^{\alpha} e^{-imk\theta_p} \frac{1}{\pi \alpha^2 J_1^2(\lambda_{0s})} J_0\left(\lambda_{0s} \frac{jr_p}{\alpha}\right) J_0\left(\lambda_{0s} \frac{r}{\alpha}\right) e^{im\theta}. \end{aligned} \quad (5.6)$$

Again we have to apply the definition of Fourier transform of a product of two radially symmetric functions for the first part and the definition of Fourier transform of a product of a radially symmetric function with a function without radial symmetry for the second part, then,

$$\begin{aligned} \tilde{\zeta}_s(\rho, \varphi) = & 2\pi \int_0^{\infty} r dr \left[\frac{1}{2\pi r} \delta(r - r_0) \cdot \sum_{s=1}^{\infty} \frac{1}{\pi \alpha^2 J_1^2(\lambda_{0s})} J_0\left(\lambda_{0s} \frac{r}{\alpha}\right) \right] J_0(2\pi \rho r) \\ & + \sum_{m=-\infty}^{\infty} 2\pi (-i)^m e^{im\varphi} \int_0^{\infty} r dr t_{pm}(r) \frac{1}{2\pi r} \delta(r - r_0) J_m(2\pi \rho r) \\ = & \sum_{s=1}^{\infty} \frac{1}{\pi \alpha^2 J_1^2(\lambda_{0s})} \int_0^{\infty} dr \delta(r - r_0) J_0\left(\lambda_{0s} \frac{r}{\alpha}\right) J_0(2\pi \rho r) \\ & + \sum_{m=-\infty}^{\infty} 2\pi (-i)^m e^{im\varphi} \sum_{s=1}^{\infty} \sum_{k=0}^{n-1} \sum_{j=1}^{\alpha} e^{-imk\theta_p} \frac{1}{\pi \alpha^2 J_1^2(\lambda_{0s})} J_0\left(\lambda_{0s} \frac{jr_p}{\alpha}\right) \\ & \times \int_0^{\infty} r dr J_0\left(\lambda_{0s} \frac{r}{\alpha}\right) \frac{1}{2\pi r} \delta(r - r_0) J_m(2\pi \rho r) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{s=1}^{\infty} \frac{1}{\pi \alpha^2 J_1^2(\lambda_{0s})} J_0\left(\lambda_{0s} \frac{r_0}{\alpha}\right) J_0(2\pi \rho r_0) \\
 &+ \sum_{m=-\infty}^{\infty} (-i)^m e^{im\varphi} \sum_{s=1}^{\infty} \sum_{k=0}^{n-1} \sum_{j=1}^{\alpha} e^{-imk\theta_p} \frac{1}{\pi \alpha^2 J_1^2(\lambda_{0s})} J_0\left(\lambda_{0s} \frac{jr_p}{\alpha}\right) \\
 &\times J_0\left(\lambda_{0s} \frac{r_0}{\alpha}\right) J_m(2\pi \rho r_0),
 \end{aligned}$$

i.e.,

$$\begin{aligned}
 \tilde{\zeta}_s(\rho, \varphi) &= \sum_{s=1}^{\infty} \frac{1}{\pi \alpha^2 J_1^2(\lambda_{0s})} J_0\left(\lambda_{0s} \frac{r_0}{\alpha}\right) J_0(2\pi \rho r_0) \\
 &+ \sum_{m=-\infty}^{\infty} \sum_{s=1}^{\infty} \sum_{k=0}^{n-1} \sum_{j=1}^{\alpha} e^{im(\varphi - \frac{\pi}{2})} e^{-imk\theta_p} \frac{J_0\left(\lambda_{0s} \frac{jr_p}{\alpha}\right)}{\pi \alpha^2 J_1^2(\lambda_{0s})} J_0\left(\lambda_{0s} \frac{r_0}{\alpha}\right) J_m(2\pi \rho r_0). \quad (5.7)
 \end{aligned}$$

Then by calculating the squared module of the previous expression, diffraction patterns are obtained for different values of n . For all figures at the bottom, on the color scale 0 represents the minimum value of intensity and 1 represents the maximum value.

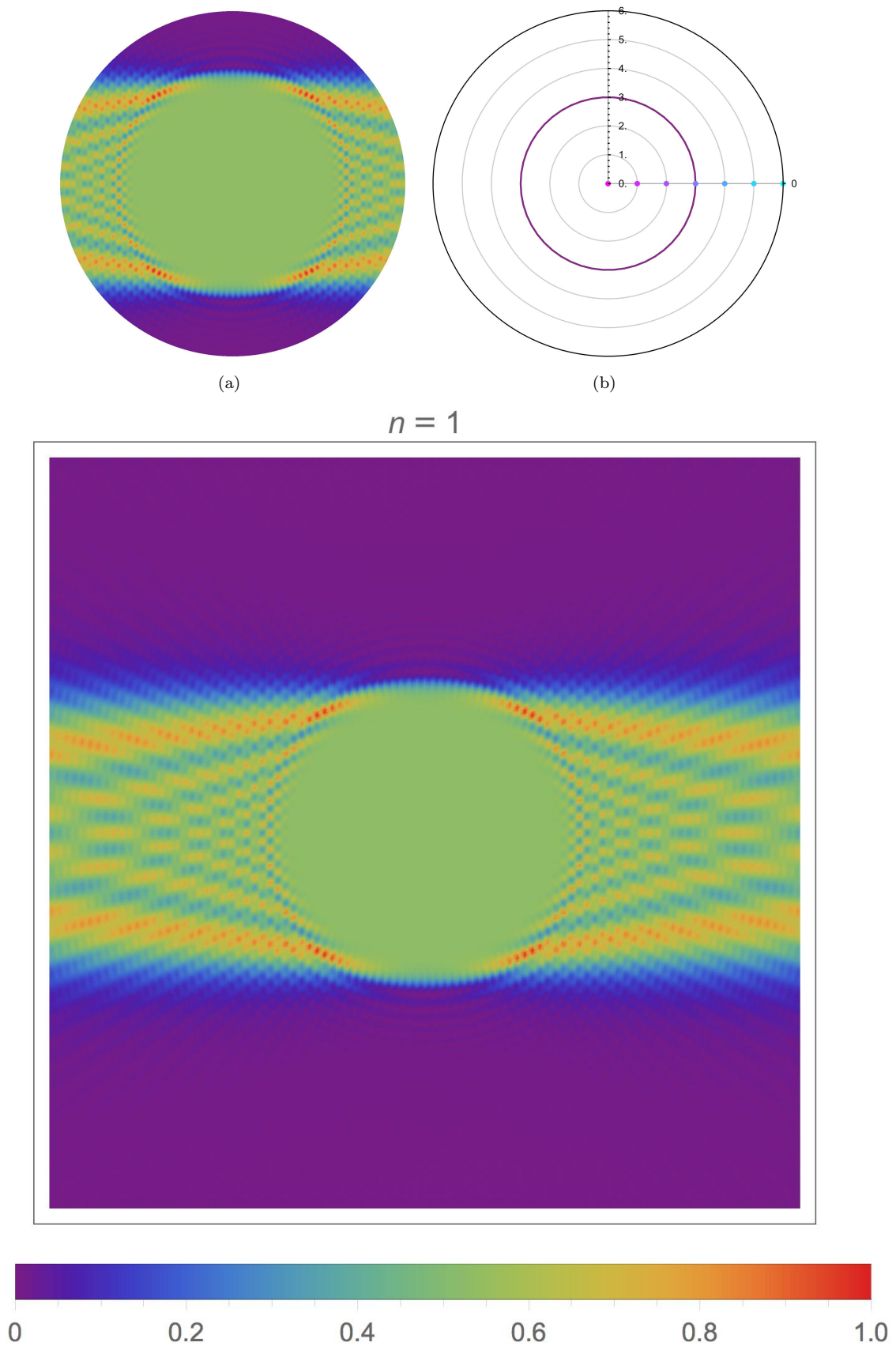


Figure 5.12: (a) Diffraction pattern of a ring delta function sampling for $m \in [-60, 60]$, $s \in [1, 50]$, $\alpha = 25$, $j = 6$ and $r_0 = 3$ (arbitrary units). (b) Graphical representation of the sampling in polar coordinates for a ring delta function with $\theta_p = 2\pi$.

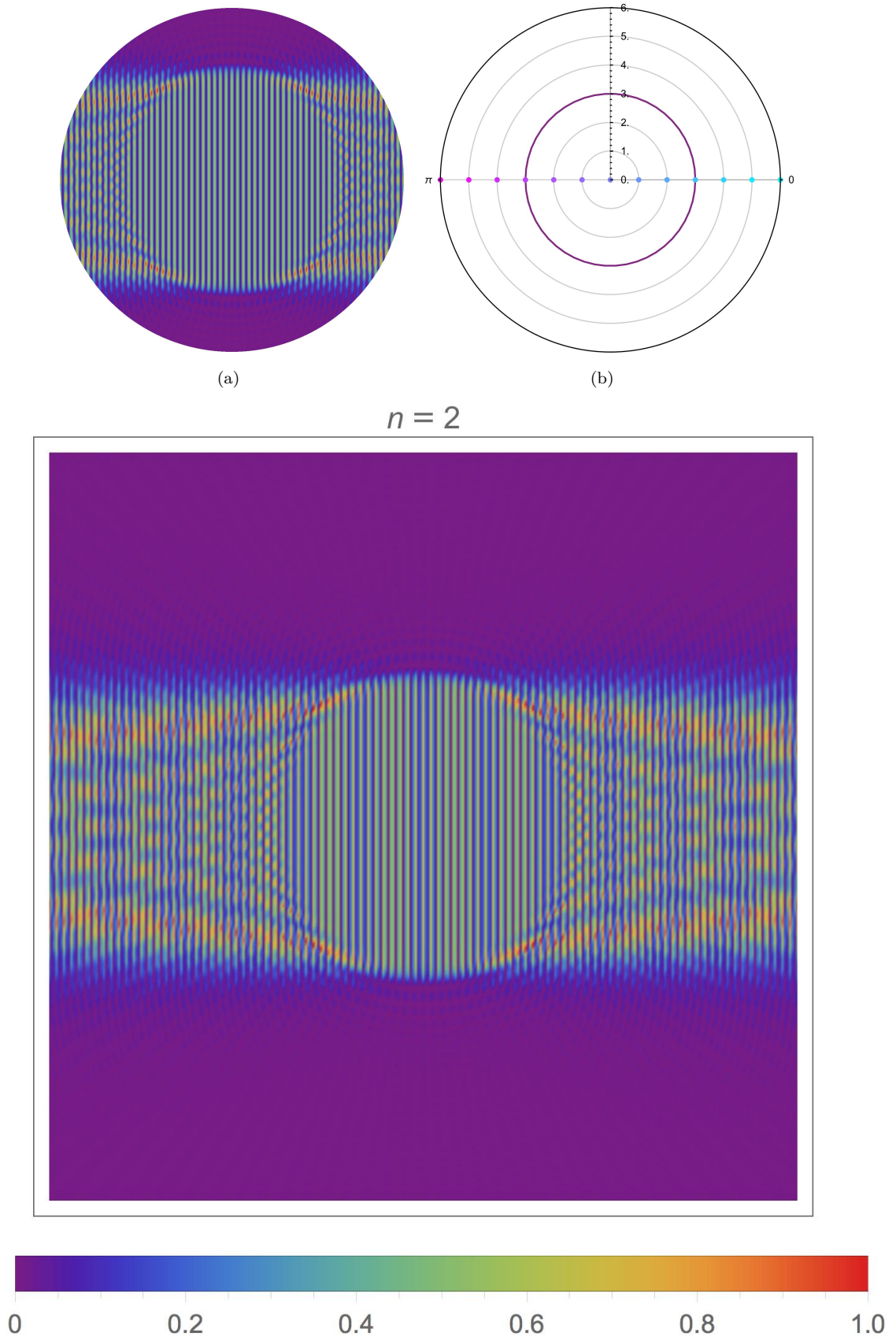


Figure 5.13: (a) Diffraction pattern of a ring delta function sampling for $n=2$, $m \in [-60, 60]$, $s \in [1, 50]$, $\alpha = 25$, $j = 6$ and $r_0 = 3$ (arbitrary units). (b) Graphical representation of the sampling in polar coordinates for a ring delta function with $\theta_p = \pi$.

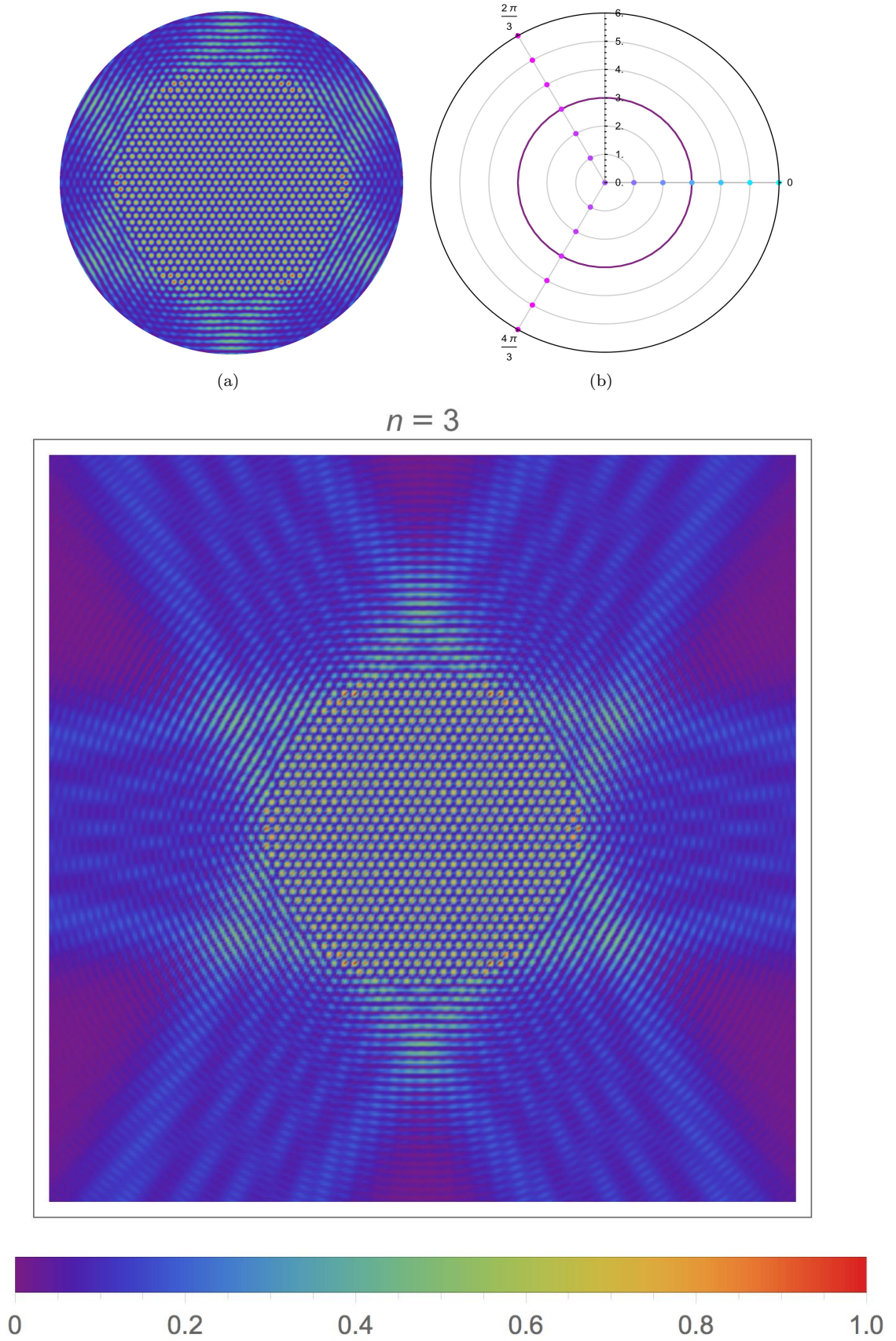


Figure 5.14: (a) Diffraction pattern of a ring delta function sampling for $n=3$, $m \in [-60, 60]$, $s \in [1, 50]$, $\alpha = 25$, $j = 6$ and $r_0 = 3$ (arbitrary units). (b) Graphical representation of the sampling in polar coordinates for a ring delta function with $\theta_p = \frac{2\pi}{3}$.

