BENEMÉRITA UNIVERSIDAD AUTÓNOMA DE PUEBLA

INSTITUTO DE FÍSICA “LUIS RIVERA TERRAZAS”

“NUEVO FORMALISMO CANÓNICO PARA TEORÍAS DE GRAVEDAD DE ALTO ORDEN: NUEVA GRAVEDAD MASIVA EN TRES DIMENSIONES Y GRAVEDAD λR EN CUATRO DIMENSIONES.”

TESIS

QUE PARA OBTENER EL GRADO DE

DOCTOR EN CIENCIAS
(FÍSICA)

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JUNIO 2025

Agradecimientos

Caminar por la ciencia es, por momentos, áspero y solitario, y en otros, emocionante y liviano. Si bien el final del camino es bastante incierto, el recorrerlo acompañado ayuda a continuar.

Por estar siempre conmigo, quiero dar las gracias a mis padres, Pedro y Beny, que han dedicado su vida a motivar la mía.

A mis hermanas, Paola y Dulce, con quienes he compartido todas las emociones de crecer.

A Juan Ramón, por ayudarme siempre a recordar el camino.

A Dania, por compartir su vida con alguien con poco tiempo y espacio.

A José Armando y Valeriano, por ser los amigos que permanecerán hasta el final de los días.

A mi asesor, el Dr. Escalante, por mostrarme constantemente la brecha por donde seguir.

Al Conacyt, que representa la contribución de todos los mexicanos, por financiar todas las tardes de café y pensamiento.

A todos ustedes, muchas gracias.

Ganesha aceptó escribir el manuscrito bajo el dictado de Viasa, únicamente a condición de que este nunca hiciera una pausa en su recitación. Viasa entonces puso otra condición: que antes de escribir, Ganesha asimilara lo que Viasa recitaba. Viasa entonces decía un verso difícil de entender para tener algún respiro en su continuo dictado.

A mis padres.

Resumen

En este trabajo se realiza el análisis Hamiltoniano a nivel perturbativo de dos teorías de gravedad de alto orden : Nueva Gravedad Masiva (NMG) en 2+1 dimensiones y Gravedad λR en 3+1 dimensiones. En ambos casos se obtienen las simetrías, en particular los conjuntos de restricciones de primera y segunda clase, y se construyen los paréntesis de Dirac. Además, se abordan cuestiones fundamentales para estas teorías utilizando el contenido de restricciones. En el caso de NMG se descarta la presencia de grados de libertad fantasma, lo que elimina una de las barreras hacia su posible cuantización. Por otra parte, para la gravedad λR, que es el límite a bajas energías de la gravedad de Hořava (HG), se analiza su cercanía con Relatividad General (GR). Esto es fundamental para cualquier teoría que pretenda generalizar a esta última, y ha sido motivo de importantes debates en la literatura debido a que HG supone un espacio-tiempo anisótropo, fundamentalmente diferente al de GR.

Abstract

In this work, the Hamiltonian analysis at the perturbative level of two higher-order gravity theories is performed, namely New Massive Gravity (NMG) in 2+1 dimensions and λR in 3+1 dimensions. In both cases, first and second-class constraint sets are obtained, and the Dirac brackets are constructed. In addition, fundamental issues for these theories are addressed using the constraint content. In the case of NMG, the presence of ghost degrees of freedom is ruled out, thus removing one of the barriers to its possible quantization. On the other hand, for λR gravity, which is the low-energy limit of Hořava gravity (HG), its closeness to General Relativity (GR) is analyzed. This is fundamental for any theory that intends to generalize the latter, and has been the subject of important debates in the literature since HG assumes an anisotropic spacetime, that is fundamentally different from that of GR.

Publicaciones

- Escalante, A., Ocaña-García, P.F. *Hamiltonian analysis for new massive gravity.* European Physical Journal C 83, 1034 (2023).
https://doi.org/10.1140/epjc/s10052-023-12157-8

- Escalante, Alberto, and P. Fernando Ocaña-García. *Hamiltonian analysis for perturbative λR gravity.* Annals of Physics 465 (2024): 169662.
https://doi.org/10.1016/j.aop.2024.169662

- Escalante, Alberto, and P. Fernando Ocaña-García. *New canonical analysis for consistent extension of λR.* Annals of Physics (2025): 169916.
https://doi.org/10.1016/j.aop.2024.169916

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Chapter 1

Introduction

From millennia of contemplating natural phenomena in deep awe, from centuries of defending the scientific method as our unique tool for generating reliable knowledge, and from decades of being able to observe at ever sharper scales, the fascinating evolution promoted in our paradigms has led to the theories that currently encompass our most profound understanding of the universe: the Standard Model (SM) and General Relativity (GR). These theories describe the fundamental electroweak, strong, and gravitational interactions; and while they agree satisfactorily with practically all experimental data, there is one important issue that we cannot overlook.

In contrast to SM theories, GR, the theory describing gravity as spacetime geometry, has a limited range of applicability generated by its difficulties in incorporating a complete quantization scheme. To mention briefly a few, in the sense of quantum field theory (QFT) we have the problem of non-renormalizability: either on its own or coupled to generic matter fields, the perturbative treatment of GR leads to the inescapable appearance of infinitely many UV divergences that spoil the predictive power of the theory [1–5]. On the other hand, by assuming that spacetime has an elementary quantum granular structure at the Planck scale, loop quantum gravity (LQG) cuts off the aforementioned UV infinities of QFT. However, this in turn raises the problem of recovering GR in some suitable continuum classical limit [6]. Another alternative, highly mathematical and fundamentally different from LQG, is string theory (ST), which considers the behavior of extended objects within a flat (Minkowskian) spacetime. This is intended as a framework for a unified theory of all interactions, but here we are faced with the impossibility of choosing one among many possible theories, resulting in a lack of predictive power [7].

All this means that spacetime phenomena occurring at high-energy and short-distance scales, such as the singularities at black holes or at the very beginning of the universe, are beyond our understanding. It is also important to mention that while a quantum gravity theory (QGT) could help us to clarify some observational data [8], the scenarios to test its most important predictions are beyond our observational and experimental capabilities. Aware of this important difficulty, the community's interest in achieving QGT is justified by the theoretical insights involved. It is well known that the underlying assumptions about the nature of spacetime made by quantum mechanics and GR are profoundly incompatible; therefore, there are fundamental questions concerning the short-distance spacetime behavior that QGT could answer. In this sense, the proposal of an effective field theory of quantum gravity that accounts for phenomena below a certain energy limit is not sufficient [9].
Concerning the QFT approach, it has been demonstrated that the problem of renormalizability of GR can be circumvented by first modifying its classical action with the aim of improving the UV behavior [10]. This has opened up an important branch of the study of modified gravity theories [11]. In this way, the present thesis work focuses on the canonical analysis of some of these alternative theories to describe gravity at the classical level. Our goal is to extract their dynamical properties and to contrast them with those of GR. Since GR is amply supported by observational evidence, any theory that claims to generalize it must be paired with it in the corresponding IR limit. For our purpose, we use the well-known Hamiltonian analysis to gauge theories [12, 14, 15].

1.1 Renormalizable Gravity Theories: Higher-Order Theories

As mentioned above, the problem of non-renormalizability of GR has to do with the impossibility of removing the divergences that appear when it is considered as a QFT in the weak-field approximation on a fixed background. The origin of this drawback is traced to the gravitational coupling constant κ that occurs in the Einstein-Hilbert (EH) action[1],

$$S_{EH}[g_{\mu\nu}] = \frac{1}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}R \tag{1.1}$$

which is related to Newton's gravitational constant G_N as $\kappa = 8\pi G_N$. Its mass dimension is negative, $[\kappa] = -2$, whereas it should be larger than or equal to zero in order for the

[1] For simplicity we set the cosmological constant Λ equal to zero.

theory to be at least power-counting renormalizable.

Remarkably, Stelle has developed a very interesting way to tackle this situation [10]. While EH action (1.1) contains terms with second-order derivatives of the metric through the Ricci scalar R, Stelle introduced covariant terms containing higher-order derivatives. As he showed, this cures the problem of the divergences, giving rise to power-counting renormalizable theories, the so-called Higher-Order Gravity Theories,

$$S_{HO}[g_{\mu\nu}] = \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{2\kappa}R + \alpha R_{\mu\nu}R^{\mu\nu} + \beta R^2\right) \tag{1.2}$$

However, not everything can be peaches and cream. It is well known that typically higher-order theories suffer from dynamical instabilities. As Ostrogradsky demonstrated in 1850 [16], theories containing time derivatives higher than two are susceptible to contain anomalous degrees of freedom characterized by having negative kinetic energy, the so-called *ghosts*. In canonical form, this can be presumed from the appearance of linear terms of some canonical momenta, thus yielding an unbounded Hamiltonian function.

The main problem that this represents is that the quantum counterpart of a theory with ghosts is non-unitary, i.e. it is not possible to establish a well-defined evolution operator [17]. The consequences of this, such as the loss of conservation of probabilities, ultimately result in a new impediment on the road to achieve a consistent quantization of gravity. There is, however, another aspect of gravity theories that can help in this regard.

GR is a gauge theory, which means, among other things, that it is not described with the minimum necessary number of variables. In other words, there is some redundancy in the characterization of the physical degrees of freedom that can be neatly handled by employing a rigorous Hamiltonian constraint analysis. An advantage of this description is that it opens the possibility of disentangling the ambiguous description of the degrees of freedom, such that the presence of ghosts could be discarded. This can be made visible either with the construction of the extended Hamiltonian, a version that incorporates all the symmetries of the theory, or with the introduction of Dirac brackets, which eliminate ambiguities in the description of the dynamics. A good symptom of relief would be the removal of the linear dependence mentioned above, so that the extended Hamiltonian becomes a bounded function [18]. Of course, all these considerations are preserved in Stelle's proposal since its action obeys the same underlying symmetry of GR, that of diffeomorphisms.