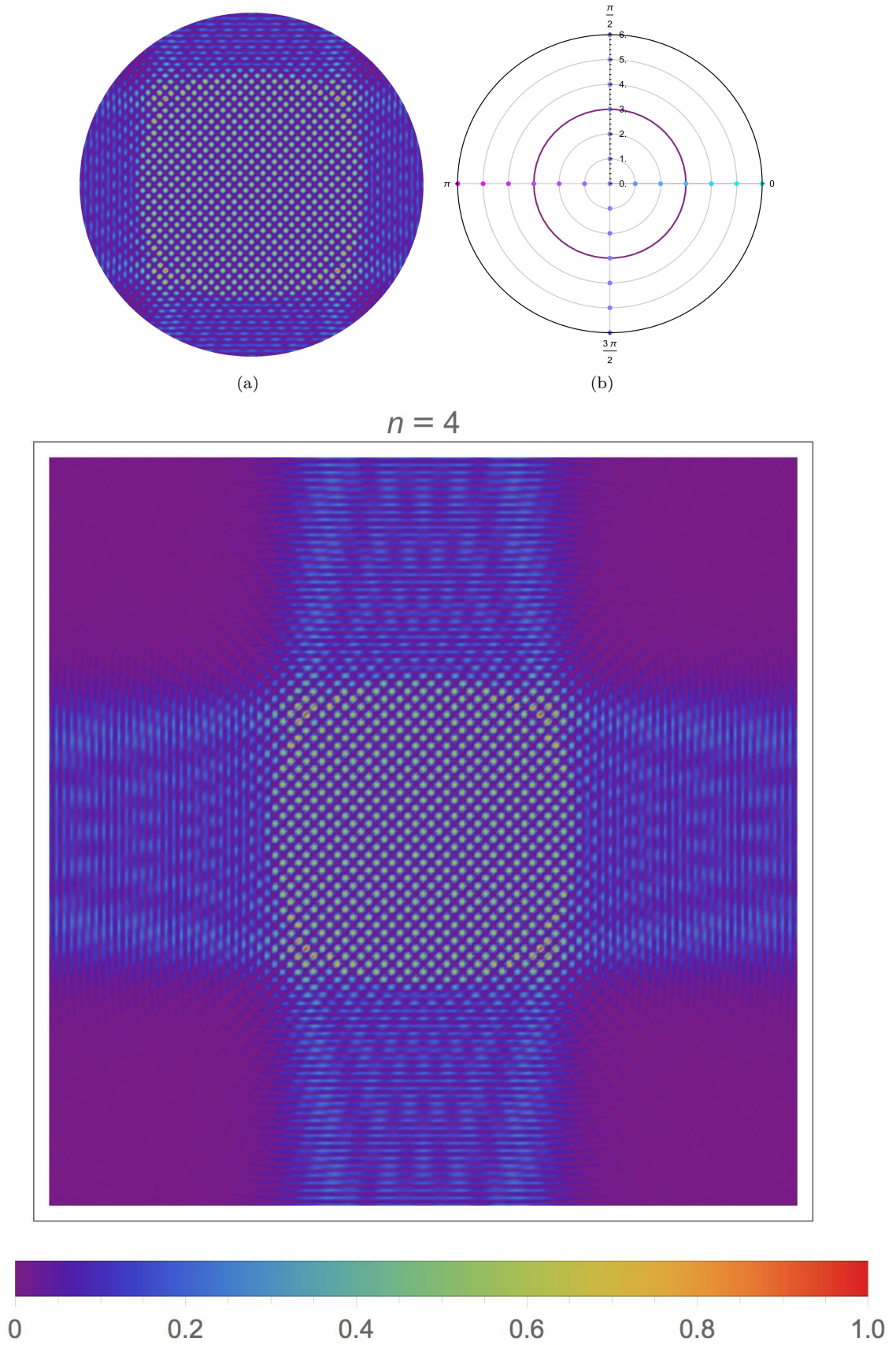


Figure 5.15: (a) Diffraction pattern of a ring delta function sampling for $n=4$, $m \in [-60, 60]$, $s \in [1, 50]$, $\alpha = 25$, $j = 6$ and $r_0 = 3$ (arbitrary units). (b) Graphical representation of the sampling in polar coordinates for a ring delta function with $\theta_p = \frac{\pi}{2}$

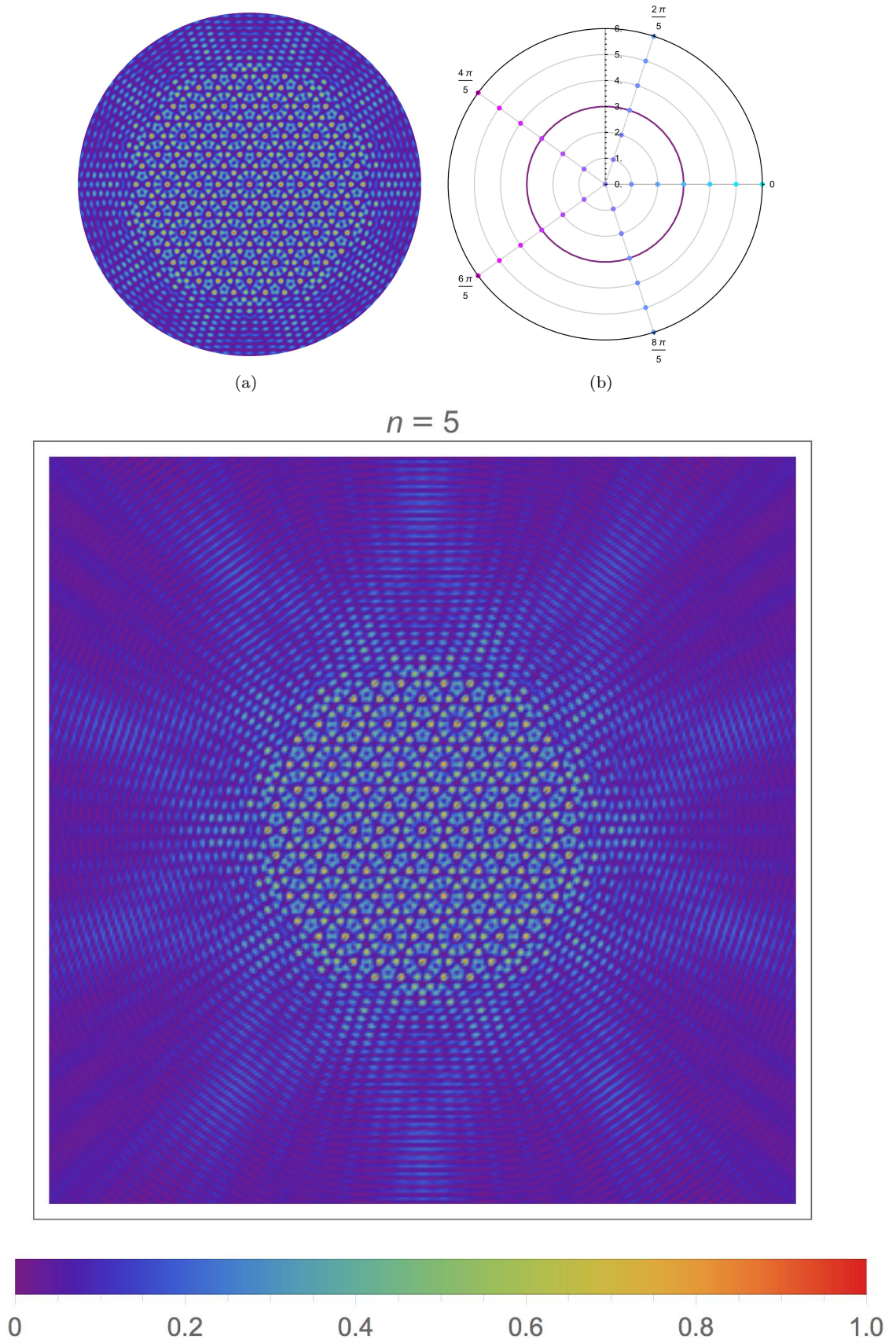


Figure 5.16: (a) Diffraction pattern of a ring delta function sampling for $n=5$, $m \in [-60, 60]$, $s \in [1, 50]$, $\alpha = 25$, $j = 6$ and $r_0 = 3$ (arbitrary units). (b) Graphical representation of the sampling in polar coordinates for a ring delta function with $\theta_p = \frac{2\pi}{5}$.

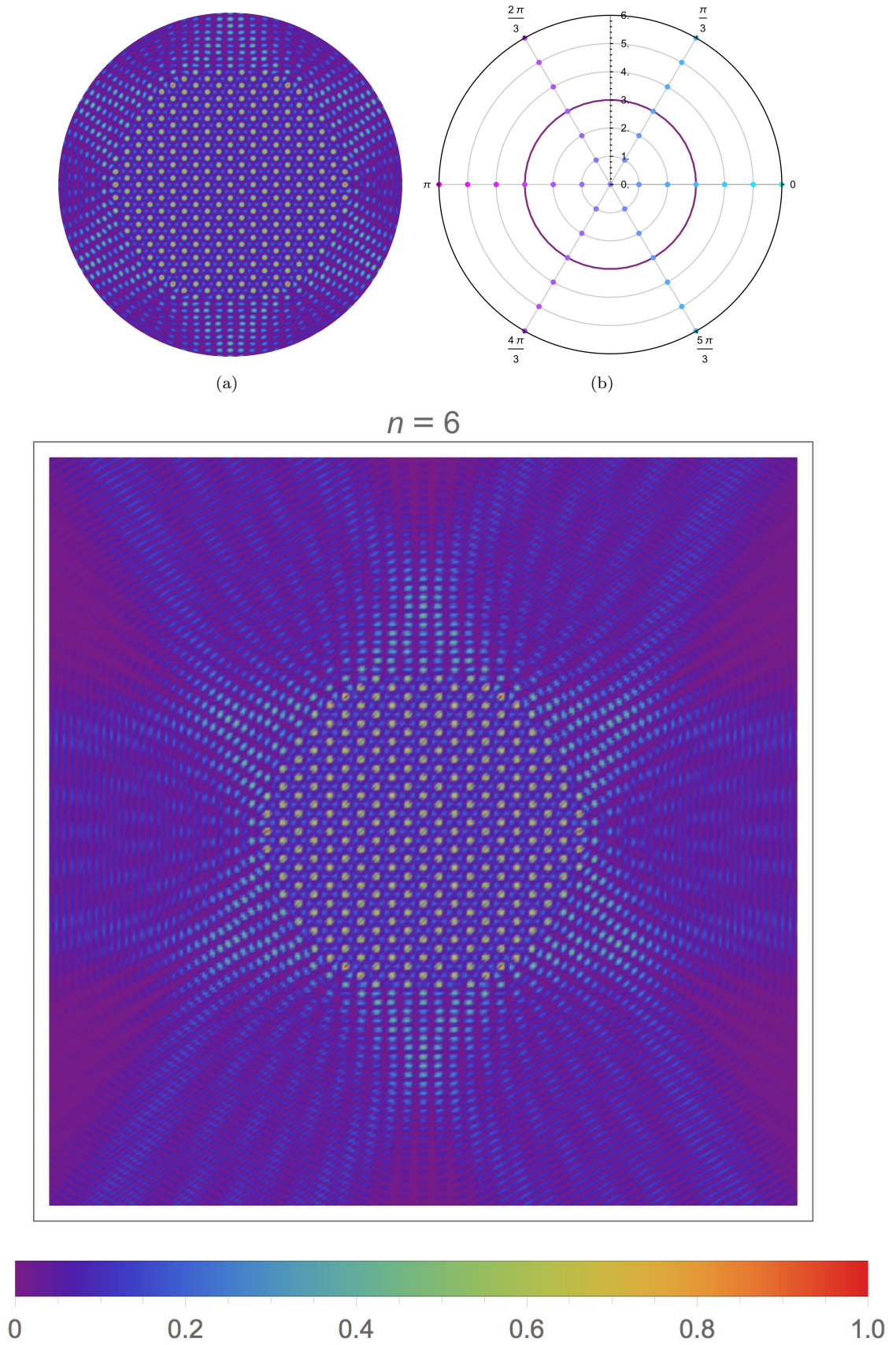


Figure 5.17: (a) Diffraction pattern of a ring delta function sampling for $n=6$, $m \in [-60, 60]$, $s \in [1, 50]$, $\alpha = 25$, $j = 6$ and $r_0 = 3$ (arbitrary units). (b) Graphical representation of the sampling in polar coordinates for a ring delta function with $\theta_p = \frac{\pi}{3}$.

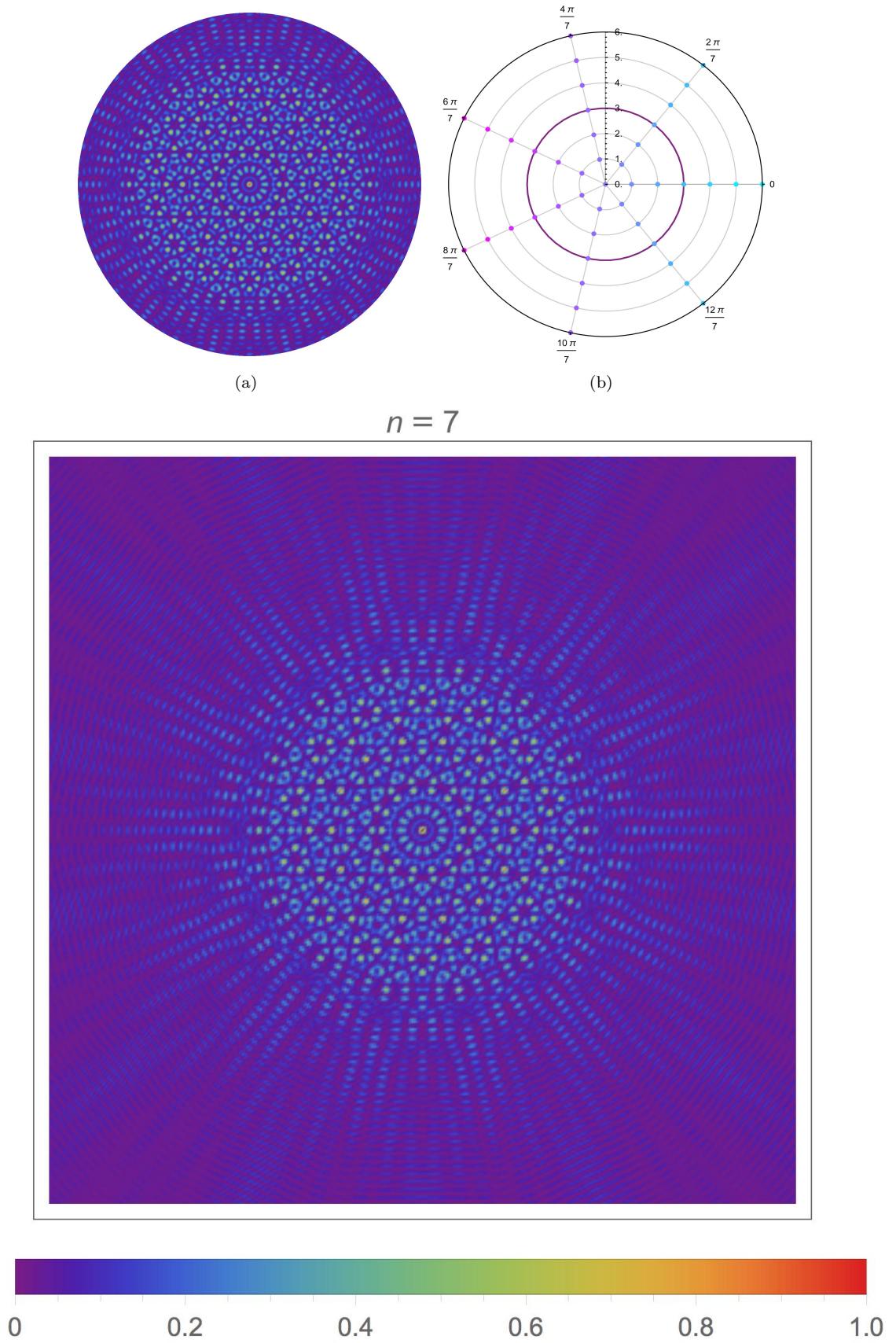


Figure 5.18: (a) Diffraction pattern of a ring delta function sampling for $n=7$, $m \in [-60, 60]$, $s \in [1, 50]$, $\alpha = 25$, $j = 6$ and $r_0 = 3$ (arbitrary units). (b) Graphical representation of the sampling in polar coordinates for a ring delta function with $\theta_p = \frac{2\pi}{7}$.

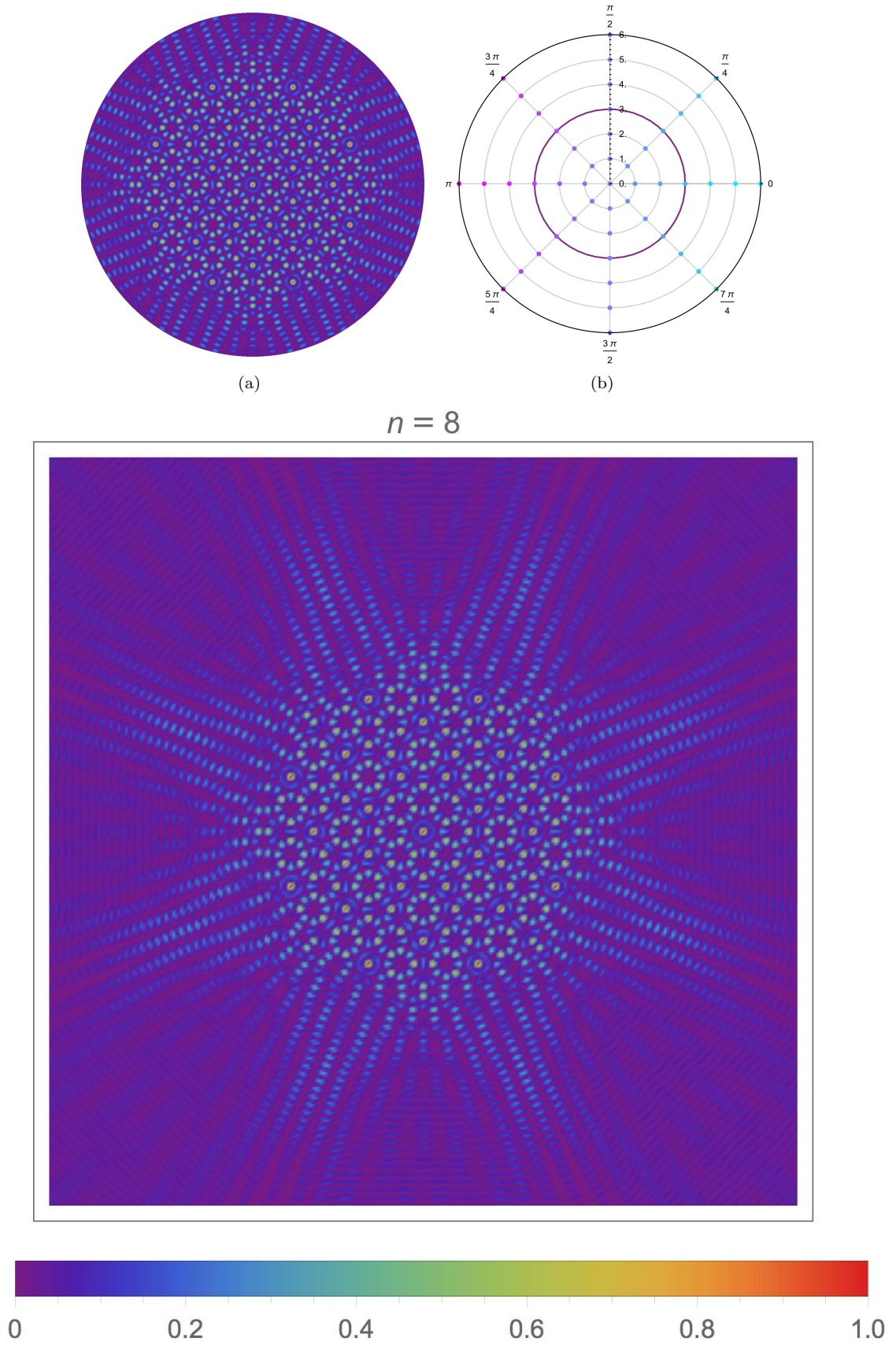


Figure 5.19: (a) Diffraction pattern of a ring delta function sampling for $n=8$, $m \in [-60, 60]$, $s \in [1, 50]$, $\alpha = 25$, $j = 6$ and $r_0 = 3$ (arbitrary units). (b) Graphical representation of the sampling in polar coordinates for a ring delta function with $\theta_p = \frac{\pi}{4}$.

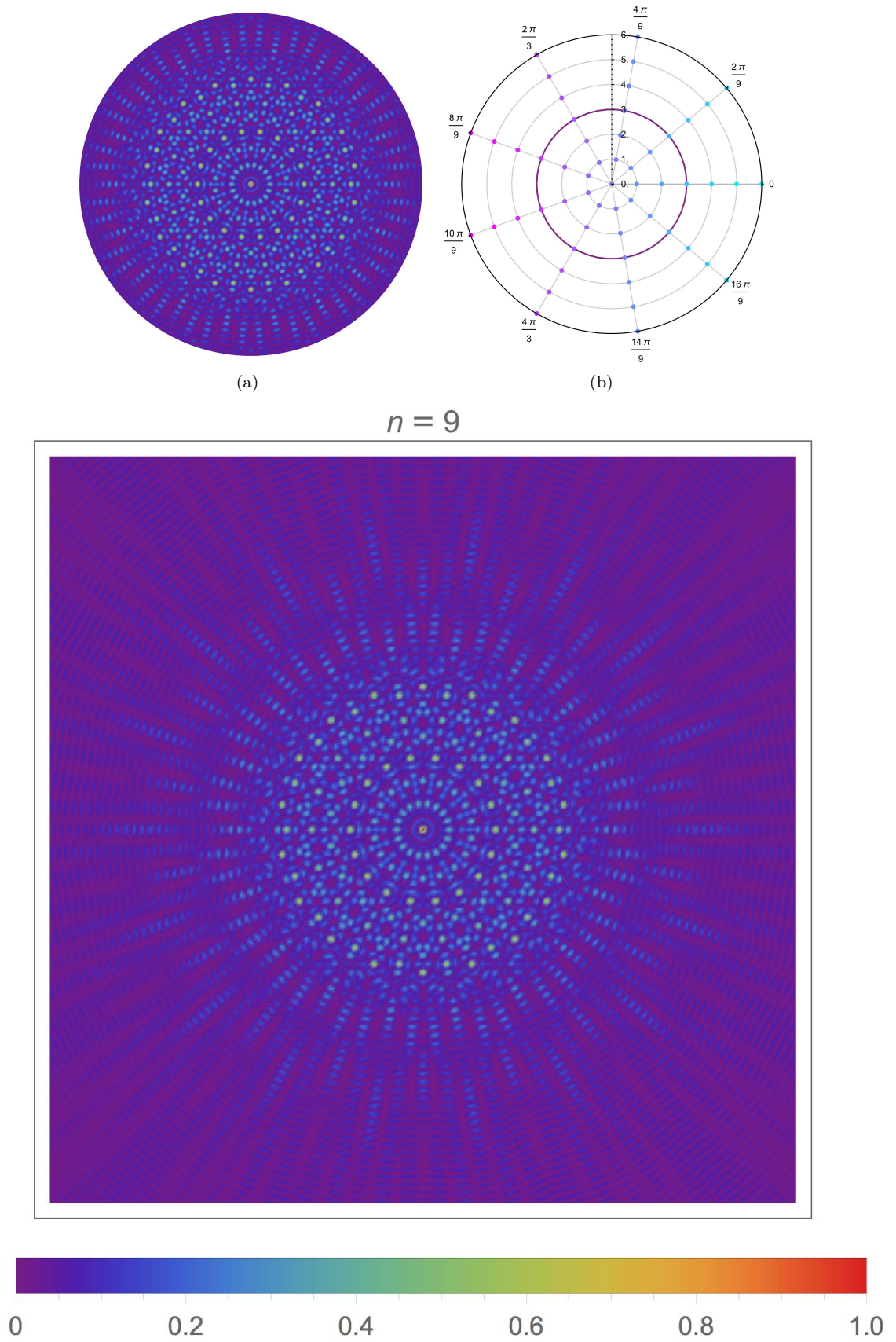


Figure 5.20: (a) Diffraction pattern of a ring delta function sampling for $n=9$, $m \in [-60, 60]$, $s \in [1, 50]$, $\alpha = 25$, $j = 6$ and $r_0 = 3$ (arbitrary units). (b) Graphical representation of the sampling in polar coordinates for a ring delta function with $\theta_p = \frac{2\pi}{9}$.

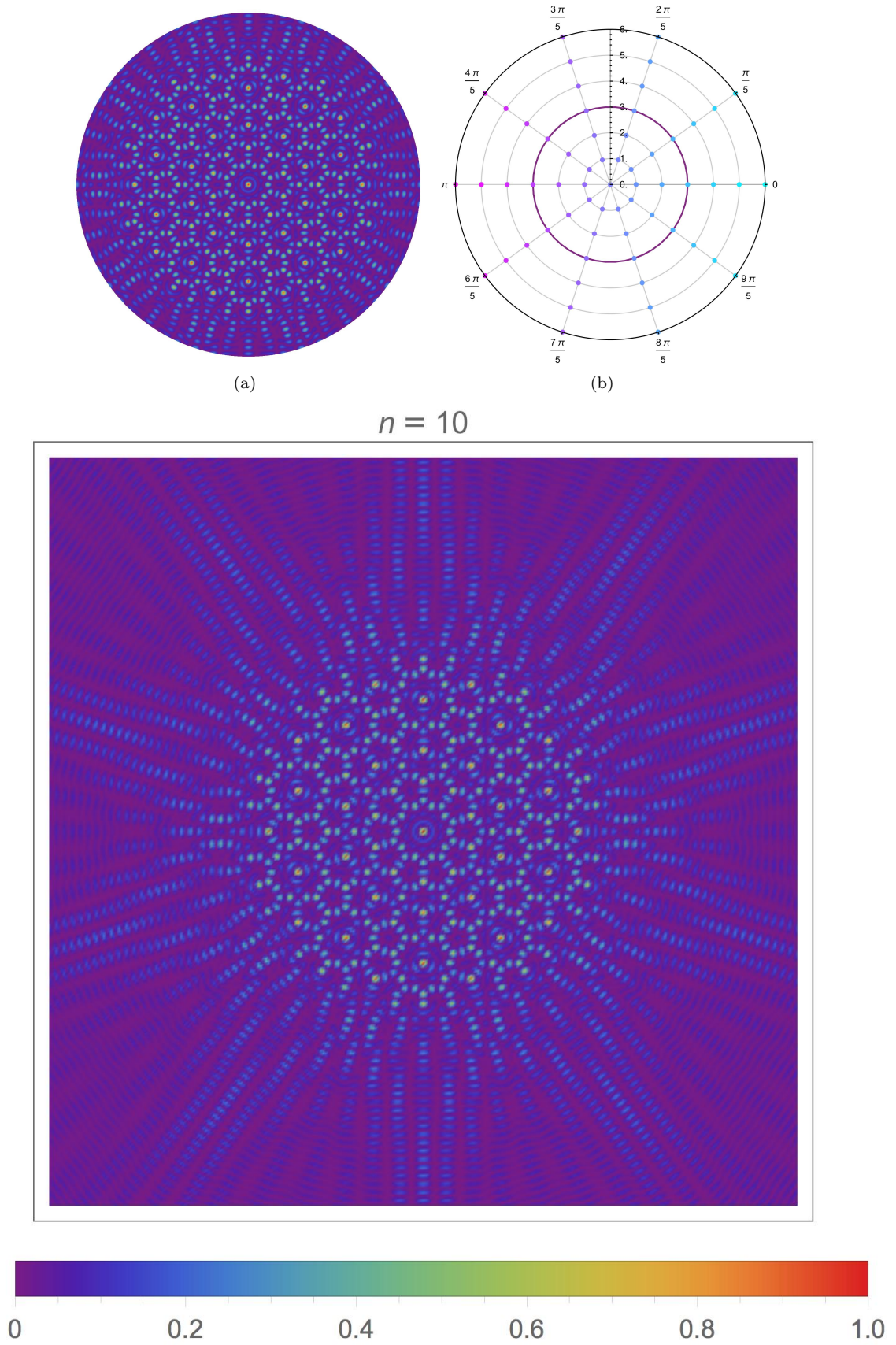


Figure 5.21: (a) Diffraction pattern of a ring delta function sampling for $n=10$, $m \in [-60, 60]$, $s \in [1, 50]$, $\alpha = 25$, $j = 6$ and $r_0 = 3$ (arbitrary units). (b) Graphical representation of the sampling in polar coordinates for a ring delta function with $\theta_p = \frac{\pi}{5}$.