Thus, it is of great interest to search for higher-order theories that do not present ghosts since they may be candidates to be fully quantized. Some proposals are promising in

3+1 [19], but the analysis of theories in 2+1 is equally appealing as a laboratory for testing ideas [20, 21].

In this thesis we have focused on two higher-order theories: NMG [26, 27], and λR gravity [28], which is the low-energy limit of HG [29,30]. The former is a 2+1 theory that comes from the specific form of Stelle's proposal, propagates two degrees of freedom and allegedly has no ghosts, although this has not been formally proven. Our work has been the complete canonical analysis where we have identified the first and second-class constraints. Likewise, with the construction of the extended Hamiltonian and the Dirac brackets, we have fully demonstrated that it contains no ghosts at the perturbative level.
On the other hand, HG is a 3+1 higher-order theory with a fundamentally different construction from GR. Once a foliation is introduced into spacetime, the underlying symmetry of this theory is reduced to that which preserves such a foliation. This allows a fundamental distinction between space and time, which, in practice, leads to the possibility of considering symmetry-compatible terms that contain temporal and spatial derivatives separately. Thus, we can consider only terms with time derivatives up to second-order to avoid the ghosts, while adding terms with higher spatial derivatives for renormalizability purposes. Of course, we will discuss this in more detail later.
It is paramount that modified gravity theories have an appropriate approximation with GR in the corresponding limit. The difference in the underlying symmetry has been the main argument for denying the closeness between HG and GR, but the implementation of a complete analysis is crucial to investigate this situation further. In this sense, our work has focused on the λR model, which, being the low energy limit of HG, must approach GR to consolidate the consistency of the Hořava proposal.

In this manner, we present our work as follows: In Chapter II we show the Hamiltonian formalism developed by Dirac and Bergmann for gauge theories. In Chapter III we present our canonical analysis of NMG, where by means of the extended Hamiltonian it is fully demonstrated that it lacks ghosts. In Chapter IV we show our analysis of the λR model, both in its original and extended versions. We conclude with Chapter VI discussing our results and making suggestions for possible future works.

Chapter 2

Hamiltonian Formalism of Gauge Theories

Let us remember that all fundamental field theories in physics are invariant with respect to some group of local symmetry transformations. For Yang -Mills theories, these are the gauge transformations, and for GR spacetime diffeomorphisms. In general, we call them *gauge theories* or, more frequently, *singular systems*.

While it is possible to deal with this type of theories in its Lagrangian form, the work of Dirac and Bergmann gives a transparent picture of how to identify and handle gauge symmetries within the framework of Hamiltonian formalism [12, 14, 15]. In this language, the local symmetries are translated into relations between the canonical variables that constrain the phase space and that degenerate the system in the sense that multiple configurations in phase space are associated with the same physical state. Hence, all gauge theories are systems with constraints.

In this chapter we introduce the Hamiltonian formalism for singular systems, starting from its description at the Lagrangian level. We also present the adjustments to be considered when working with higher-order time derivative theories. We recall that the gravity theories we deal with in the present research contain such derivatives.

2.1 Singular Lagrangian systems and primary Hamiltonian

We first consider systems whose Lagrangians depend at most on first derivatives[1]. As we know the equation of motions of these systems are derivable from Hamilton's variational principle of the action functional

$$S = \int_{t_1}^{t_2} dt \;\, L\left(q_i, \dot{q}_i\right), \tag{2.1}$$

where $q = (q_1, .., q_N)$ are the generalized coordinates. Then the classical trajectories are solutions of the *Euler-Lagrange equations*

$$-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \frac{\partial L}{\partial q_i} = 0. \tag{2.2}$$

This second-order equation determines the accelerations at a given time uniquely by q and $\dot{q}$ at that time. To see this we rewrite this as

$$-\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_j - \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}\dot{q}_j + \frac{\partial L}{\partial q_i} \equiv -W^{ij}(q,\dot{q})\ddot{q}_j + V^i(q,\dot{q}) = 0. \tag{2.3}$$

Then the only necessary condition to yield $\ddot{q}$ as a function of q and $\dot{q}$ is that the Hessian matrix W^{ij} can be inverted, i.e. $\det W^{ij} \neq 0$. If this is possible, such systems are called *regular*. In our case we are interested in the opposite, when $\det W^{ij} = 0$. Here the complete set of accelerations is not uniquely determined by q and $\dot{q}$ and its solutions will contain arbitrary functions of time. These systems are called *singular*. It is mandatory to mention that gauge theories are necessarily singular. However, the converse is not true. Not all conceivable singular systems are gauge theories.

In the following we show the process of turning to the Hamiltonian formalism in the case of singular systems. As usual, it starts by defining the canonical momenta,

$$p^i = \frac{\partial L}{\partial \dot{q}_i}. \tag{2.4}$$

Remember that in Hamiltonian form, we describe the dynamics of a mechanical system with a set of $2n$ first-order equations instead of n second-order equations of the Lagrangian formalism. These equations are obtained from the Hamiltonian function that depends on the canonical variables: the generalized coordinates q and their canonically conjugated

[1]Theories with higher derivatives will be considered in section 2.5.

momenta p, that span the *phase space*. On the other hand, the condition $\det W^{ij} = 0$ means that we cannot express all the velocities in terms of p's and q's. In fact, given the rank $\mathcal{R}$ of W, we will find $M = N - \mathcal{R}$ relations involving the p's and q's that follow from the definition (2.4)

$$\phi_l(p, q) = 0, \qquad\qquad l = 1, .., M. \tag{2.5}$$

These are the *primary constraints*[2] that define the $2N - M$-dimensional *primary constraint surface*, denoted by Γ_p. While the canonical Hamiltonian for regular systems is simply $H_c = \dot{q}_i p^i - L|_{\dot{q}=\dot{q}(p,q)}$, for singular systems the Hamiltonian will be restricted to Γ_p as

$$H_p = H_c + u^i \phi_i, \tag{2.6}$$

where u^i are Lagrange multipliers. This is the *primary Hamiltonian*.
Now we introduce the Poisson bracket of phase space functions

$$\{F, G\} = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i}, \tag{2.7}$$

by means of which the Hamiltonian equations of motion are expressed as

$$\begin{aligned} \dot{q}^i &= \{q^i, H_p\} = \{q^i, H_c\} + \{q^i, \phi_m\}u^m, \\ \dot{p}_i &= \{p_i, H_p\} = \{p_i, H_c\} + \{p_i, \phi_m\}u^m. \end{aligned} \tag{2.8}$$

In general, for any phase space function $F(q, p)$ the time evolution is

$$\dot{F} = \{F, H_p\} = \{F, H_c\} + \{F, \phi_m\}u^m. \tag{2.9}$$

It is useful at this stage to introduce the following equality symbol. A function $F(q, p)$ defined in a neighborhood of Γ_p is called *weakly zero* if

$$F|_{\Gamma_p} = 0, \qquad \text{abbreviated as} \qquad F \approx 0, \tag{2.10}$$

and *strongly zero* if

$$F|_{\Gamma_p} = 0 \;\; \text{and} \;\; \left(\frac{\partial F}{\partial q^i}, \frac{\partial F}{\partial p_i}\right)|_{\Gamma_p} = 0, \qquad \text{written as} \qquad F \simeq 0, \tag{2.11}$$

[2] For simplicity, we consider these expressions to be independent.

2.2 Dirac-Bergmann Algorithm

As the equations of motion are not used to obtain the primary constraints, they imply no restriction on the coordinates q^n and their velocities $\dot{q}^n$. However, a basic consistency requirement is that the equations of motion preserve these constraints. Thus we require

$$\dot{\phi}_m = \{\phi_m, H_c\} + \{\phi_m, \phi_n\}u^n \equiv h_m + C_{mn}u^n = 0. \tag{2.12}$$

This is known as the *consistency condition.* We distinguish the following cases:
i) det $C \not\approx 0$: In this case u is weakly uniquely fixed by (2.12) to be $u^n \approx -C^{nm}h_m$, where C^{mn} is the inverse of C_{mn}. The time evolution of a phase space function becomes

$$\dot{F} \approx \{F, H_c\} - \{F, \phi_m\}C^{mn}\{\phi_n, H_c\}. \tag{2.13}$$

No additional conditions appear. For any initial data (q, p) on Γ_p, the time evolution stemming from (2.13) is unambiguous and stays on Γ_p.
ii) det $C \approx 0$: In this case u is not fixed and (2.12) is only solvable if $h_m w_a^m \approx 0$ for all left null-eigenvectors w_a of C. Either these equations are fulfilled or they lead to a certain number K_1 of new constraints

$$\phi_k \approx 0, \qquad k = M+1, ..., M+K_1 \equiv J_1, \tag{2.14}$$

called *secondary constraints.* The primary and secondary constraints $\phi_j \approx 0$, $j = 1, ..., J_1$ define a submanifold $\Gamma_1 \subset \Gamma_p$.
This procedure is applied repeatedly on the new constraints until we reach either an inconsistency, or to a discrete phase space, or relations involving Lagrangian multipliers u. In the latter case, as a final result we obtain a submanifold $\Gamma_c \subset \Gamma$ defined by

$$\phi_j \approx 0, \qquad j = 1, ..., M+K_1 \equiv J, \tag{2.15}$$

and a set of equations for the multiplier fields u^m:

$$\{\phi_j, H_c\} + \{\phi_j, \phi_m\}u^m \approx 0, \tag{2.16}$$

where $\approx$ now means equality on Γ_c.

2.3 First and Second-Class Constraints

Concerning the final set of constraints (2.15), the distinction between primary and secondary constraints will be of minor importance. A different classification of constraints, namely into first and second-class, will play a central part in the final form of the Hamiltonian theory. For this classification we observe that the general solution for the multipliers u in (2.16) has the form

$$u^m = \tilde{u}^m + \mu^a v_a^m, \tag{2.17}$$

where $\tilde{u}$ is a particular solution and v_a is a basis of the right-kernel of C,

$$\{\phi_j, \phi_m\} v_a^m \approx 0, \qquad a = 1, ..., \dim \mathrm{Ker}\, C = M - \mathrm{rank}\, C. \tag{2.18}$$

The part of u that remains undetermined by the consistency conditions contains $M-\mathrm{rank}\, C$ free functions μ^a. We observe that the combinations of primary constraints

$$\phi_a = v_a^m \phi_m \tag{2.19}$$

weakly commute with all other constraints,

$$\{\phi_a, \phi_j\} \approx 0, \qquad j = 1, ..., J. \tag{2.20}$$

This leads to the concept of *first-class functions* and in particular *first-class constraints* (FCC). A function $F(q,p)$ is said to be first-class if its Poisson brackets with all constraints vanish weakly (on Γ_c),

$$\{F, \phi_j\} \approx 0, \qquad j = 1, ..., J. \tag{2.21}$$

The set of first-class functions is closed under Poisson brackets. Also, as a result of the Dirac-Bergmann algorithm H_p is first-class.

On the other hand, a function that is not first-class is called *second-class*. We denote a basis for FCC by γ_a. The remaining constraints are called second-class constraints (SCC), denoted by χ_α. Of course, the matrix of SCC

$$\Delta_{\alpha\beta} = \{\chi_\alpha, \chi_\beta\} \tag{2.22}$$

is non-singular. For counting degrees of freedom it is important to note that the number of SCC must be even. Otherwise the antisymmetric Δ would be singular.

Now consider the consistency conditions (2.16). They are identically fulfilled for the γ_a, whereas for the SCC we have

$$\{\chi_\alpha, H\} + \Delta_{\alpha\beta} u^\beta \approx 0. \tag{2.23}$$

Inserting this into the equations of motion (2.9) we end up with

$$\dot{F} \approx \{F, H\} + \{F, \phi_a\}\mu^a - \{F, \chi_\alpha\}\Delta^{\alpha\beta}\{\chi_\beta, H\}, \tag{2.24}$$

where the ϕ_a are the primary FCC. One can easily check that all constraints are preserved in time.

2.4 The Reduced Phase Space

2.4.1 Second-Class Constraints and Dirac Bracket

First we shall consider SC systems, for which no multipliers remain in the evolution (2.24). There is no ambiguity in the dynamics. The term in (2.24) containing the inverse of Δ forces the system to stay on Γ_c. This surface will be the *reduced phase space* for SC systems. In this way, based on (2.24), the *Dirac bracket* for two phase space functions is defined as

$$\{F, G\}_D \equiv \{F, G\} - \{F, \chi_\alpha\}\Delta^{\alpha\beta}\{\chi_\beta, G\}, \tag{2.25}$$

in terms of which

$$\dot{F} \approx \{F, H\}_D, \tag{2.26}$$

for SC systems. This bracket possesses the same properties as the Poisson bracket, i.e. it is antisymmetric, bilinear, and obeys the Jacobi identity and product rule. In addition we have

$$\{F, \chi_\alpha\}_D = 0, \quad \{F, G\}_D = \{F, G\}, \qquad \{F, \{G, K\}_D\}_D = \{F, \{G, K\}\}, \tag{2.27}$$

for arbitrary F and first-class G, K.

2.4.2 First-Class Constraints and Gauge Transformations

Now we consider FC systems. These are the most important systems since all gauge theories are of this type. Gauge related points should be identified and this leads us to the problem of gauge invariant functions and/or the gauge fixing problem. The FCC together with a complete set of gauge fixing, define a subset $\Gamma_r \subset \Gamma_c$ and this set is the reduced phase space. For purely FC systems the time evolution is governed by

$$\dot{F} \approx \{F, H\} + \{F, \phi_a\}\mu^a, \tag{2.28}$$

where ϕ_a are the primary FCC. For the same initial conditions we get different evolutions, depending on the multipliers μ^a. The presence of arbitrary functions μ^a in the primary Hamiltonian tell us that not all $x = (q, p)$ are observable, i.e. there are several x representing a given physical state. By assuming that the initial value $x(0)$ is given and represents a certain state, the equation of motion should fully determine the physical state at later times. So if $x'(t) \neq x(t)$ stem from the same physical state $x(0)$, they should be identified.
Now we consider two infinitesimal time evolutions of $F = F(0)$ given by H_p with different values of the multipliers,

$$F_i(t) = F(0) + t\{F, H\} + t\{F, \phi_a\}\mu_i^a, \quad i = 1, 2. \tag{2.29}$$

The difference $\delta F = F_2(t) - F_1(t)$ between the values is then

$$\delta_\mu F = \{F, \mu^a\phi_a\}, \quad \mu = t(\mu^2 - \mu^1). \tag{2.30}$$

Such transformation does not alter the physical state at time t, and hence is called an *infinitesimal gauge transformation*. The commutator of any two primary FCC

$$[\delta_\mu, \delta_\nu]F = \{\{\mu^a\phi_a, v^b\phi_b\}, F\} \tag{2.31}$$

also generate gauge transformations. Also, performing a gauge transformation at $t = 0$ with multipliers v and then time evolve with multipliers μ should lead to the same state as doing these transformations in the reverse order. We find

$$[\delta_{t,\mu}, \delta_\nu]F = -t\{\{v^a\phi_a, H\}, F\} - t\{\{v^a\phi_a, \mu^b\phi_b\}, F\} \tag{2.32}$$

and conclude that the commutators $\{\phi_a, H\}$ also generate gauge transformations.
We have seen that the set of first-class functions is closed with respect to the Poisson bracket and thus the $\{\phi_a, \phi_b\}$ and $\{\phi_a, H\}$ are linear combinations of the FCC. However, in general there will appear secondary FCC in these combinations. Also, if we compared the higher-order terms in the time evolution (2.29) we would find that time derivatives of $\{\phi_a, H_p\}$ generate gauge transformations. This way secondary FCC show up as gauge

transformations in all relevant systems. However, it is not possible to show that every FCC is of the form $\{\phi_a, \phi_b\}$ or $\{\phi_a, H\}$ for some first-class primary constraints (ϕ_a, ϕ_b) and thus is a gauge generator. Dirac conjectured that this might be the case, that is that all FCC γ_a show up this way and generate gauge transformations. We shall assume this conjecture to hold in what follows (although there are some exotic counterexamples).

We conclude that the most general physically permissible motion should allow for an arbitrary gauge transformation to be performed during the time evolution. But H_p contains only the primary FCC. We thus have to add to H_p the secondary FCC multiplied by arbitrary functions. This led Dirac to introduce the *extended Hamiltonian*

$$H_p \longrightarrow H_e = H + \mathcal{N}^a \gamma_a \tag{2.33}$$

which contains all FCC. H_e accounts for all the gauge freedom.

Clearly, H_p and H_e should imply the same time evolution for the classical observables. *Observables* are gauge invariant functions on Γ_c, that is phase space functions that weakly commute with the gauge generators,

$$F \quad \text{observable} \quad \leftrightarrow \{F, \gamma_a\} \approx 0 \quad \text{for all} \quad a. \tag{2.34}$$

Since $H_e - H_p$ is a combination of the secondary FCC, we have

$$\dot{F} \approx \{F, H_p\} \approx \{F, H_e\} \tag{2.35}$$

for any observable F, as required. In the extended formalism one makes no distinction between primary and secondary FCC since they are treated symmetrically. The introduction of H_e is a new feature of the Hamiltonian scheme. It does not follow from the Lagrangian formalism.

In a next step one wants to eliminate the gauge degrees of freedom, that is identify points on the same gauge orbit. This can in principle be achieved by introducing gauge invariant variables, e.g. the transverse potential or holonomies in electrodynamics, or alternatively by fixing the gauge. We can fix the gauge by imposing the independent conditions

$$F_a(q,p) = 0, \qquad\qquad a = 1, .., M \tag{2.36}$$

A *gauge fixing* must obey two conditions: first it must be attainable and second it should fix the gauge uniquely. The first requirement guarantees that (2.36) does not affect the physically relevant (gauge-invariant) properties of the theory but merely restricts the gauge freedom. The second condition means that there must exist no gauge transformation other than the identity that preserves (2.36). In other words

$$\det\ \{\gamma_a, F_b\} \equiv \det\ F_{ab} \neq 0. \tag{2.37}$$

These both requirements together imply that in order to fully fix the gauge, *the number of independent gauge conditions must be equal to the number of independent first-class constraints.* Likewise, due to (2.37), the FCC together with the gauge fixings form a SC system. Thus, after completely gauge fixing, no first-class constraint is left.

The reduced phase space Γ_r consists of the points fulfilling the constraints and gauge fixings. Denoting the γ_a and F_a collectively as Ω_p, $p = 1, ..., 2M$, we find for the Hamiltonian equation of motion for any phase space function

$$\dot{F} = \{F, H_c\} - \{F, \Omega_p\}\ G^{pq}\ \{\Gamma_q, H_c\}, \tag{2.38}$$

where G^{pq} is the inverse matrix of the Poisson bracket matrix of the constraints and gauge fixings, $G_{pq} = \{\Omega_p, \Omega_q\}$.