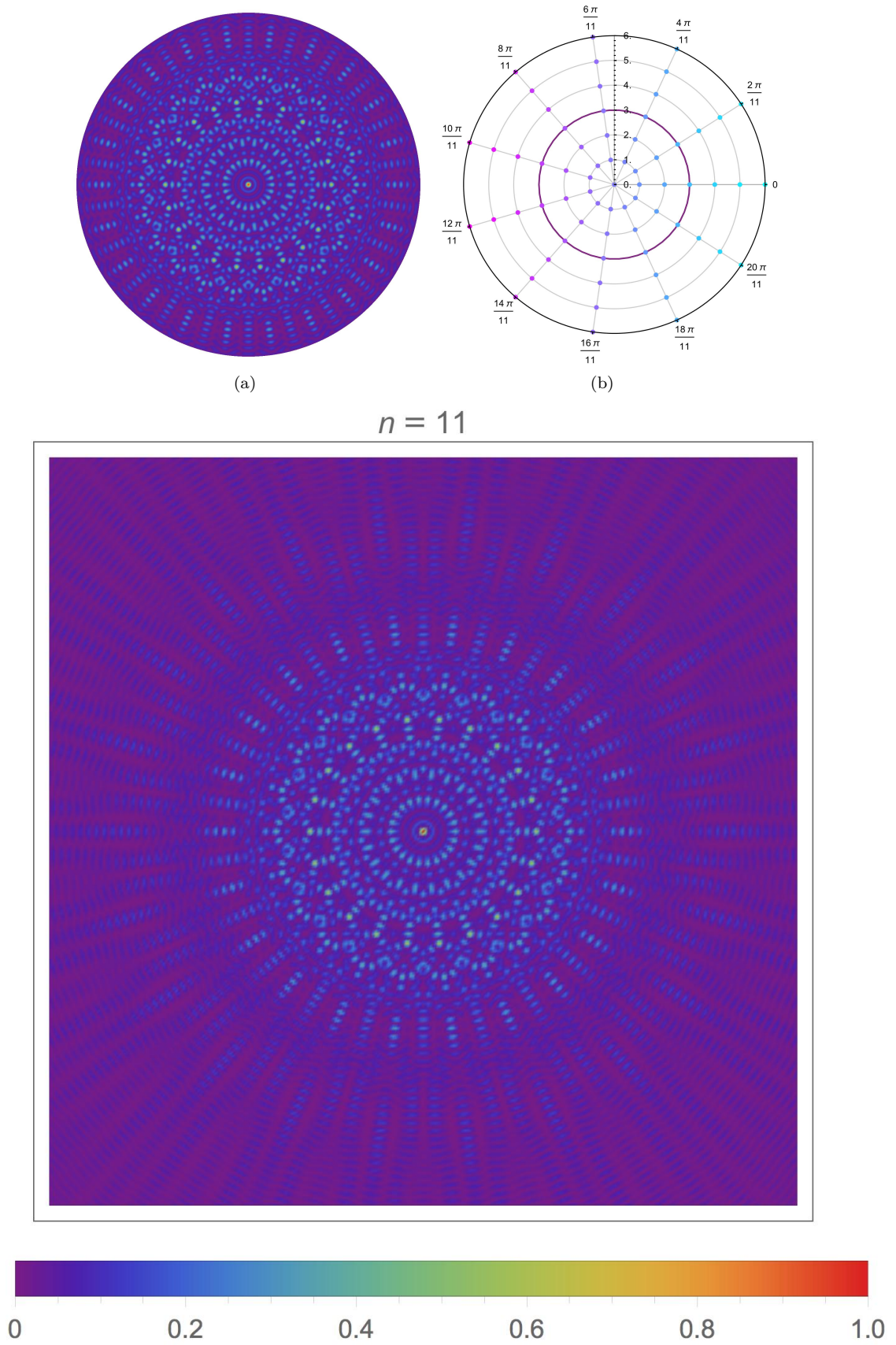


Figure 5.22: (a) Diffraction pattern of a ring delta function sampling for $n=11$, $m \in [-60, 60]$, $s \in [1, 50]$, $\alpha = 25$, $j = 6$ and $r_0 = 3$. (b) Graphical representation of the sampling in polar coordinates for a ring delta function with $\theta_p = \frac{2\pi}{11}$.

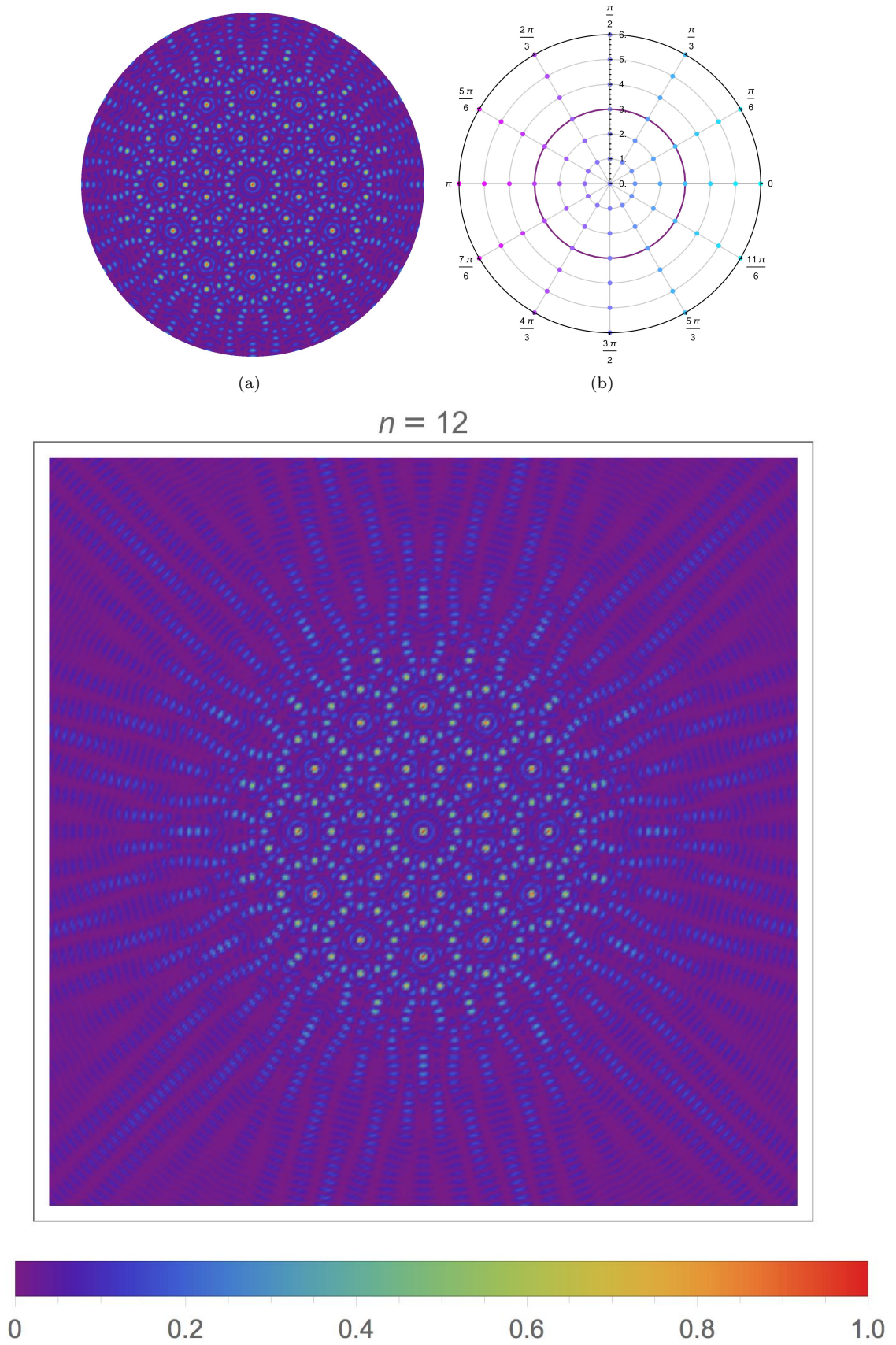


Figure 5.23: (a) Diffraction pattern of a ring delta function sampling for $n=12$, $m \in [-60, 60]$, $s \in [1, 50]$, $\alpha = 25$, $j = 6$ and $r_0 = 3$ (arbitrary units). (b) Graphical representation of the sampling in polar coordinates for a ring delta function with $\theta_p = \frac{\pi}{6}$.

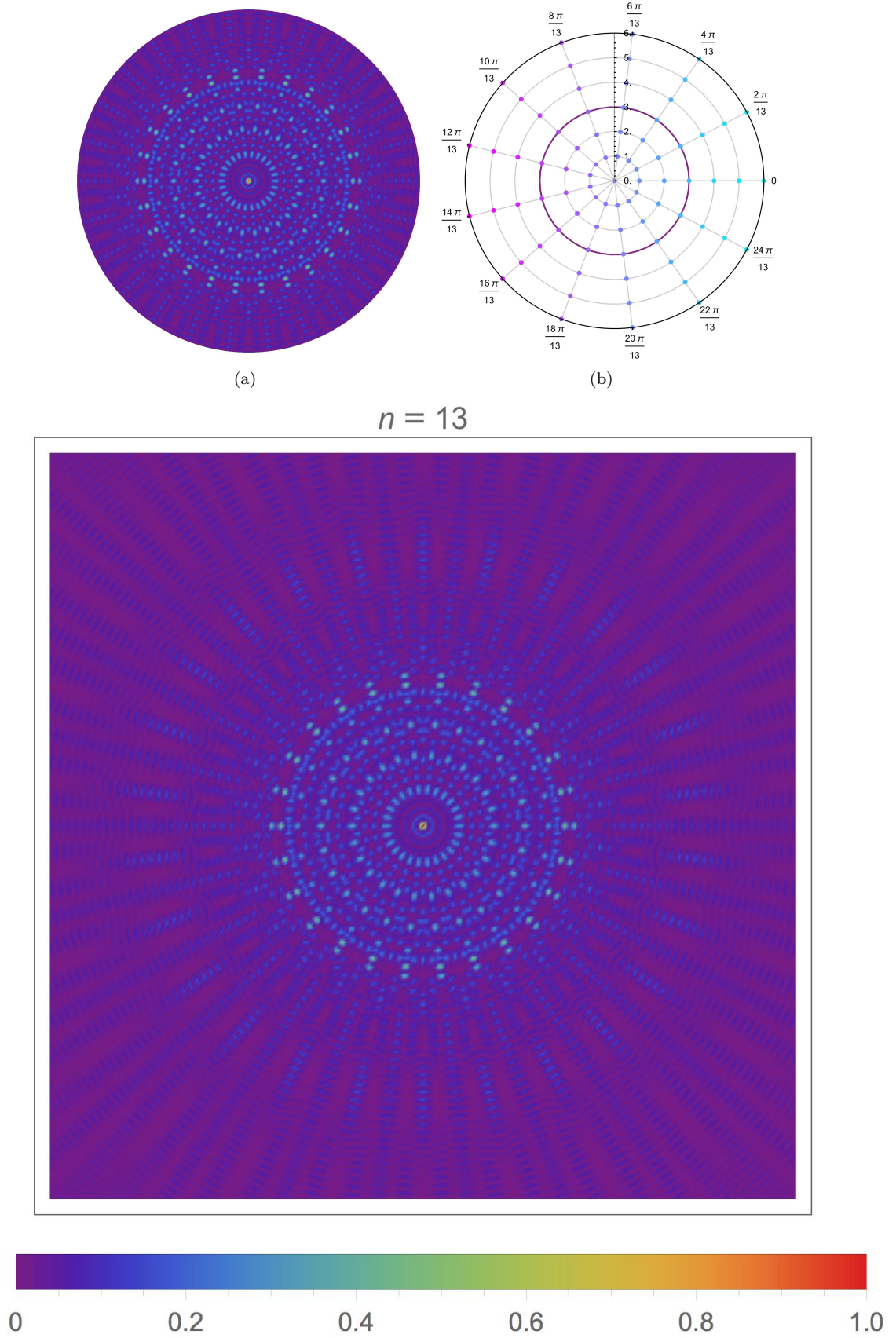


Figure 5.24: (a) Diffraction pattern of a ring delta function sampling for $n=13$, $m \in [-60, 60]$, $s \in [1, 50]$, $\alpha = 25$, $j = 6$ and $r_0 = 3$ (arbitrary units). (b) Graphical representation of the sampling in polar coordinates for a ring delta function with $\theta_p = \frac{2\pi}{13}$.

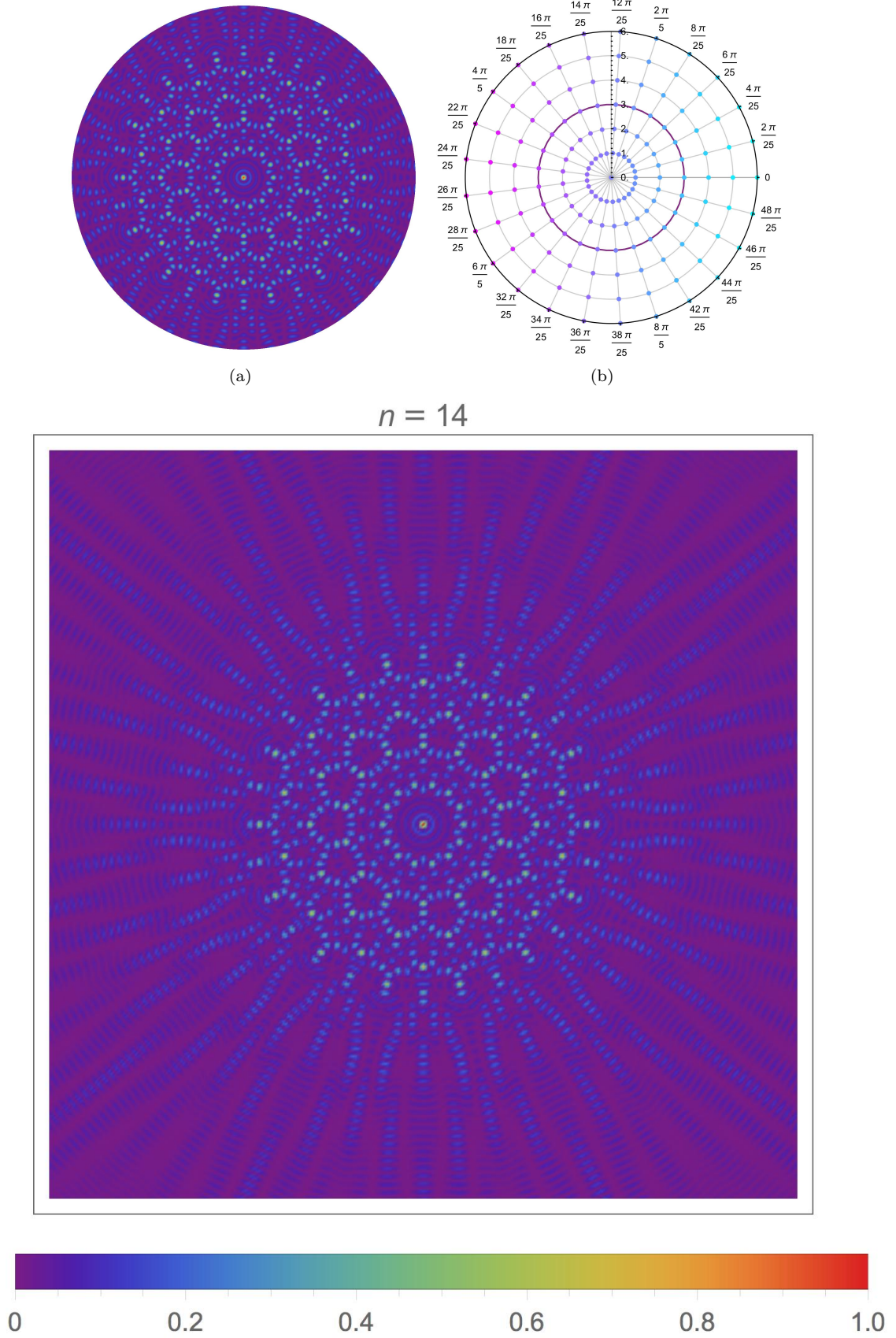


Figure 5.25: (a) Diffraction pattern of a ring delta function sampling for $n=14$, $m \in [-60, 60]$, $s \in [1, 50]$, $\alpha = 25$, $j = 6$ and $r_0 = 3$ (arbitrary units). (b) Graphical representation of the sampling in polar coordinates for a ring delta function with $\theta_p = \frac{\pi}{7}$.

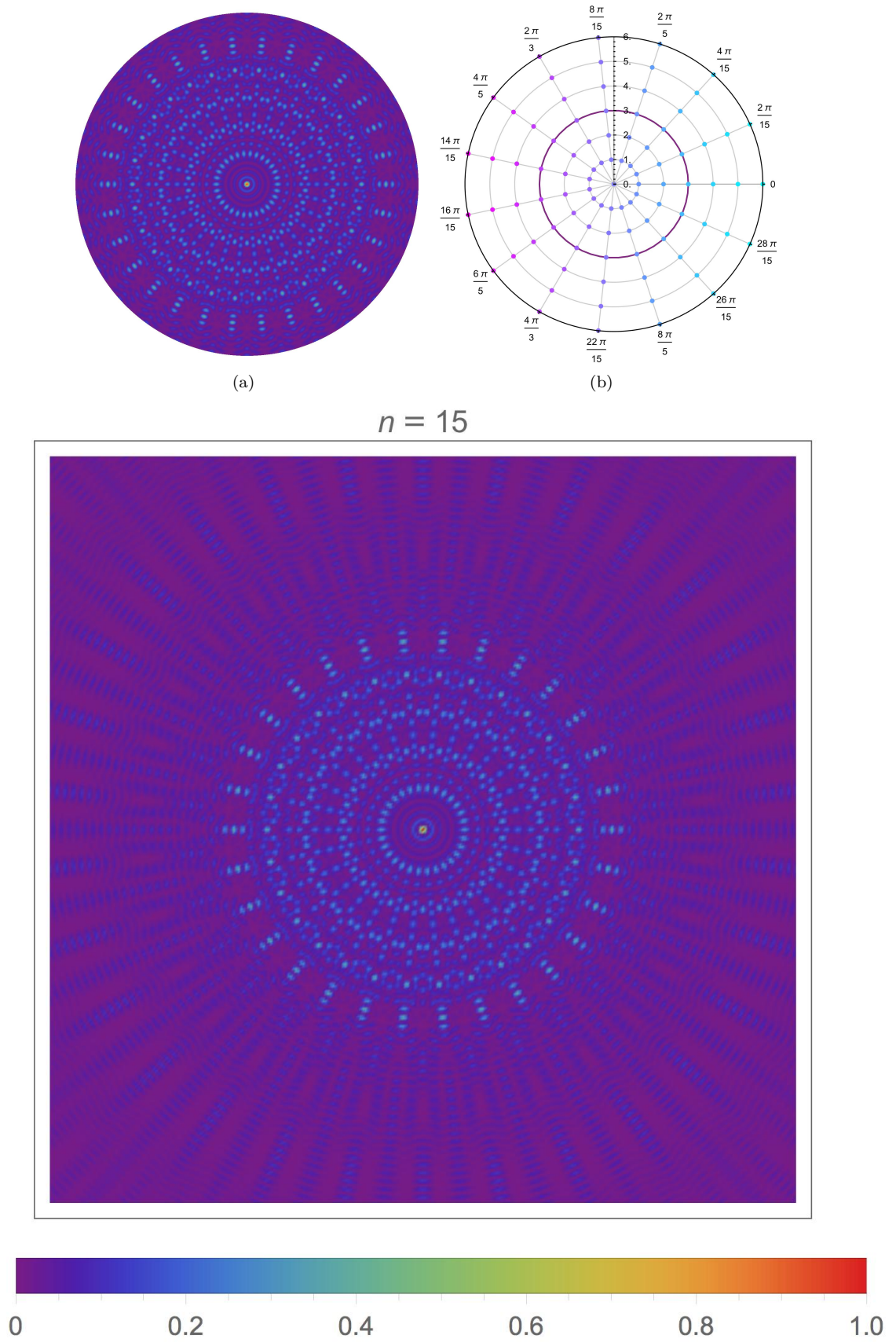


Figure 5.26: (a) Diffraction pattern of a ring delta function sampling for $n=15$, $m \in [-60, 60]$, $s \in [1, 50]$, $\alpha = 25$, $j = 6$ and $r_0 = 3$ (arbitrary units). (b) Graphical representation of the sampling in polar coordinates for a ring delta function with $\theta_p = \frac{2\pi}{15}$.

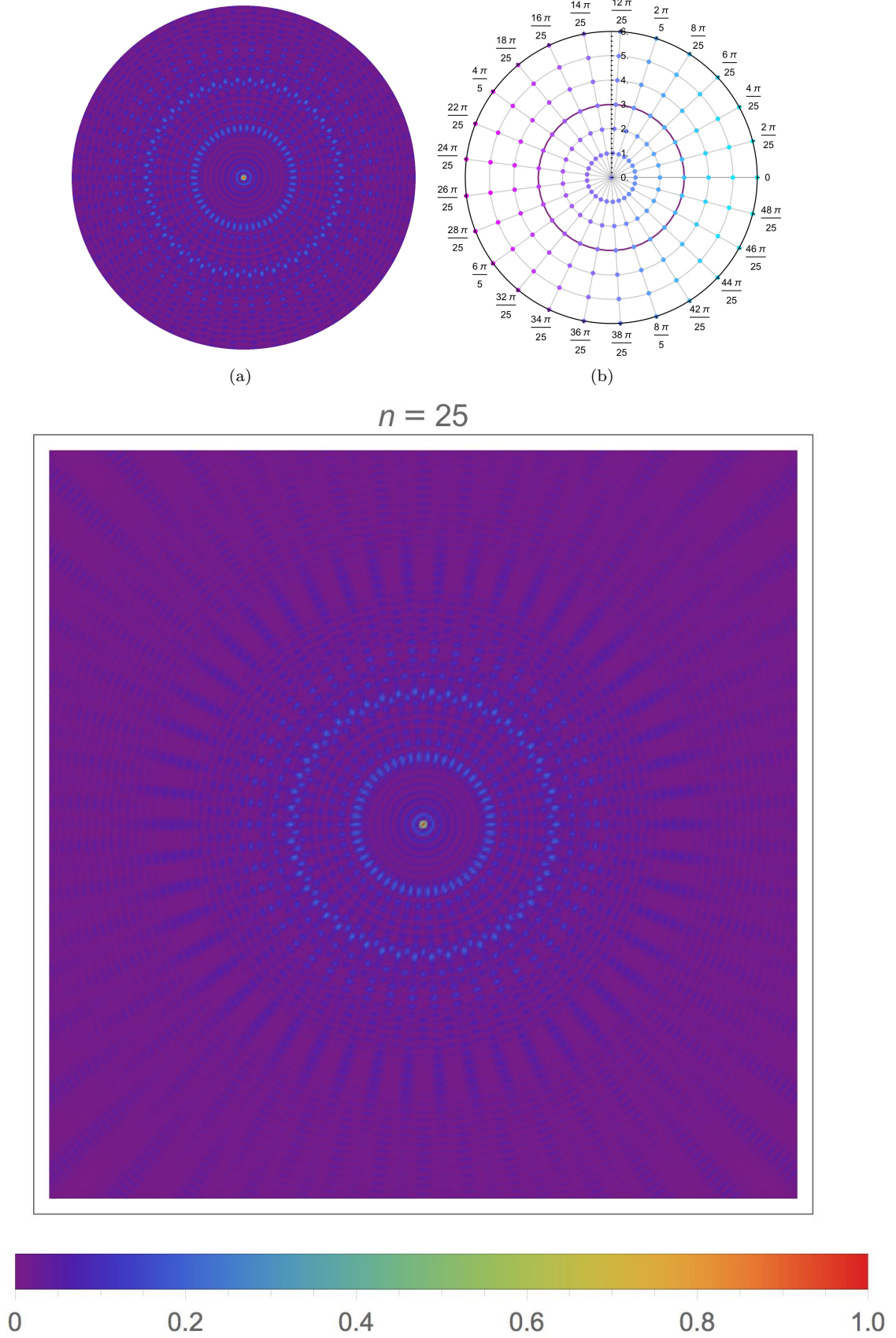


Figure 5.27: (a) Diffraction pattern of a ring delta function sampling for $n=25$, $m \in [-60, 60]$, $s \in [1, 50]$, $\alpha = 25$, $j = 6$ and $r_0 = 3$ (arbitrary units). (b) Graphical representation of the sampling in polar coordinates for a ring delta function with $\theta_p = \frac{2\pi}{25}$.

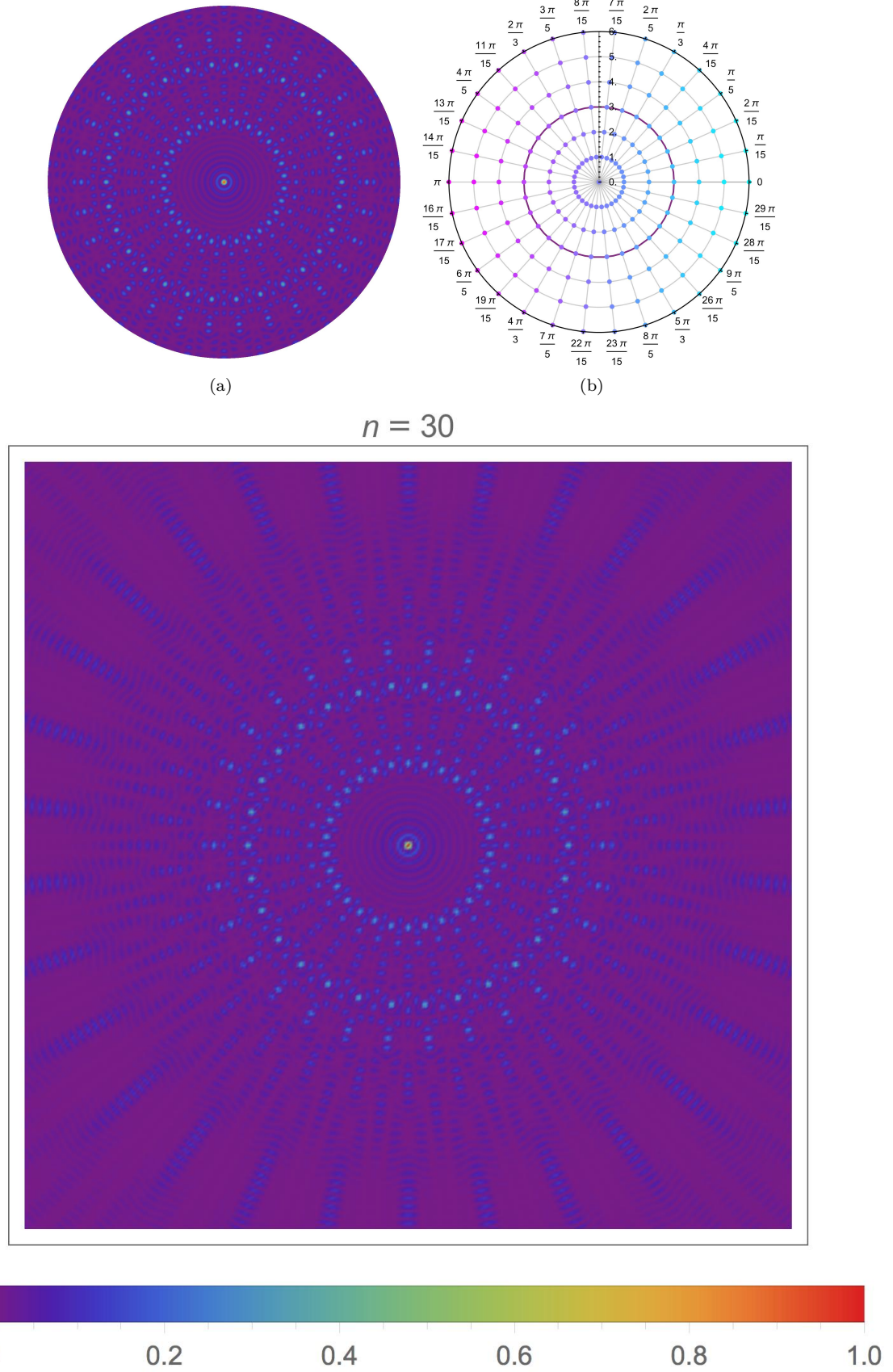


Figure 5.28: (a) Diffraction pattern of a ring delta function sampling for $n=30$, $m \in [-60, 60]$, $s \in [1, 50]$, $\alpha = 25$, $j = 6$ and $r_0 = 3$ (arbitrary units). (b) Graphical representation of the sampling in polar coordinates for a ring delta function with $\theta_p = \frac{\pi}{15}$.