In a theory possessing only SCC no arbitrary functions appear in the Hamiltonian. A set of canonical variables that satisfies the constraint equations determines then one and only one physical state. Since after fixing the gauge there are only second-class constraints left, we arrive at the counting of physical degrees of freedom (DOF):

$$2 \times \text{DOF} = \text{Number of canonical variables} - \text{SCC} - 2 \times (\text{FCC}). \tag{2.39}$$

2.4.3 Systems with Second and First-Class Constraints

Before gauge fixing, the evolution is governed by the *first-class partner of the extended Hamiltonian*

$$H_E = H_c + \mathcal{N}^a \gamma_a - \chi_\alpha \Delta^{\alpha\beta} \{\chi_\beta, H_c\} \tag{2.40}$$

since for observables F,

$$\dot{F} = \{F, H_E\} \approx \{F, H_c\} + \{F, \gamma_a\}\mathcal{N}^a - \{F, \chi_\alpha\}\Delta^{\alpha\beta}\{\chi_\beta, H_c\}. \tag{2.41}$$

2.5 Hamiltonian Formalism of Perturbative GR

It is useful to illustrate the machinery developed in the previous pages with GR itself. Furthermore, this will pave the way for the analysis of gravity theories in the following chapters.

The Hamiltonian analysis of GR is carried out entirely at the non-perturbative level using the ADM formalism [31]; and at the perturbative level by fixing a background metric [32], often the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$. Moreover, the perturbed version of a field theory is mandatory in the QFT approach of QGT, which is the one we are concerned with. Thus, using deviations $h_{\mu\nu}$ that represent the gravitational interaction around the Minkoswki metric by means of

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{2.42}$$

the EH action (1.1), to second order, becomes

$$\mathcal{L}_{FP} = \partial^\alpha h^{\mu\nu}\partial_\mu h_{\nu\alpha} - \frac{1}{2}\partial^\alpha h^{\mu\nu}\partial_\alpha h_{\mu\nu} - \partial_\nu h \partial_\mu h^{\mu\nu} + \frac{1}{2}\partial_\alpha h \partial^\alpha h. \tag{2.43}$$

This expression is known as the Fierz-Pauli Lagrangian for massless particles of spin two and propagates two degrees of freedom [33].

In order to perform the Legendre transformation, we will define velocities with the help of notion of time provided by $\eta_{\mu\nu}$. Then, we decompose each term of (2.43) considering separately the temporal and spatial components. For instance, the first term takes the form

$$\partial^\alpha h^{\mu\nu}\partial_\mu h_{\nu\alpha} = -\dot{h}^{00}\dot{h}_{00} - \dot{h}^{0i}\dot{h}_{0i} + 2\partial^i h^{00}\dot{h}_{i0} - 2\dot{h}^{ij}\partial_i h_{0j} + \partial^i h^{j0}\partial_j h_{i0} + \partial^i h^{jk}\partial_j h_{ik}. \tag{2.44}$$

where $i, j, k, .. = 1, 2, 3$; are purely spatial. The decomposition of all the terms is condensed in the following expression

$$\begin{aligned}\mathcal{L}_{FP} =& \frac{1}{4}\dot{h}_{ij}\dot{h}^{ij} - \dot{h}^{ij}\partial_i h_{0j} - \dot{h}^j_j \partial_i h^{0i} - \frac{1}{4}(\dot{h}^i_i)^2 - \frac{1}{2}\partial_i h_{0j}\partial^i h^{0j} + \frac{1}{2}\partial^i h^{j0}\partial_j h_{i0} + \frac{1}{2}\partial_i h_{00}\partial_j h^{ij} \\ &- \frac{1}{2}\partial_i h^k_k \partial_j h^{ij} - \frac{1}{2}\partial_i h_{00}\partial^i h^k_k + \frac{1}{4}\partial_i h^j_j \partial^i h^k_k + \frac{1}{2}\partial^i h^{jk}\partial_j h_{ik} - \frac{1}{4}\partial_i h_{jk}\partial^i h^{jk}.\end{aligned} \tag{2.45}$$

The canonical form of this Lagrangian, as well as the construction of the Dirac brackets, is discussed in [32]. However, as we will see in the following chapters, there are important advantages if we first introduce the variable

$$K_{ij} = \frac{1}{2}\left(\dot{h}_{ij} - \partial_i h_{0j} - \partial_j h_{0i}\right), \tag{2.46}$$

the so-called extrinsic curvature [34, 35]. In this way, the above Lagrangian takes the form

$$\mathcal{L} = K_{ij}K^{ij} - K^2 - \frac{1}{2}h^{00}R_{ij}{}^{ij} - \frac{1}{2}h^{ij}\left(R_{ikj}{}^k - \frac{1}{2}\delta_{ij}R_{lm}{}^{lm}\right), \tag{2.47}$$

where

$$\begin{aligned} R_{ikj}{}^{k} &= \frac{1}{2}\left(\partial_k\partial_i h^k_j - \partial^k\partial_k h_{ij} - \partial_j\partial_i h^k_k + \partial_j\partial^k h_{ik}\right), \\ R_{ij}{}^{ij} &= \partial_i\partial_j h^{ij} - \nabla^2 h^j_j. \end{aligned} \tag{2.48}$$

and $\nabla^2 = \partial_i\partial^i$. Now we start the Hamiltonian construction by calculating the canonical momenta of the action (2.47). The dynamical variables are h_{00}, h_{0i} and h_{ij}, then it is easy to see that

$$\pi^{00} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{00}} = 0, \tag{2.49}$$

$$\pi^{0i} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{0i}} = 0, \tag{2.50}$$

$$\pi^{ij} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{ij}} = K^{ij} - \delta^{ij}K. \tag{2.51}$$

We can immediately identify the primary constraints given by $\pi^{00} = 0$, $\pi^{0i} = 0$. On the other hand, to construct the canonical Hamiltonian, we need an expression for the velocity $\dot{h}_{ij}$ in terms of the canonical variables. We achieve this by considering (2.51) and its trace. The result is

$$\dot{h}_{ij} = 2\pi_{ij} - \delta_{ij}\pi + \partial_i h_{j0} + \partial_j h_{i0}, \tag{2.52}$$

where $\pi = \delta_{ij}\pi^{ij}$. Now we arrive to the primary Hamiltonian

$$\mathcal{H}_p = \pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2 - 2h_{j0}\partial_i\pi^{ij} + \frac{1}{2}h^{00}R_{ij}{}^{ij} + \frac{1}{2}h^{ij}\left(R_{ikj}{}^{k} - \frac{1}{2}\delta_{ij}R_{lm}{}^{lm}\right) + u\pi^{00} + u_i\pi^{0i}. \tag{2.53}$$

where u and u_i are the Lagrange multipliers enforcing the primary constraints. On the other hand, by considering the relevant Poisson brackets

$$\left\{h_{ij}(x), \pi^{kl}(y)\right\} = \frac{1}{2}\left(\delta^k_i\delta^l_j + \delta^l_i\delta^k_j\right)\delta^3(x-y), \tag{2.54}$$

we are in position to apply the consistency condition to the primary constraints. This leads us to four secondary constraints, given by

$$\begin{aligned} \mathcal{H} &: \dot{\pi}^{00} = \left\{\pi^{00}, \int d^3x\, \mathcal{H}_p\right\} = R_{ij}{}^{ij} \approx 0, \\ \mathcal{H}^i &: \dot{\pi}^{0i} = \left\{\pi^{0i}, \int d^3x\, \mathcal{H}_p\right\} = \partial_j\pi^{ji} \approx 0. \end{aligned} \tag{2.55}$$

The process continues by applying the same criteria to these secondary constraints, but no new constraints are produced. The consistency of $\mathcal{H}$ is followed from $\mathcal{H}^i = 0$, while it is identically satisfied for $\mathcal{H}^i$. In this way, once achieved the total set of constraints, the next step is the classification into first and second-class. Since the Poisson brackets between all the constraints is null, all the constraints are first-class. We list them as follows.

$$\begin{aligned}
\gamma_1 &: \pi^{00} \approx 0, \\
\gamma_2^i &: \pi^{0i} \approx 0, \\
\gamma_3 &: \mathcal{H} \approx 0, \\
\gamma_4^i &: \mathcal{H}^i \approx 0.
\end{aligned} \tag{2.56}$$

$\mathcal{H} \approx 0$ is known in the literature as the Hamiltonian constraint, and $\mathcal{H}^i \approx 0$ is the so-called momentum constraint which generates spatial diffeomorphisms. Furthermore, with this classification we can count the degrees of freedom. We have no SCC, but only eight FCC. Thus

$$DOF = \frac{1}{2}(20 - 2 * 8) = 2. \tag{2.57}$$

2.5.1 Gauge Fixing and Dirac Brackets

For this purpose let us consider $h^{00} \approx 0$ together with $h^{0i} \approx 0$, $\pi \approx 0$, and the Coulomb gauge $\partial_i h^{ij} \approx 0$, as gauge fixing conditions for the first-class constraints listed above respectively. In this way we have a set of only SCC,

$$\begin{aligned}
\chi_1 &: \pi^{00} \approx 0, \\
\chi_2 &: h^{00} \approx 0, \\
\chi_3 &: \pi^{0i} \approx 0, \\
\chi_4 &: h^{0i} \approx 0, \\
\chi_5 &: h_i^i \approx 0, \\
\chi_6 &: \pi \approx 0, \\
\chi_7 &: \partial_j \pi^{ji} \approx 0, \\
\chi_8 &: \partial_j h^{ji} \approx 0.
\end{aligned} \tag{2.58}$$

Considering now (2.25), we can establish the Dirac brackets for this theory. Of course the Dirac brackets $\{h^{00}, \pi^{00}\}_D$ and $\{h^{0i}, \pi^{0j}\}_D$ are null, since h^{00} and h_{0j} can be considered just as the Lagrange multipliers associated with $\mathcal{H}$ and $\mathcal{H}^i$ respectively (see (2.53)). Thus they are not associated with degrees of freedom. On the other hand, after a long algebraic work, we obtain the Dirac brackets for the pair h_{ij}, π^{lm}.

$$\begin{aligned}\left\{h_{ij},\pi^{lm}\right\}_D =& \frac{1}{2}\left(\delta_i^l\delta_j^m+\delta_i^m\delta_j^l\right)\delta^3(x-y)+\frac{1}{2\nabla^2}\left(\delta_{ij}\partial^l\partial^m+\delta^{lm}\partial_i\partial_j\right)\delta^3(x-y)\\ &-\frac{1}{2\nabla^2}\left(\delta_i^m\partial_j\partial^l+\delta_i^l\partial_j\partial^m+\delta_j^m\partial_i\partial^l+\delta_j^l\partial_i\partial^m\right)\delta^3(x-y)+\frac{1}{2}\frac{\partial_i\partial_j\partial^l\partial^m}{\nabla^4}\delta^3(x-y)\\ &-\frac{1}{2}\delta_{ij}\delta^{lm}\delta^3(x-y).\end{aligned} \tag{2.59}$$

These brackets correspond to those found in [32].

2.6 Higher-Order Theories

So far, we have considered singular systems described by Lagrangians, which are at most quadratic in the velocities. This means that the time derivatives are by far of first-order. However, it is possible to generate dynamical systems whose Lagrangians contain higher time derivatives. Without going any further, the theories of gravity considered here belong to this category. The Hamiltonian treatment of these systems can be easily realized by extending the phase space.

Consider a higher-order system described by the Lagrangian $L=L\left(q,\dot{q},\ddot{q},...,\dot{q}^{(n)}\right)$, with $\dot{q}^{(n)}$ representing the nth-order time derivative of q. Now we choose to introduce auxiliary fields with the following identification.

$$q_1=\dot{q},\qquad q_2=\dot{q}_1,\quad ...\quad q_{n-1}=\dot{q}_{n-2}. \tag{2.60}$$

Since the auxiliary fields are not true degrees of freedom, we should consider the above equations as a set of $n-1$ constraints.

$$\psi_1=q_1-\dot{q}\approx 0,\qquad \psi_2=q_2-\dot{q}_1\approx 0,\quad ...\quad \psi_{n-1}=q_{n-1}-\dot{q}_{n-2}\approx 0. \tag{2.61}$$

In this way, the system would be described in a similar but simpler way by the Lagrangian

$$L^*=L\left(q,q_1,...,q_{n-1},\dot{q}_{n-1}\right)+\lambda^1\psi_1+...+\lambda^{n-1}\psi_{n-1}. \tag{2.62}$$

In this description we have extended the configuration space, which implies an extended phase space as well. However, the main advantage is that the Lagrangian depends only on first-order time derivatives. Then all the previously developed apparatus is again applicable.

2.6.1 Ghost Degrees of Freedom and Non-unitarity

There is an important result due to Ostrogradsky on the dynamical systems with higher-order time derivatives [16,17]. To discuss it in a simple way, let us consider a system whose Lagrangian depends on $\ddot{q}$, i.e. $L = L\left(q, \dot{q}, \ddot{q}\right)$. Now, by using the first-order form introduced above, we cast this Lagrangian as

$$L^{*} = L\left(q, q_{1}, \dot{q}_{1}\right) + \lambda\psi, \tag{2.63}$$

where $\psi = q_{1} - \dot{q}$. The passing to the Hamiltonian form starts by establishing the canonical momenta,

$$p = \frac{\partial L^{*}}{\partial \dot{q}} = -\lambda, \qquad p^{1} = \frac{\partial L^{*}}{\partial \dot{q}_{1}}. \tag{2.64}$$

We ask for non-degeneracy such that $p^{1} \neq 0$, thus it is possible to express $\dot{q}_{1}$ as $\dot{q}_{1} = \dot{q}_{1}(q, q_{1}, p^{1})$. Since there is a primary constraint given by $\phi = p + \lambda = 0$, the primary Hamiltonian is

$$H_{p}^{*} = \dot{q}p + \dot{q}_{1}p^{1} - L^{*} + u\phi = q_{1}p + \dot{q}_{1}(q, q_{1}, p^{1})p^{1} - L(q, q_{1}, p^{1}) + u\phi. \tag{2.65}$$

Thus $H_{p}^{*} = H_{p}^{*}(q, p, q_{1}, p^{1})$, but the dependence on p is necessarily linear. In this way, the description of (q, p) makes the Hamiltonian naturally unbounded. This is the so-called *ghost degree of freedom*. We can find an interesting dynamical system containing a ghost in the widely studied Pais-Uhlenbeck oscillator [36].

While there are many things to say about a classical system with ghosts, the most important part for us is the quantization of such a system. As we know, in canonical quantization we establish an evolution operator from the classical Hamiltonian after promoting the canonical variables to operators. The requirement of unitarity, i.e. that the probabilities be conserved during temporal evolution, is satisfied when the Hamiltonian operator is Hermitian. However, when the classical Hamiltonian is not bounded, there can exist some troubles with the Hermeticity. Some of these troubles are related to the possibility of arriving to states with negative norm, which naturally breaks the probabilistic interpretation.

It is important to remember that in the second-order Lagrangian discussed here, the basic requirement is the non-degeneracy with respect to the auxiliary field representing the higher derivative. In fact, this is the basis of the Ostrogradsky discussion. For us, in regard with the gauge systems, this non-degeneracy is equivalent with non-singularity. On the other hand, the connection between singular systems and ghost is further complex such that there are works showing that it is possible to overcome the problems of non-unitarity in this kind of systems [18]. This is very relevant for us in the study of higher-order gravity theories. In the next chapter, we show a no-ghost singular system.

Chapter 3

New Massive Gravity

The 2+1 dimensions models are a good framework to implement ideas around the project of modifying the EH action to alleviate its drawbacks preventing quantization. Of course, we must mention that the quantization of GR in 2+1 is possible [37]; however, since its classical counterpart does not propagate degrees of freedom[1], its usefulness to explore the quantum implications is limited.

In this sense, a very attractive proposal in 2+1 is that of Topologically Massive Gravity (TMG) [21, 22], constructed by coupling the topological terms of EH and Chern-Simons. Remarkably, this theory describes the propagation of a single massive state of helicity ± 2 on a Minkowski background [23–25]. However, this theory breaks parity, and it is not unitary. On the other hand, New Massive Gravity (NMG) is a parity-conserving theory and describes the propagation of two massive degrees of freedom of helicity ± 2 [26, 27]. These properties make the theory very interesting because, although in three dimensions, it shares the same number of degrees of freedom with GR, with the clear exception that they are massive. Massive gravity theories have been found to be useful in addressing questions such as the accelerating expansion of the universe and the cosmological constant problem [38].

The form of the NMG action stems from Stelle's proposal of higher-order theories (1.2), and is given by

$$S[g_{\mu\nu}] = \frac{1}{\kappa^2} \int d^3x \, \sqrt{-g} \left(R + \frac{1}{m^2} J \right) , \tag{3.1}$$

[1]Recall that GR in 3+1 propagates two degrees of freedom.

where $g_{\mu\nu}$ is the metric tensor, $\mu, \nu, ... = 0, 1, 2$; κ is a constant with mass dimension in fundamental units $[\kappa] = -1/2$, m is a "relative" mass parameter and J is given by

$$J = R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2. \tag{3.2}$$

Through the variational principle the equations of motion obtained from this action are

$$J_{\mu\nu} + 2m^2 G_{\mu\nu} = 0, \tag{3.3}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor and

$$J_{\mu\nu} = 2\Box R_{\mu\nu} - \frac{1}{2}\left(\nabla_\nu \nabla_\mu R + g_{\mu\nu}\Box R\right) - 8R^\rho_\mu R_{\nu\rho} + \frac{9}{2}RR_{\mu\nu} + g_{\mu\nu}\left(3R_{\rho\sigma}R^{\rho\sigma} - \frac{13}{8}R^2\right). \tag{3.4}$$

$\Box = \nabla_\mu \nabla^\mu$ is the D'Alambertian operator. We observe that the trace of (3.3) implies

$$J = m^2 R. \tag{3.5}$$

3.1 Perturbative Theory

As mentioned in the introduction, our work is the canonical analysis at the perturbative level of gravitational theories. The results of our research on NMG, shown below, are published in [39]. In fact, this theory had only been analyzed at the Lagrangian level [26, 27], but without a complete identification of symmetries.