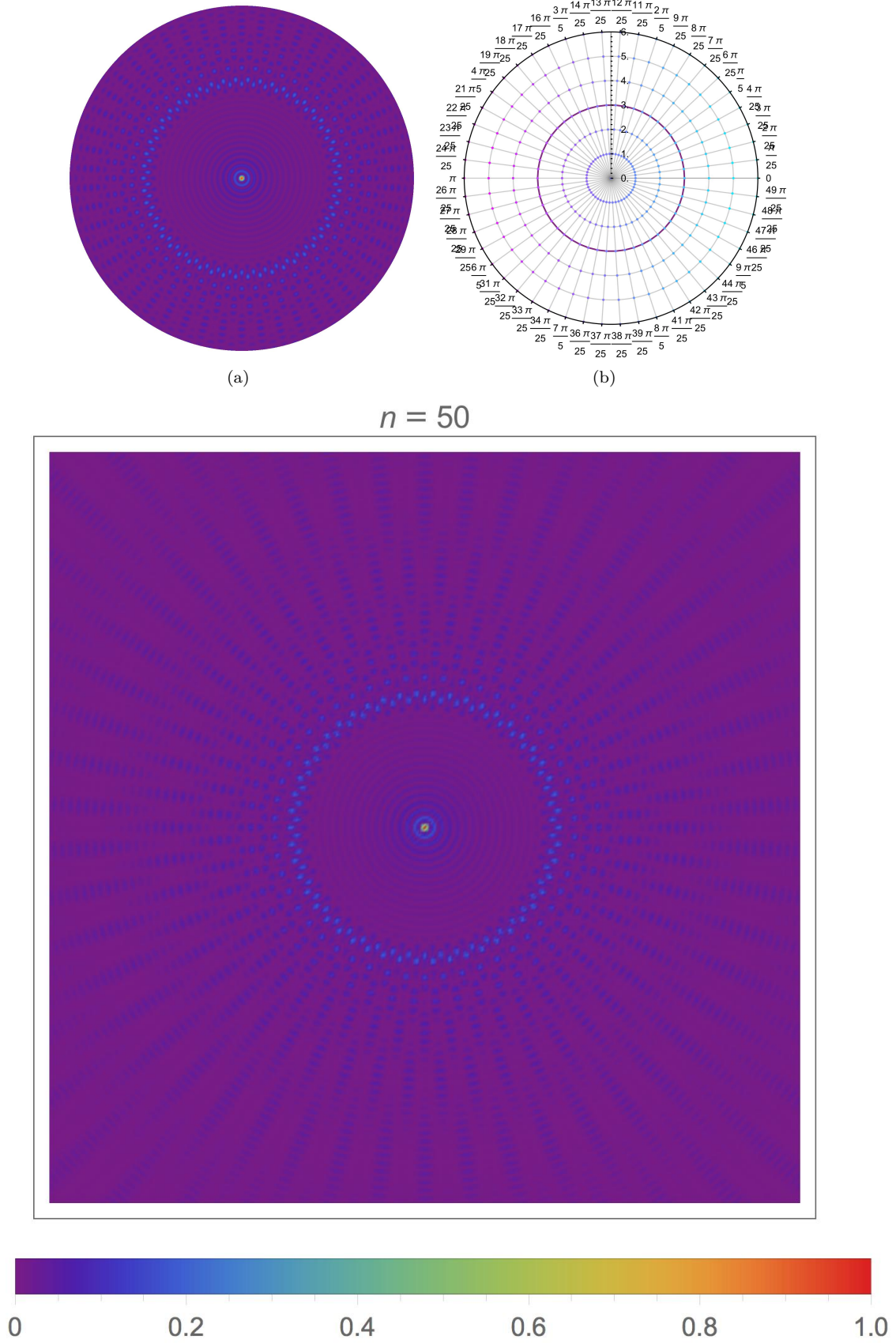


Figure 5.29: (a) Diffraction pattern of a ring delta function sampling for $n=50$, $m \in [-60, 60]$, $s \in [1, 50]$, $\alpha = 25$, $j = 6$ and $r_0 = 3$ (arbitrary units). (b) Graphical representation of the sampling in polar coordinates for a ring delta function with $\theta_p = \frac{\pi}{25}$.

However, the simulation of Eq.(5.7) doesn't seem to have variation with $j = 3, 4, \dots$, this is because in the sampling only for $j = 3$ the deltas of the third radial period of the polar train coincide with the ring. This can be seen clearly using Eq.(4.8), where the polar train is expressed in series but only for the angular part, i.e.,

$$\begin{aligned} t_p(r, \theta) &= \frac{1}{2\pi r} \delta(r) + \sum_{m=-\infty}^{\infty} t_{pm}(r) e^{im\theta} \\ &= \frac{1}{2\pi r} \delta(r) + \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \sum_{j=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{r} \delta(r - jr_p) e^{-imk\theta_p} e^{im\theta} \end{aligned}$$

This time, the sampling is represented by

$$\begin{aligned} \zeta_s(r, \theta) &= \frac{1}{2\pi r} \delta(r - r_0) \cdot \frac{1}{2\pi r} \delta(r) \\ &\quad + \frac{1}{2\pi r} \delta(r - r_0) \cdot \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \sum_{j=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{r} \delta(r - jr_p) e^{-imk\theta_p} e^{im\theta}. \end{aligned} \quad (5.8)$$

The first term is canceled and the Fourier transform is given by

$$\begin{aligned} \tilde{\zeta}_s(\rho, \varphi) &= \sum_{m=-\infty}^{\infty} 2\pi (-i)^m e^{im\varphi} \int_0^{\infty} r dr t_{pm}(r) \frac{1}{2\pi r} \delta(r - r_0) J_m(2\pi\rho r) \\ &= \sum_{m=-\infty}^{\infty} 2\pi (-i)^m e^{im\varphi} \\ &\quad \times \int_0^{\infty} r dr \frac{1}{2\pi} \sum_{j=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{r} \delta(r - jr_p) e^{-imk\theta_p} \frac{1}{2\pi r} \delta(r - r_0) J_m(2\pi\rho r) \\ &= \sum_{m=-\infty}^{\infty} \sum_{k=0}^{n-1} \frac{1}{2\pi} (-i)^m e^{-imk\theta_p} e^{im\varphi} \int_0^{\infty} dr \frac{1}{r} \sum_{j=1}^{\infty} \delta(r - jr_p) \delta(r - r_0) J_m(2\pi\rho r) \\ &= \sum_{m=-\infty}^{\infty} \sum_{k=0}^{n-1} \frac{1}{2\pi} (-i)^m e^{-imk\theta_p} e^{im\varphi} \int_0^{\infty} dr \frac{1}{r} \delta(r - r_0) \delta(r - r_0) J_m(2\pi\rho r) \\ &= \sum_{m=-\infty}^{\infty} \sum_{k=0}^{n-1} \frac{1}{2\pi} (-i)^m e^{-imk\theta_p} e^{im\varphi} \int_0^{\infty} dr \frac{1}{r} \delta(r - r_0) J_m(2\pi\rho r) \\ &= \sum_{m=-\infty}^{\infty} \sum_{k=0}^{n-1} \frac{1}{2\pi r_0} (-i)^m e^{-imk\theta_p} e^{im\varphi} J_m(2\pi\rho r_0). \end{aligned} \quad (5.9)$$

This result shows that there is no dependence on the radial period, it only depends on the

ring's radius and the angular period but this doesn't happen for non-null width functions.

It must be mentioned that the simulations of Eq.(5.9) give the same result as Eq. (5.7).

Given the form of Eq.(5.9) it is not easy to understand if there is a relation between the sampling spectrum and the spectrum of the unsampled object (as in cartesian coordinates).

However, an analysis can be done to try to simplify the result and express it in a more friendly way.

Since $k\theta_p = k\frac{2\pi}{n}$ and the sum goes from 0 to $n - 1$ then

$$\sum_{k=0}^{n-1} e^{-imk\theta_p} = \begin{cases} n & \text{if } m = ln, l \in \mathbb{Z} \\ 0 & \text{if } m \neq ln. \end{cases} \quad (5.10)$$

Thereby Eq.(5.9) turns into

$$\begin{aligned} \tilde{\zeta}_s(\rho, \varphi) &= \frac{n}{2\pi r_0} \sum_{l=-\infty}^{\infty} (-i)^{ln} e^{iln\varphi} J_{ln}(2\pi\rho r_0) \\ &= \frac{n}{2\pi r_0} \left[J_0(2\pi\rho r_0) + \sum_{l=-\infty}^{-1} (-i)^{ln} e^{iln\varphi} J_{ln}(2\pi\rho r_0) + \sum_{l=1}^{\infty} (-i)^{ln} e^{iln\varphi} J_{ln}(2\pi\rho r_0) \right] \\ &= \frac{n}{2\pi r_0} \left[J_0(2\pi\rho r_0) + \sum_{l=1}^{\infty} (-i)^{-ln} e^{-iln\varphi} J_{-ln}(2\pi\rho r_0) + \sum_{l=1}^{\infty} (-i)^{ln} e^{iln\varphi} J_{ln}(2\pi\rho r_0) \right] \\ &= \frac{n}{2\pi r_0} \left[J_0(2\pi\rho r_0) + \sum_{l=1}^{\infty} (i)^{ln} e^{-iln\varphi} (-1)^{ln} J_{ln}(2\pi\rho r_0) + \sum_{l=1}^{\infty} (-i)^{ln} e^{iln\varphi} J_{ln}(2\pi\rho r_0) \right] \\ &= \frac{n}{2\pi r_0} \left[J_0(2\pi\rho r_0) + \sum_{l=1}^{\infty} (-i)^{ln} e^{-iln\varphi} J_{ln}(2\pi\rho r_0) + \sum_{l=1}^{\infty} (-i)^{ln} e^{iln\varphi} J_{ln}(2\pi\rho r_0) \right] \\ &= \frac{n}{2\pi r_0} \left[J_0(2\pi\rho r_0) + 2 \sum_{l=1}^{\infty} (-i)^{ln} J_{ln}(2\pi\rho r_0) \cos(ln\varphi) \right] \end{aligned}$$

Chapter 6

Method to form first-order Bessel beams

6.1 Overlap of Circ functions

The Fourier transform of an circular aperture gives, as a result first-order Bessel function over its argument. Let's consider n circular slits for the interferometer, which are distributed over an imaginary circle with radio $r_1 + R_1$ as shown in Fig. 6.1

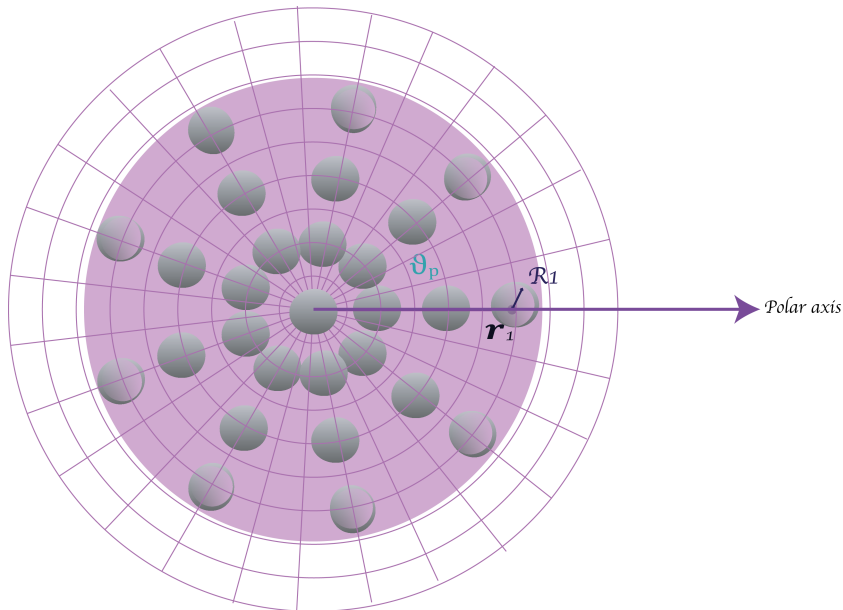


Figure 6.1: Overlap of circ functions.

Just like the previous chapter, every slit can be described as a shifted circle function. How-

ever, due to the configuration given, this time both the radius and the angle change. For Fig. 6.1 let's consider the radial variation as r_{1p} , $2r_{1p}$ and $3r_{1p}$, where $3r_{1p} = r_1$, but in general we can consider lr_{1p} , $l = 1, 2, \dots, \beta$ with $\beta r_{1p} = r_1$ for some β . Also

$$k\theta_p = k \frac{2\pi}{n}; \quad k = 0, \dots, n-1. \quad (6.1)$$

Then, for each slit shifted according to $\vec{r}_k = \vec{r}_{lk}(lr_{1p}, k\theta_p)$ we have

$$\begin{aligned} \mathcal{F} \left\{ \text{circ} \left(\frac{r - \vec{r}_{lk}}{R_1} \right) \right\} &= e^{-i2\pi \rho l r_{1p} \cos(k\theta_p - \varphi)} 2\pi R_1^2 \frac{J_1(2\pi \rho R_1)}{2\pi \rho R_1} \\ &= \sum_{m=-\infty}^{\infty} (-i)^m e^{im\varphi} e^{-imk\theta_p} J_m(2\pi \rho l r_{1p}) 2\pi R_1^2 \frac{J_1(2\pi \rho R_1)}{2\pi \rho R_1}. \end{aligned} \quad (6.2)$$

If the configuration of the Fig. (6.1) is followed, one slit at the origin needs to be added, whose Fourier transform is $2\pi R_1^2 \frac{J_1(2\pi \rho R_1)}{2\pi \rho R_1}$. Both results can be divided by a factor πR_1^2 , then the Fourier transform of the complete configuration is given by

$$\begin{aligned} \mathcal{F} \left\{ \sum_{l=0}^{\beta} \sum_{k=0}^{n-1} \text{circ} \left(\frac{r - \vec{r}_{lk}}{R_1} \right) \right\} &= 2 \frac{J_1(2\pi \rho R_1)}{2\pi \rho R_1} \left[\sum_{l=0}^{\beta} \sum_{k=0}^{n-1} e^{-i2\pi \rho l r_{1p} \cos(k\theta_p - \varphi)} + 1 \right] \\ &= 2 \frac{J_1(2\pi \rho R_1)}{2\pi \rho R_1} \left[\sum_{m=-\infty}^{\infty} \sum_{l=0}^{\beta} \sum_{k=0}^{n-1} (-i)^m e^{im\varphi} e^{-imk\theta_p} J_m(2\pi \rho l r_{1p}) + 1 \right]. \end{aligned} \quad (6.3)$$

Using the squared module of Eq.(6.3) in exponential form, the diffraction pattern for different values of n is shown below.