For our analysis we will follow the same line as in the previous chapter for perturbative GR. Recall that we consider the perturbation as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. In this case, we start by looking at the linearized equations of motion. We obtain, to second-order in derivatives, the equations

$$\begin{aligned}(\Box + m^2)\underset{Lin}{G_{\mu\nu}} &= 0,\\ \underset{Lin}{R} &= 0,\end{aligned} \tag{3.6}$$

for (3.3) and (3.5) respectively. Here the linearized versions of the Ricci tensor and scalar curvature are

$$\begin{aligned}\underset{Lin}{R_{\mu\nu}} &= \frac{1}{2}\left(\partial_\alpha \partial_\mu h^\alpha_\nu + \partial_\alpha \partial_\nu h^\alpha_\mu - \partial_\mu \partial_\nu h - \partial_\alpha \partial^\alpha h_{\mu\nu}\right),\\ \underset{Lin}{R} &= \partial_\mu \partial_\nu h^{\mu\nu} - \partial_\alpha \partial^\alpha h,\end{aligned} \tag{3.7}$$

and $\Box = \partial_\mu \partial^\mu$. The first equation of motion in (3.6) is a Klein-Gordon type for the linearized Einstein tensor, while the second equation is easily explained from the fact that (3.5) consider terms of different orders on each side. Then, after the perturbation, only the right-hand side remains of second order. Likewise, we can anticipate from this the presence of a second-class constraints at the Hamiltonian level. This will be shown in the following. Another way for obtaining the linearized equations (3.6) is by the variation of the linearized version of (3.1). This is

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left(\partial_\mu h^{\mu\nu}\partial_\alpha h^\alpha_\nu - \frac{1}{2}\partial^\alpha h^{\mu\nu}\partial_\alpha h_{\mu\nu} - \partial_\nu h \partial_\mu h^{\mu\nu} + \frac{1}{2}\partial_\alpha h \partial^\alpha h\right) + \frac{1}{4m^2}\Big(\frac{1}{2}\partial_\mu\partial_\nu h^{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} \\ &\quad +\Box h_{\mu\nu}\Box h^{\mu\nu} - \frac{1}{2}\Box h \Box h + \partial_\mu\partial_\nu h^{\mu\nu}\Box h - 2\partial_\alpha\partial_\mu h^\alpha_\nu \Box h^{\mu\nu}\Big). \qquad (3.8)\end{aligned}$$

In order to perform the Hamiltonian analysis, we must first consider its 2+1 decomposition. This is (see (2.44)),

$$\begin{aligned}\mathcal{L} &= \frac{1}{4}\dot{h}_{ij}\dot{h}^{ij} - \frac{1}{4}\dot{h}^i_i\dot{h}^j_j - \dot{h}_{ij}\partial^i h^j_0 - \dot{h}^i_i\partial_k h^{0k} + \frac{1}{4m^2}\Big(\ddot{h}^{ij}\ddot{h}_{ij} - 2\ddot{h}_{ij}\nabla^2 h^{ij} - \frac{1}{2}\ddot{h}^i_i\ddot{h}^j_j + \ddot{h}^i_i\nabla^2 h^j_j \\ &\quad -4\ddot{h}_{ij}\partial^j\dot{h}^i_0 + 2\ddot{h}_{ij}\partial_l\partial^j h^{il} + 4\partial^j\partial^k h^i_0\partial_k\dot{h}_{ij} - 2\partial_j\dot{h}_{0i}\partial^i\partial_l h^{jl} - 2\partial_j\dot{h}_{0i}\partial^i\partial^j h^k_k - \ddot{h}_{00}\nabla^2 h^i_i \\ &\quad -2\partial_k\dot{h}^{0k}\ddot{h}^i_i - \partial_l\partial_m h^{lm}\ddot{h}^i_i - 2\ddot{h}_{0i}\nabla^2 h^{0i} - 2\partial^i\partial^j h_{00}\partial_j\dot{h}_{0i} + 2\partial_i\partial_j h^{ij}\ddot{h}_{00}\Big) - V, \qquad (3.9)\end{aligned}$$

where

$$\begin{aligned}V &= \frac{1}{2}\partial_i h_{0j}\partial^i h^{0j} - \frac{1}{2}\partial^i h^{j0}\partial_j h_{i0} - \frac{1}{2}\partial_i h_{00}\partial_j h^{ij} + \frac{1}{2}\partial_i h^k_k\partial_j h^{ij} + \frac{1}{2}\partial_i h_{00}\partial^i h^k_k - \frac{1}{4}\partial_i h^j_j\partial^i h^k_k \\ &\quad -\frac{1}{2}\partial^i h^{jk}\partial_j h_{ik} + \frac{1}{4}\partial_i h_{jk}\partial^i h^{jk} - \frac{1}{4m^2}\Big(\partial_i\partial_j h^{ij}\nabla^2 h^k_k + 2\nabla^2 h^{0i}\nabla^2 h_{0i} - 2\nabla^2 h^{ij}\partial_k\partial_i h^k_j \\ &\quad +\frac{1}{2}\left(\nabla^2 h^{00}\right)^2 + \nabla^2 h_{00}\nabla^2 h^i_i - \partial_i\partial_j h^{ij}\nabla^2 h_{00} + \nabla^2 h^{ij}\nabla^2 h_{ij} - \frac{1}{2}(\nabla^2 h^i_i)^2 + \frac{1}{2}(\partial_i\partial_j h^{ij})^2 \\ &\quad -2\nabla^2 h^{j0}\partial_i\partial_j h^i_0\Big), \qquad (3.10)\end{aligned}$$

here $\nabla^2 = \partial_i\partial^i$. In this representation, the Lagrangian is composed of a kinetic part, which contains up to quadratic terms of second-order time derivatives, and a potential part, which contains only spatial derivatives.

Once defined the time derivatives, the Hamiltonian treatment of these higher-order systems starts by introducing auxiliary fields in order to cast the Lagrangian in a familiar first-order form, as commented in chapter II. In this way, since we have the fields h_{00}, h_{0i} and h_{ij} with higher-time derivatives, it would be necessary to incorporate three additional auxiliary fields. Of course, this treatment often makes the generation of constraints difficult due to the significant increase in dynamical variables. However, if we choose to reformulate the above Lagrangian by introducing the extrinsic curvature (2.46), we obtain

$$\begin{aligned}
\mathcal{L} \;=\; & K_{ij}K^{ij} - K^2 - \frac{1}{2}h^{00}R_{ij}{}^{ij} - \frac{1}{2}h^{ij}\left(R_{ikj}{}^{k} - \frac{1}{2}\delta_{ij}R_{lm}{}^{lm}\right) + \frac{1}{m^2}\left(\dot{K}^{ij}\dot{K}_{ij} - \frac{1}{2}\dot{K}^2\right. \\
& -\frac{3}{2}\dot{K}R_{ij}{}^{ij} - \frac{1}{2}\dot{K}\nabla^2 h_{00} + \dot{K}^{ij}\partial_i\partial_j h_{00} + 2\dot{K}_{ij}R^{i}{}_{l}{}^{jl} - 2\partial^l K_{il}\partial^j K^i_j + 4\partial^j K_{ij}\partial^i K \\
& \left.-2\partial_i K\partial^i K + R_{ikj}{}^{k}R^{i}{}_{l}{}^{jl} + \partial_i\partial_j h_{00}R^{i}{}_{k}{}^{jk} - \frac{3}{8}R_{ij}{}^{ij}R_{lm}{}^{lm} - \frac{3}{4}\nabla^2 h_{00}R_{ij}{}^{ij} + \frac{1}{8}(\nabla^2 h_{00})^2\right) \\
& +\alpha^{ij}\left(\dot{h}_{ij} - 2\partial_i h_{0j} - 2K_{ij}\right),
\end{aligned} \tag{3.11}$$

where α^{ij} are Lagrange multipliers enforcing the definition of K_{ij}. As we see, this representation greatly simplifies the kinetic part compared to (3.9). The extrinsic curvature makes it possible to compact the time derivatives in such a way that the Lagrangian only depends significantly on the velocity of a single field. In fact, this representation of the Lagrangian as an alternative to perform the Hamiltonian analysis is an original contribution of this thesis. Similarly, we can express the equations of motion (3.6) through the components of the Ricci tensor as

$$\begin{aligned}
\underset{Lin}{R_{00}} &= -\frac{1}{2}\Delta h_{00} - \dot{K}, \\
\underset{Lin}{R_{0i}} &= \partial^j K_{ij} - \partial_i K, \\
\underset{Lin}{R_{ij}} &= \dot{K}_{ij} + \frac{1}{2}\partial_i\partial_j h_{00} + R_{ikj}{}^{k},
\end{aligned} \tag{3.12}$$

and the scalar curvature as

$$\underset{Lin}{R} = \Delta h_{00} + 2\dot{K} + R_{ij}{}^{ij}. \tag{3.13}$$

The definitions of $R_{ij}{}^{ij}$ and $R_{ikj}{}^{k}$ are given in (2.48).

3.2 Hamiltonian Analysis of NMG

Having established the appropriate Lagrangian, we now begin the implementation of the canonical formalism. The canonical variables we consider are given by $h_{\mu\nu}$, K_{ij}, α_{ij} and their corresponding canonical momenta are $\pi^{\mu\nu}$, P^{ij} and τ^{ij}. Thus the fundamental Poisson brackets between the canonical variables are

$$
\begin{aligned}
\left\{h_{\mu\nu}, \pi^{\alpha\beta}\right\} &= \frac{1}{2}\left(\delta_\mu^\alpha \delta_\nu^\beta + \delta_\mu^\beta \delta_\nu^\alpha\right)\delta^2(x-y), \\
\left\{K_{ij}, P^{lm}\right\} &= \frac{1}{2}\left(\delta_i^l \delta_j^m + \delta_i^m \delta_j^l\right)\delta^2(x-y), \\
\left\{\alpha_{ij}, \tau^{lm}\right\} &= \frac{1}{2}\left(\delta_i^l \delta_j^m + \delta_i^m \delta_j^l\right)\delta^2(x-y). \qquad (3.14)
\end{aligned}
$$

In this way, since the Lagrangian (3.11) does not depend on the velocities $\dot{h}_{00}$, $\dot{h}_{0i}$ and $\dot{\alpha}_{ij}$, their respective conjugate momenta are $\pi^{00} = 0$, $\pi^{0i} = 0$, and $\tau^{ij} = 0$, i.e., they represent primary constraints. On the other hand, for the velocities $\dot{h}_{ij}$ and $\dot{K}_{ij}$ we have $\pi^{ij} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{ij}} = \alpha^{ij}$ and

$$
P^{ij} = \frac{\partial\mathcal{L}}{\partial\dot{K}_{ij}} = \frac{1}{m^2}\left(-\delta^{ij}\dot{K} + 2\dot{K}^{ij} - \frac{1}{2}\delta^{ij}\nabla^2 h_{00} + \partial^i\partial^j h_{00} + 2R^i{}_l{}^{jl} - \frac{3}{2}\delta^{ij}R_{lm}{}^{lm}\right). \quad (3.15)
$$

By contracting (3.15) we obtain an additional primary constraint

$$
\psi : P + \frac{1}{m^2}R_{lm}{}^{lm} \approx 0, \qquad (3.16)
$$

where $P = \delta_{ij}P^{ij}$. This constraint is not trivial and will be useful at the end of the analysis for removing the apparent Ostrogradski's instability. Thus the set of primary constraints is given by

$$
\begin{aligned}
\phi^{00} &: \pi^{00} \approx 0, \\
\phi^{0i} &: \pi^{0i} \approx 0, \\
\phi^{ij} &: \pi^{ij} - \alpha^{ij} \approx 0, \\
\psi^{ij} &: \tau^{ij} \approx 0, \\
\psi &: P + \frac{1}{m^2}R_{lm}{}^{lm} \approx 0, \qquad (3.17)
\end{aligned}
$$

and the non-zero Poisson brackets between them are given by

$$
\begin{aligned}
\left\{\psi^{ij}, \phi^{lm}\right\} &= \frac{1}{2}\left(\delta^{il}\delta^{jm} + \delta^{im}\delta^{jl}\right)\delta^2(x-y), \\
\left\{\psi, \phi^{ij}\right\} &= \frac{1}{m^2}\left(\partial^i\partial^j - \delta^{ij}\nabla^2\right)\delta^2(x-y). \qquad (3.18)
\end{aligned}
$$

These brackets are relevant for applying the consistency condition, but first we need to establish the primary Hamiltonian. After the Legrendre transformation we obtain

$$
\begin{aligned}
\mathcal{H}_p &= \frac{m^2}{4}P^{ij}P_{ij} - \frac{1}{2}P^{ij}\partial_i\partial_j h_{00} - P^{ij}R_{ilj}{}^{l} + \frac{3}{4}PR_{lm}{}^{lm} + \frac{2}{m^2}\partial^l K_{il}\partial^j K^i_j - \frac{4}{m^2}\partial^j K_{ij}\partial^i K \\
&\quad + \frac{2}{m^2}\partial_i K\partial^i K + K^2 - K_{ij}K^{ij} + \frac{1}{2}h^{ij}\left(R_{ikj}{}^{k} - \frac{1}{2}\delta_{ij}R_{lm}{}^{lm}\right) + \frac{1}{2}h_{00}R_{ij}{}^{ij} + 2\pi^{ij}K_{ij} \\
&\quad + 2\pi^{ij}\partial_i h_{0j} + u_{\mu\nu}\phi^{\mu\nu} + \zeta_{ij}\psi^{ij} + \xi\psi, \qquad (3.19)
\end{aligned}
$$

where $u_{\mu\nu}$, ζ_{ij} and ξ are Lagrange multipliers enforcing the primary constraints. It is essential to observe the presence of linear terms in the conjugate momenta π^{ij}, such as $2\pi^{ij}K_{ij}$ and $2\pi^{ij}\partial_i h_{0j}$, which could be related to an Ostrogradski's instability. However, in further lines we will remove this apparently instability by means of the introduction of the Dirac brackets and the second-class constraints.

For now, the consistency condition on the primary constraints results in

$$
\begin{aligned}
\mathcal{H} &: \quad \dot{\phi}^{00} = \frac{1}{2}\left(\partial_i\partial_j P^{ij} - R_{ij}{}^{ij}\right) \approx 0, \\
\mathcal{H}^i &: \quad \dot{\phi}^{0i} = \partial_j \pi^{ij} \approx 0, \\
S^{lm} &: \quad \dot{\phi}^{lm} = \frac{1}{2}\left(\partial^l\partial_i P^{im} + \partial^m\partial_i P^{il} - \nabla^2 P^{lm} - \delta^{lm}\partial_i\partial_j P^{ij}\right) - \left(\partial^l\partial^m - \delta^{lm}\nabla^2\right)\left(\frac{3}{4}P + \frac{1}{2}h_{00}\right) \\
&\qquad - \left(R^l{}_k{}^{mk} - \frac{1}{2}\delta^{lm}R_{ij}{}^{ij}\right) - \frac{1}{m^2}\left(\partial^l\partial^m - \delta^{lm}\nabla^2\right)\xi - \zeta^{lm} \approx 0, \\
Q^{ij} &: \quad \dot{\psi}^{ij} = u^{ij} \approx 0, \\
W &: \quad \dot{\psi} = \frac{2}{m^2}\left(\nabla^2 K - \partial_i\partial_j K^{ij}\right) - 2K - 2\pi^i_i \approx 0. \qquad (3.20)
\end{aligned}
$$

We observe that only $\mathcal{H}, \mathcal{H}^i$ and W are secondary constraints, while S^{ij} and Q^{ij} represent relations involving the Lagrange multipliers. According to the algorithm, we should again apply the consistency condition on the secondary constraints. We find

$$
\begin{aligned}
\dot{\mathcal{H}} &= -\partial_i\partial_j\pi^{ij} \approx 0, \\
\dot{\mathcal{H}}^i &\approx 0, \\
\dot{W} &= -\partial_i\partial_j P^{ij} - 4\xi \approx 0. \qquad (3.21)
\end{aligned}
$$

Finally, since there are no more constraints from the time evolution of (3.21), the algorithm ends, and the complete set of constraints is given by the following 14 expressions:

$$
\begin{aligned}
\phi^{00} &: \quad \pi^{00} \approx 0, \\
\phi^{0i} &: \quad \pi^{0i} \approx 0, \\
\phi^{ij} &: \quad \pi^{ij} - \alpha^{ij} \approx 0, \\
\psi^{ij} &: \quad \tau^{ij} \approx 0, \\
\psi &: \quad P + \frac{1}{m^2} R_{lm}{}^{lm} \approx 0, \\
\mathcal{H} &: \quad \partial_i \partial_j P^{ij} - R_{ij}{}^{ij} \approx 0, \\
\mathcal{H}^i &: \quad \partial_j \pi^{ij} \approx 0, \\
W &: \quad \frac{1}{m^2} \left(\partial_i \partial_j K^{ij} - \nabla^2 K \right) + K + \pi^i_i \approx 0.
\end{aligned} \tag{3.22}
$$

At the Hamiltonian level the relevant constraint classification is into first-class and second-class. Recall that for this we need to calculate the Poisson's brackets between all the constraints. It is helpful to set up the following matrix, whose entries are the Poisson brackets between all the constraints

$$
M = \begin{array}{c} \\ \phi^{00} \\ \phi^{0l} \\ \phi^{lm} \\ \psi^{lm} \\ \psi \\ \mathcal{H} \\ \mathcal{H}^l \\ W \end{array}
\begin{array}{c}
\begin{array}{cccccccc} \phi^{00} & \phi^{0i} & \phi^{ij} & \psi^{ij} & \psi & \mathcal{H} & \mathcal{H}^i & W \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \{\phi^{lm}, \psi^{ij}\} & \{\phi^{lm}, \psi\} & \{\phi^{lm}, \mathcal{H}\} & 0 & 0 \\
0 & 0 & \{\psi^{lm}, \phi^{ij}\} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \{\psi, \phi^{ij}\} & 0 & 0 & 0 & 0 & \{\psi, W\} \\
0 & 0 & \{\mathcal{H}, \phi^{ij}\} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \{W, \psi\} & 0 & 0 & 0
\end{pmatrix}
\end{array} \tag{3.23}
$$

where

$$
\begin{aligned}
\left\{\psi^{ij}, \phi^{lm}\right\} &= \frac{1}{2} \left(\delta^{il} \delta^{jm} + \delta^{im} \delta^{jl} \right) \delta^2(x-y), \\
\left\{\psi, \phi^{ij}\right\} &= \frac{1}{m^2} \left(\partial^i \partial^j - \delta^{ij} \nabla^2 \right) \delta^2(x-y), \\
\left\{\mathcal{H}, \phi^{ij}\right\} &= -\left(\partial^i \partial^j - \delta^{ij} \nabla^2 \right) \delta^2(x-y), \\
\{\psi, W\} &= -2\delta^2(x-y).
\end{aligned} \tag{3.24}
$$

After a long calculation we observe that the matrix (3.23) has rank = 8 and thus 6 null vectors. From the null vectors we find the following 6 first-class constraints

$$
\begin{aligned}
\gamma_1 \quad &: \quad \pi^{00} \approx 0, \\
\gamma_2^i \quad &: \quad \pi^{0i} \approx 0, \\
\gamma_3^i \quad &: \quad \partial_j \pi^{ij} \approx 0, \\
\gamma_4 \quad &: \quad \partial_i \partial_j P^{ij} - R_{ij}^{\ \ ij} + \left(\partial_i \partial_j - \delta_{ij} \nabla^2\right) \tau^{ij} \approx 0,
\end{aligned} \tag{3.25}
$$

where γ_3^i is the so-called Gauss constraint, or momentum constraint. On the other hand, we identify the following 8 second-class constraints

$$
\begin{aligned}
\chi_1^{ij} \quad &: \quad \pi^{ij} - \alpha^{ij} \approx 0, \\
\chi_2^{ij} \quad &: \quad \tau^{ij} \approx 0, \\
\chi_3 \quad &: \quad P + \frac{1}{m^2} R_{ij}^{\ \ ij} \approx 0, \\
\chi_4 \quad &: \quad K + \frac{1}{m^2}\left(\partial_i \partial_j K^{ij} - \nabla^2 K\right) + \pi_i^i \approx 0.
\end{aligned} \tag{3.26}
$$