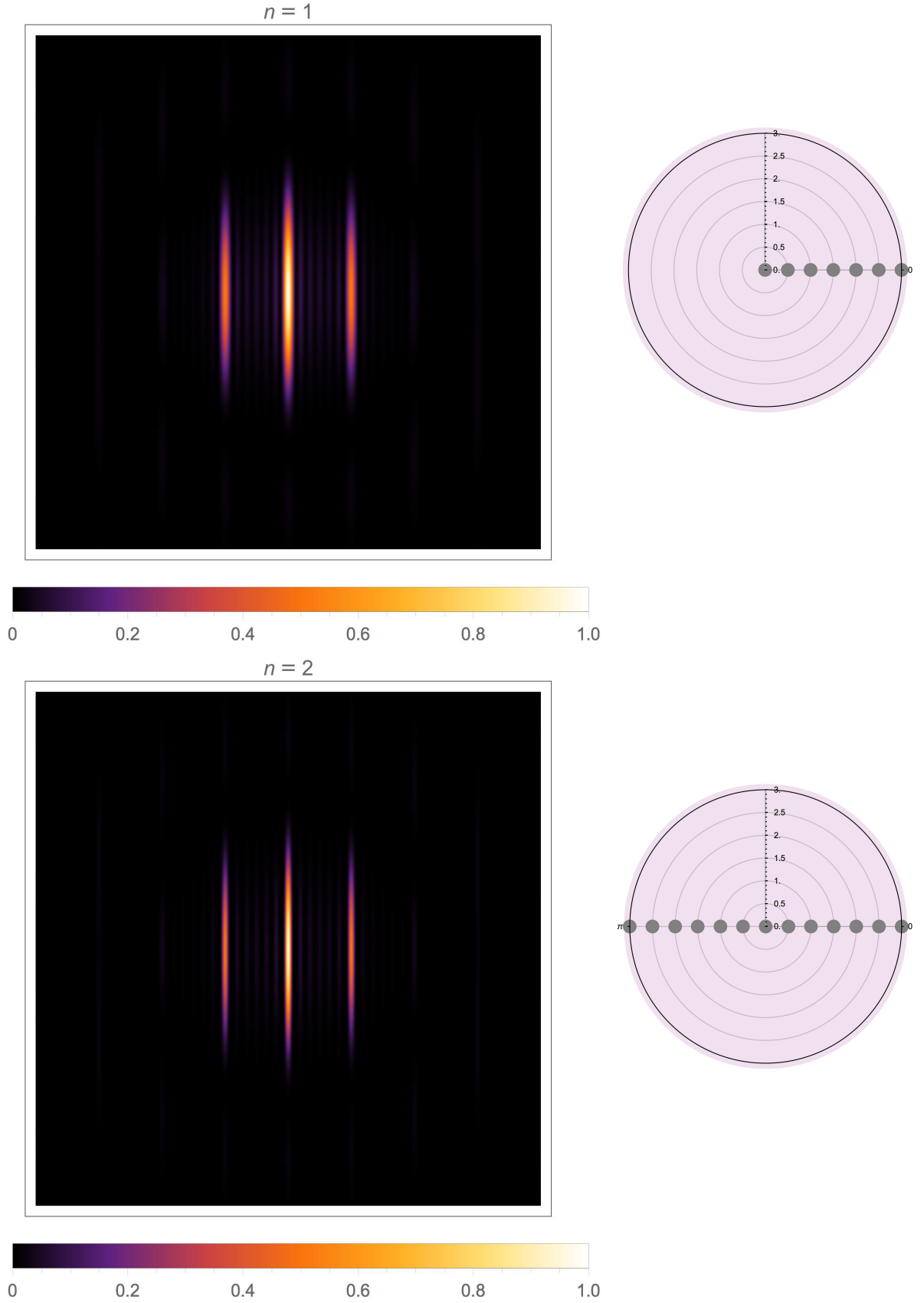


Figure 6.2: The figures on the left show the diffraction patterns for $R_1 = 0.125$, $r_{1p} = 0.5$ and $\beta = 6$ (arbitrary units). On the color scale 0 represents the minimum value of intensity and 1 represents the maximum value. The figures on the right show the distribution of the circular slits on an imaginary circle (in purple).

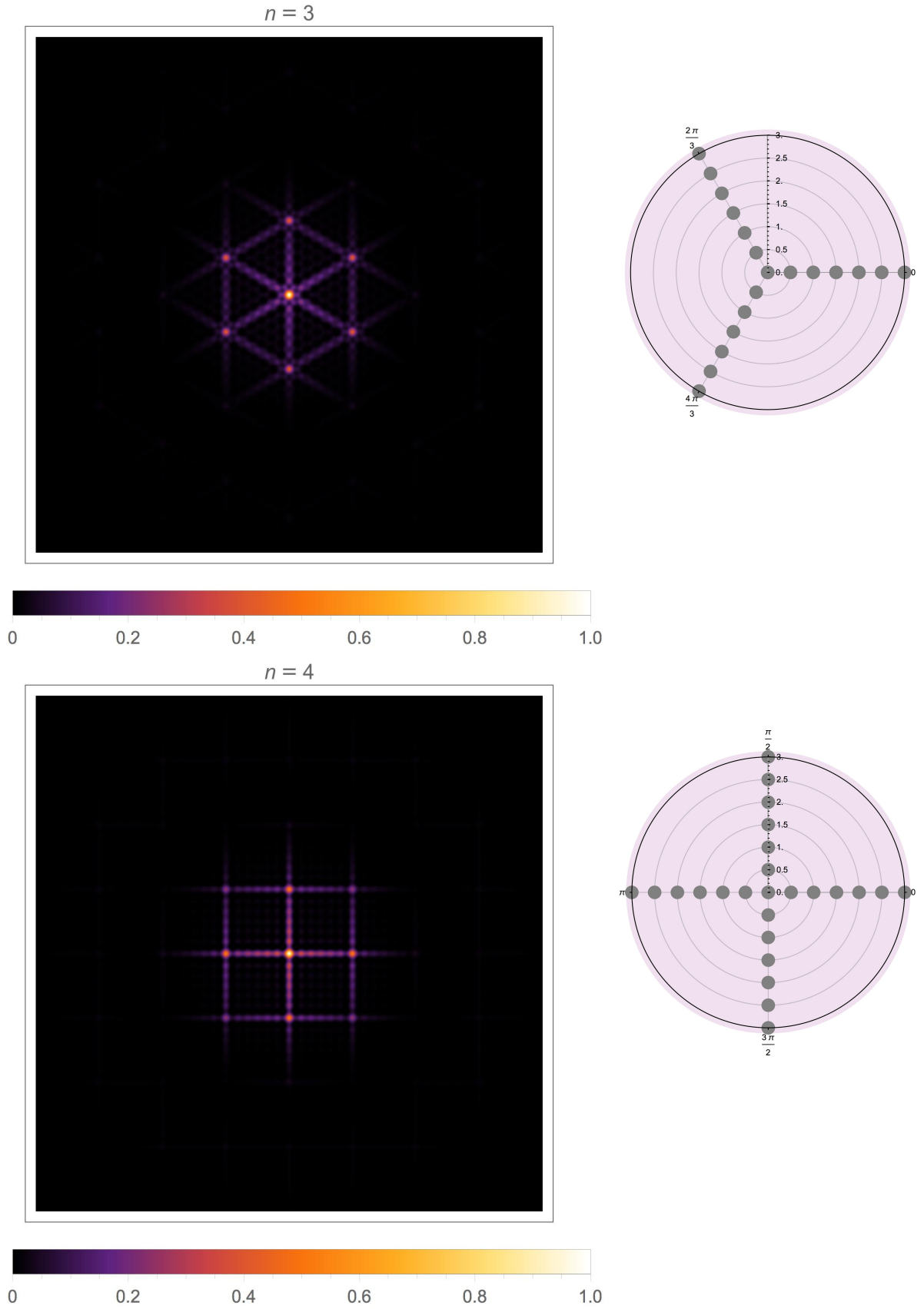


Figure 6.3: The figures on the left show the diffraction patterns for $R_1 = 0.125$, $r_{1p} = 0.5$ and $\beta = 6$ (arbitrary units). On the color scale 0 represents the minimum value of intensity and 1 represents the maximum value. The figures on the right show the distribution of the circular slits on an imaginary circle (in purple).

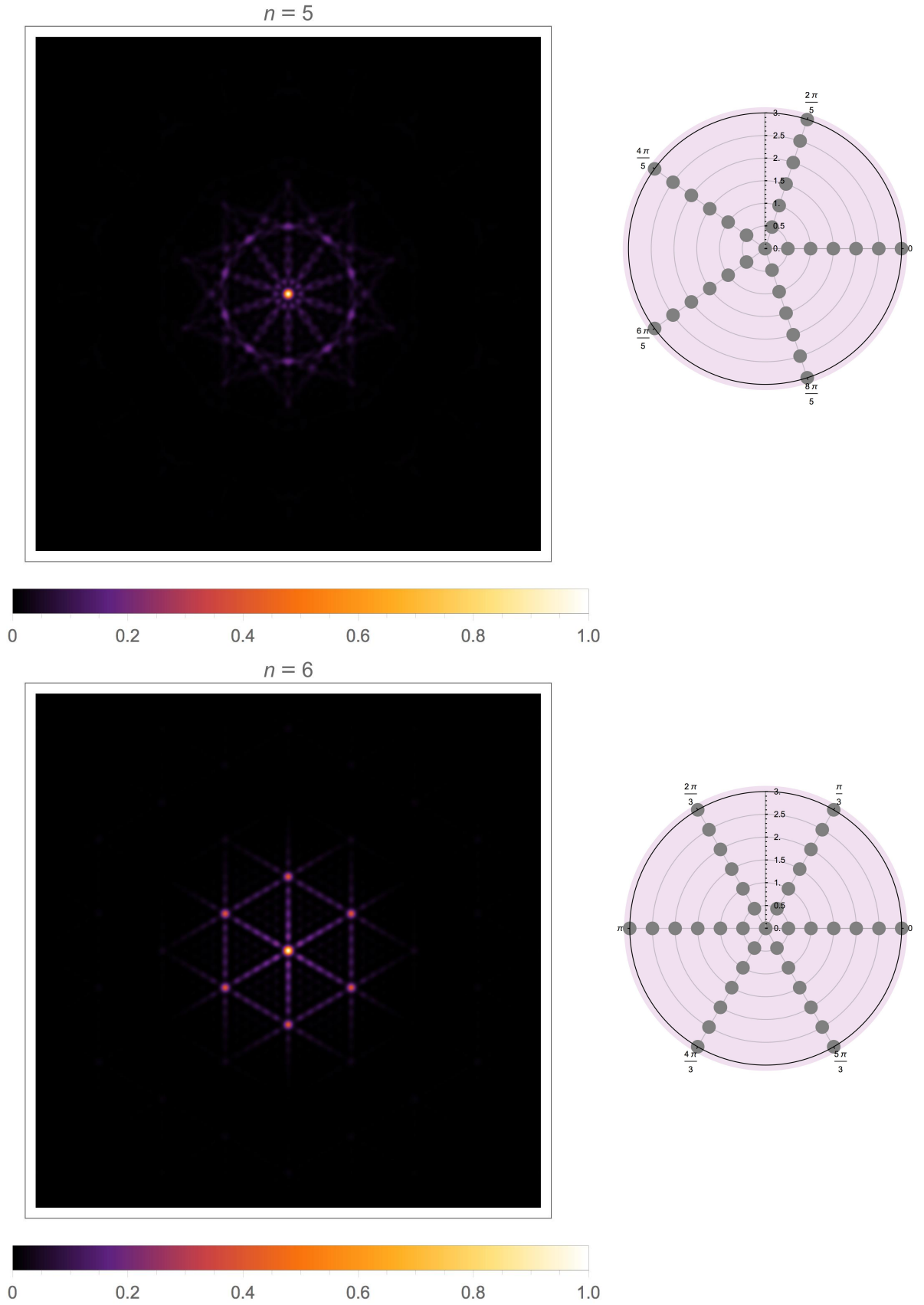


Figure 6.4: The figures on the left show the diffraction patterns for $R_1 = 0.125$, $r_{1p} = 0.5$ and $\beta = 6$ (arbitrary units). On the color scale 0 represents the minimum value of intensity and 1 represents the maximum value. The figures on the right show the distribution of the circular slits on an imaginary circle (in purple).

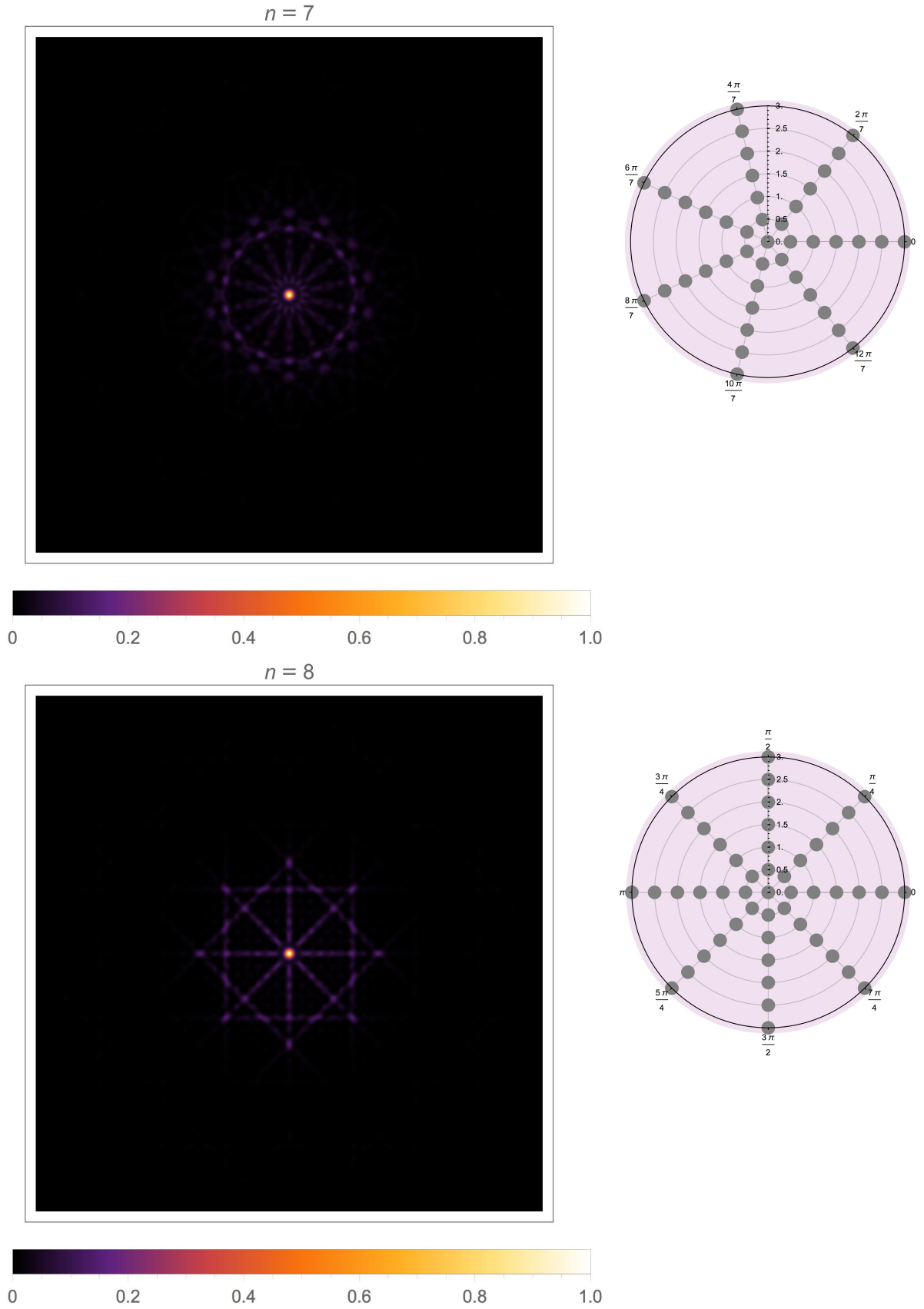


Figure 6.5: The figures on the left show the diffraction patterns for $R_1 = 0.125$, $r_{1p} = 0.5$ and $\beta = 6$ (arbitrary units). On the color scale 0 represents the minimum value of intensity and 1 represents the maximum value. The figures on the right show the distribution of the circular slits on an imaginary circle (in purple).

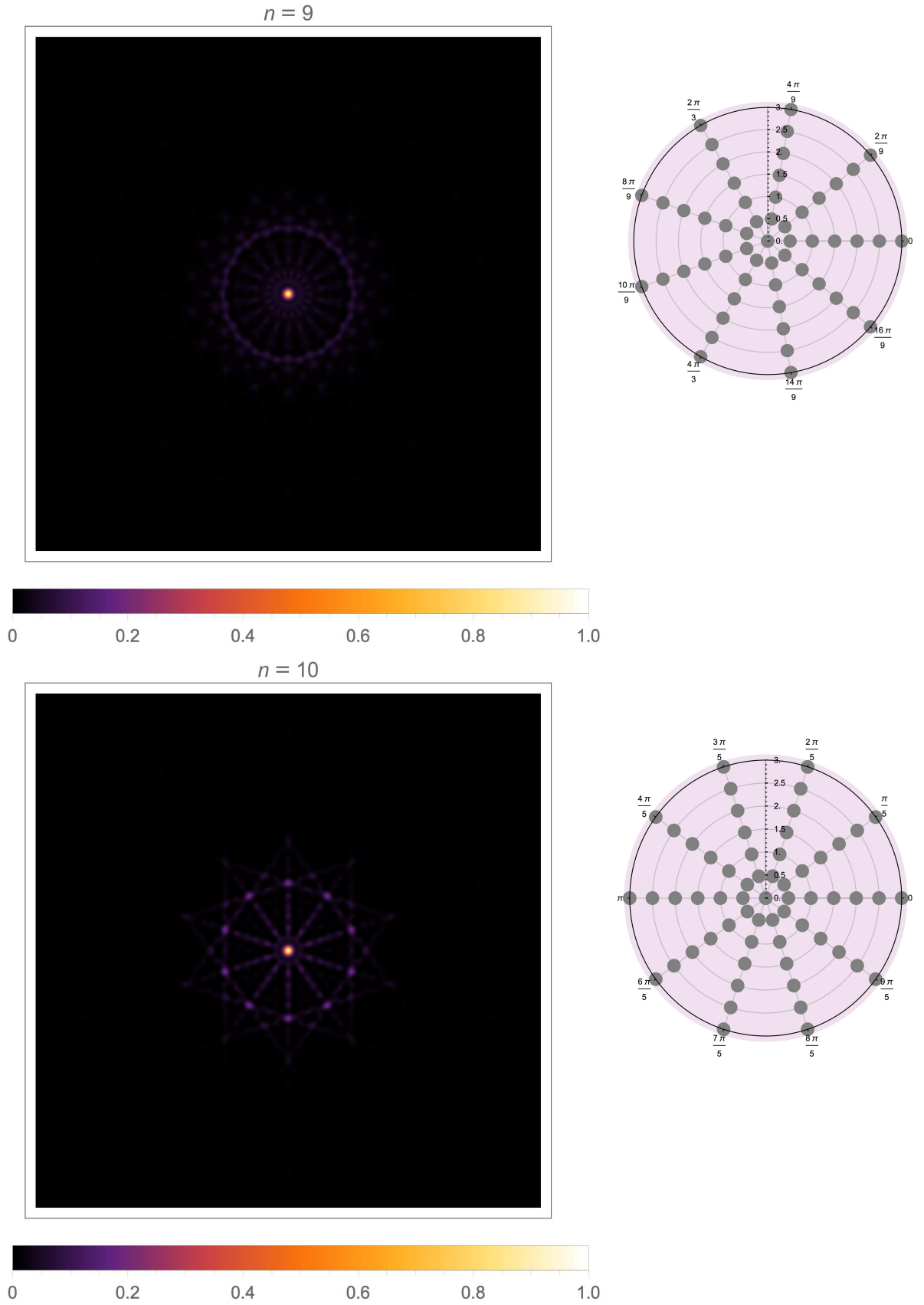


Figure 6.6: The figures on the left show the diffraction patterns for $R_1 = 0.125$, $r_{1p} = 0.5$ and $\beta = 6$ (arbitrary units). On the color scale 0 represents the minimum value of intensity and 1 represents the maximum value. The figures on the right show the distribution of the circular slits on an imaginary circle (in purple).

6.2 Sampling of a circ function

The sampling of a circ function is expressed as

$$\zeta_s(r, \theta) = \text{circ}\left(\frac{r}{r_1 + R_1}\right) \cdot t_p(r, \theta),$$

where, by Eq.(4.16)

$$\begin{aligned} t_p(r, \theta) &= \sum_{s=1}^{\infty} \frac{1}{\pi \alpha^2 J_1^2(\lambda_{0s})} J_0\left(\lambda_{0s} \frac{r}{\alpha}\right) \\ &\quad + \sum_{m=-\infty}^{\infty} \sum_{s=1}^{\infty} \sum_{k=0}^{n-1} \sum_{j=1}^{\alpha} e^{-imk\theta_p} \frac{1}{\pi \alpha^2 J_1^2(\lambda_{0s})} J_0\left(\lambda_{0s} \frac{jr_p}{\alpha}\right) J_0\left(\lambda_{0s} \frac{r}{\alpha}\right) e^{im\theta} \\ &= \sum_{s=1}^{\infty} \frac{1}{\pi \alpha^2 J_1^2(\lambda_{0s})} J_0\left(\lambda_{0s} \frac{r}{\alpha}\right) + \sum_{m=-\infty}^{\infty} t_{pm}(r) e^{im\theta}. \end{aligned}$$

Then,

$$\begin{aligned} \zeta_s(r, \theta) &= \text{circ}\left(\frac{r}{r_1 + R_1}\right) \cdot \sum_{s=1}^{\infty} \frac{1}{\pi \alpha^2 J_1^2(\lambda_{0s})} J_0\left(\lambda_{0s} \frac{r}{\alpha}\right) \\ &\quad + \text{circ}\left(\frac{r}{r_1 + R_1}\right) \cdot \sum_{m=-\infty}^{\infty} \sum_{s=1}^{\infty} \sum_{k=0}^{n-1} \sum_{j=1}^{\alpha} e^{-imk\theta_p} \frac{1}{\pi \alpha^2 J_1^2(\lambda_{0s})} J_0\left(\lambda_{0s} \frac{jr_p}{\alpha}\right) J_0\left(\lambda_{0s} \frac{r}{\alpha}\right) e^{im\theta}. \end{aligned} \tag{6.4}$$

Just like in the previous section we have to apply the definition of Fourier transform of a product of two radially symmetric functions for the first part and the definition of Fourier transform of a product of a radially symmetric function with a function without radial symmetry for the second part, this is,

$$\begin{aligned} \tilde{\zeta}_s(\rho, \varphi) &= 2\pi \int_0^{\infty} r dr \left[\text{circ}\left(\frac{r}{r_1 + R_1}\right) \cdot \sum_{s=1}^{\infty} \frac{1}{\pi \alpha^2 J_1^2(\lambda_{0s})} J_0\left(\lambda_{0s} \frac{r}{\alpha}\right) \right] J_0(2\pi \rho r) \\ &\quad + \sum_{m=-\infty}^{\infty} 2\pi (-i)^m e^{im\varphi} \int_0^{\infty} r dr t_{pm}(r) \text{circ}\left(\frac{r}{r_1 + R_1}\right) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{s=1}^{\infty} \frac{2}{\alpha^2 J_1^2(\lambda_{0s})} \int_0^{r_1+R_1} r dr J_0 \left(\lambda_{0s} \frac{r}{\alpha} \right) J_0(2\pi\rho r) \\
 &+ \sum_{m=-\infty}^{\infty} 2\pi(-i)^m e^{im\varphi} \sum_{s=1}^{\infty} \sum_{k=0}^{n-1} \sum_{j=1}^{\alpha} e^{-imk\theta_p} \frac{1}{\pi\alpha^2 J_1^2(\lambda_{0s})} J_0 \left(\lambda_{0s} \frac{jr_p}{\alpha} \right) \\
 &\times \int_0^{r_1+R_1} r dr J_0 \left(\lambda_{0s} \frac{r}{\alpha} \right) J_m(2\pi\rho r). \tag{6.5}
 \end{aligned}$$

The first integral in Eq.(6.5) can be evaluated with the help of a formula given by [11], namely,

$$\int_0^x t dt J_\nu(kt) J_\nu(lt) = \frac{x}{k^2 - l^2} [kJ_{\nu+1}(kx)J_\nu(lx) - lJ_\nu(kx)J_{\nu+1}(lx)]. \tag{6.6}$$

Letting $x = r_1 + R_1$, $t = r$, $k = \frac{\lambda_{0s}}{\alpha}$, $l = 2\pi\rho$, $\nu = 0$ etc., enables us to write

$$\begin{aligned}
 &\int_0^{r_1+R_1} r dr J_0 \left(\lambda_{0s} \frac{r}{\alpha} \right) J_0(2\pi\rho r) \\
 &= \frac{r_1 + R_1}{\left(\frac{\lambda_{0s}}{\alpha}\right)^2 - (2\pi\rho)^2} \left[\frac{\lambda_{0s}}{\alpha} J_1 \left(\frac{\lambda_{0s}}{\alpha} (r_1 + R_1) \right) J_0(2\pi\rho(r_1 + R_1)) - 2\pi\rho J_0 \left(\frac{\lambda_{0s}}{\alpha} (r_1 + R_1) \right) J_1(2\pi\rho(r_1 + R_1)) \right], \tag{6.7}
 \end{aligned}$$

therefore

$$\begin{aligned}
 \tilde{\zeta}_s(\rho, \varphi) &= \sum_{s=1}^{\infty} \frac{2}{\alpha^2 J_1^2(\lambda_{0s})} \frac{r_1 + R_1}{\left(\frac{\lambda_{0s}}{\alpha}\right)^2 - (2\pi\rho)^2} \\
 &\times \left[\frac{\lambda_{0s}}{\alpha} J_1 \left(\frac{\lambda_{0s}}{\alpha} (r_1 + R_1) \right) J_0(2\pi\rho(r_1 + R_1)) - 2\pi\rho J_0 \left(\frac{\lambda_{0s}}{\alpha} (r_1 + R_1) \right) J_1(2\pi\rho(r_1 + R_1)) \right] \\
 &+ \sum_{m=-\infty}^{\infty} 2\pi(-i)^m e^{im\varphi} \sum_{s=1}^{\infty} \sum_{k=0}^{n-1} \sum_{j=1}^{\alpha} e^{-imk\theta_p} \frac{1}{\pi\alpha^2 J_1^2(\lambda_{0s})} J_0 \left(\lambda_{0s} \frac{jr_p}{\alpha} \right) \\
 &\times \int_0^{r_1+R_1} r dr J_0 \left(\lambda_{0s} \frac{r}{\alpha} \right) J_m(2\pi\rho r). \tag{6.8}
 \end{aligned}$$

Due to the complexity of the theoretical result, simulations aren't presented for this case.

Chapter 7

Conclusions and future work

The conclusions of the present work and some future work options that may continue to be developed as a result of the investigation are presented below.

7.1 Conclusions

- Two methods in polar coordinates to form both zero-order and first-order Bessel beams were developed.
- For the theoretical part of zero-order Bessel beams a numerical simulation was implemented.
- The simulations for the two methods to form zero-order Bessel beams show significant similarities.
- Due to the complexity of the theoretical development results of the first-order Bessel beams for the sampling case, only the simulation results for the first method are shown.

7.2 Future work

- To express the results in simpler ways such that the behavior observed in the simulations can be deduced in a simple way from the theoretical part.
- To know with what value of n and the dimensions of the slits, the sought beams can be found and thus, proceed with the experimental part.

- To look for the possibility of creating another type of beam.
- To evaluate the possibility of modulating amplitude and phase of the slits.

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