Considering this classification we carry out the counting of physical degrees of freedom as follows: there are 24 canonical variables, eight second-class constraints and six first-class constraints, thus

$$
DOF = \frac{1}{2}\left(24 - 8 - 2 * 6\right) = 2, \tag{3.27}
$$

which correspond to the massive modes of helicity ± 2 [27]. On the other hand, since we have the complete set of second-class constraints at our disposal, we can construct the Dirac brackets (2.25). Taking into account that the matrix between the second-class constraints is

$$
C_{\alpha\beta} = \begin{array}{c} \\ \chi_1^{11} \\ \chi_1^{12} \\ \chi_1^{22} \\ \chi_2^{11} \\ \chi_2^{12} \\ \chi_2^{22} \\ \chi_3 \\ \chi_4 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} \chi_1^{11} & \chi_1^{12} & \chi_1^{22} & \chi_2^{11} & \chi_2^{12} & \chi_2^{22} & \chi_3 & \chi_4 \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & -m^2 & 0 & 0 & \partial^2\partial^2 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2}m^2 & 0 & -\partial^1\partial^2 & 0 \\
0 & 0 & 0 & 0 & 0 & -m^2 & \partial^1\partial^1 & 0 \\
m^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2}m^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & m^2 & 0 & 0 & 0 & 0 & 0 \\
-\partial^2\partial^2 & \partial^1\partial^2 & -\partial^1\partial^1 & 0 & 0 & 0 & 0 & -2m^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 2m^2 & 0
\end{pmatrix}
\end{array}
\frac{1}{m^2}\delta^2(x-y). \tag{3.28}
$$

and its inverse is given by

$$
C^{\alpha\beta}=\begin{array}{c c}
 & \begin{array}{cccccccc} \chi_1^{11} & \chi_1^{12} & \chi_1^{22} & \chi_2^{11} & \chi_2^{12} & \chi_2^{22} & \chi_3 & \chi_4 \end{array} \\
\begin{array}{c} \chi_1^{11} \\ \chi_1^{12} \\ \chi_1^{22} \\ \chi_2^{11} \\ \chi_2^{12} \\ \chi_2^{22} \\ \chi_3 \\ \chi_4 \end{array} &
\begin{pmatrix}
0 & 0 & 0 & m^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2m^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & m^2 & 0 & 0 \\
-m^2 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\partial_2\partial_2 \\
0 & -2m^2 & 0 & 0 & 0 & 0 & 0 & -\partial_1\partial_2 \\
0 & 0 & -m^2 & 0 & 0 & 0 & 0 & \frac{1}{2}\partial_1\partial_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}m^2 \\
0 & 0 & 0 & -\frac{1}{2}\partial_2\partial_2 & \partial_1\partial_2 & -\frac{1}{2}\partial_1\partial_1 & -\frac{1}{2}m^2 & 0
\end{pmatrix}
\end{array}\frac{1}{m^2}\delta^2(x-y),
\tag{3.29}
$$

we obtain the following non-trivial Dirac brackets

$$
\begin{aligned}
\left\{h_{ij},\pi^{lm}\right\}_D &= \frac{1}{2}\left(\delta_i^l\delta_j^m+\delta_i^m\delta_j^l\right)\delta^2(x-y)+\frac{1}{2m^2}\delta_{ij}\left(\partial^l\partial^m-\delta^{lm}\nabla^2\right)\delta^2(x-y),\\
\left\{h_{ij},\alpha^{lm}\right\}_D &= \frac{1}{2}\left(\delta_i^l\delta_j^m+\delta_i^m\delta_j^l\right)\delta^2(x-y)+\frac{1}{2m^2}\delta_{ij}\left(\partial^l\partial^m-\delta^{lm}\nabla^2\right)\delta^2(x-y),\\
\left\{K_{ij},P^{lm}\right\}_D &= \frac{1}{2}\left(\delta_i^l\delta_j^m+\delta_i^m\delta_j^l\right)\delta^2(x-y)-\frac{1}{2}\delta_{ij}\left(\delta^{lm}+\frac{1}{m^2}\left(\partial^l\partial^m-\delta^{lm}\nabla^2\right)\right)\delta^2(x-y),\\
\left\{\pi_{ij},P^{lm}\right\}_D &= \frac{1}{2m^2}\left(\partial_i\partial_j-\delta_{ij}\nabla^2\right)\left(\delta^{lm}+\frac{1}{m^2}\left(\partial^l\partial^m-\delta^{lm}\nabla^2\right)\right)\delta^2(x-y),\\
\left\{\alpha_{ij},P^{lm}\right\}_D &= \frac{1}{2m^2}\left(\partial_i\partial_j-\delta_{ij}\nabla^2\right)\left(\delta^{lm}+\frac{1}{m^2}\left(\partial^l\partial^m-\delta^{lm}\nabla^2\right)\right)\delta^2(x-y),\\
\left\{h_{ij},K^{lm}\right\}_D &= -\frac{1}{2}\delta_{ij}\delta^{lm}\delta^2\left(x-y\right).
\end{aligned}
\tag{3.30}
$$

3.3 Extended NMG Hamiltonian

Now, we shall construct the extended Hamiltonian which is a fundamental element in the canonical formulation. In fact, the extended Hamiltonian is a first-class function and it is used in the quantization program because it contains all the relevant information from the theory. It is given by

$$
\mathcal{H}_E=\mathcal{H}+w^\alpha\chi_\alpha,
\tag{3.31}
$$

where w^α are the Lagrange multipliers associated with the second-class constraints, these multipliers can be determined through [13–15]

$$
w^\alpha=C^{\beta\alpha}\left\{\chi_\beta,H\right\}_D.
\tag{3.32}
$$

In this manner the following expressions for the Lagrange multipliers are obtained

$$
\begin{aligned}
w_1 &= 0, \\
w_2 &= 0, \\
w_3 &= 0, \\
w_4 &= \left(\partial^1\partial_i P^{i1} - \frac{1}{2}\nabla^2 P^{11} + \frac{3}{4}\partial_2\partial_2 P + \frac{1}{2}\partial_2\partial_2 h_{00} - R^1{}_k{}^{1k} - \frac{1}{2}\partial_i\partial_j P^{ij} + \frac{1}{2}R_{ij}{}^{ij}\right. \\
&\quad \left. -\frac{1}{4m^2}\partial_2\partial_2\partial_i\partial_j P^{ij}\right)\delta^2(x-y), \\
w_5 &= \left(\partial^1\partial_i P^{i2} + \partial^2\partial_i P^{i1} - \nabla^2 P^{12} - \frac{3}{2}\partial_1\partial_2 P - \partial_1\partial_2 h_{00} - 2R^1{}_k{}^{2k}\right. \\
&\quad \left. +\frac{1}{2m^2}\partial_1\partial_2\partial_i\partial_j P^{ij}\right)\delta^2(x-y), \\
w_6 &= \left(\partial^2\partial_i P^{i2} - \frac{1}{2}\nabla^2 P^{22} + \frac{3}{4}\partial_1\partial_1 P + \frac{1}{2}\partial_1\partial_1 h_{00} - R^2{}_k{}^{2k} - \frac{1}{2}\partial_i\partial_j P^{ij} + \frac{1}{2}R_{ij}{}^{ij}\right. \\
&\quad \left. -\frac{1}{4m^2}\partial_1\partial_1\partial_i\partial_j P^{ij}\right)\delta^2(x-y), \\
w_7 &= \frac{1}{4}\partial_i\partial_j P^{ij}\delta^2(x-y), \\
w_8 &= \left(\frac{1}{m^2}\left(\nabla^2 K - \partial_i\partial_j K^{ij}\right) - K - \pi^i_i\right)\delta^2(x-y).
\end{aligned} \tag{3.33}
$$

Hence, by using the second-class constraints (3.26) and the Lagrange multipliers (3.33), the extended Hamiltonian takes the following form

$$
\begin{aligned}
\mathcal{H}_E &= \frac{m^2}{4}P^{ij}P_{ij} - P^{ij}R_{ilj}{}^l + \frac{3}{4}PR_{lm}{}^{lm} + \frac{2}{m^2}\partial^l K_{il}\partial^j K^i_j - \frac{4}{m^2}\partial^j K_{ij}\partial^i K + \frac{2}{m^2}\partial_i K\partial^i K \\
&\quad + K^2 - K_{ij}K^{ij} + \frac{1}{2}h^{ij}\left(R_{ikj}{}^k - \frac{1}{2}\delta_{ij}R_{lm}{}^{lm}\right) + 2\pi^{ij}K_{ij} + 2\pi^{ij}\partial_i h_{0j} \\
&\quad - \left(\frac{1}{m^2}\left(\nabla^2 K - \partial_i\partial_j K^{ij}\right) - K - \pi^i_i\right)\left(K + \frac{1}{m^2}\left(\partial_i\partial_j K^{ij} - \nabla^2 K\right) + \pi^i_i\right) \\
&\quad - \frac{1}{2}h_{00}\left(\partial_i\partial_j P^{ij} - R_{ij}{}^{ij} + \left(\partial_i\partial_j - \delta_{ij}\nabla^2\right)\tau^{ij}\right) - \left(R^l{}_k{}^{mk} - \frac{1}{2}\delta^{lm}R_{ij}{}^{ij}\right)\tau_{lm} \\
&\quad + \frac{1}{4}\partial_i\partial_j P^{ij}\left(P + \frac{1}{m^2}R_{lm}{}^{lm} + \frac{1}{m^2}\left(\partial_l\partial_m - \delta_{lm}\nabla^2\right)\tau^{lm}\right) \\
&\quad + \left(\partial^i\partial_l P^{lj} - \frac{1}{2}\partial_l\partial_m P^{lm}\delta^{ij} - \frac{1}{2}\nabla^2 P^{ij} - \frac{3}{4}P\left(\partial^i\partial^j - \delta^{ij}\nabla^2\right)\right)\tau_{ij}.
\end{aligned} \tag{3.34}
$$

In the third line we observe a positive quadratic contribution of the momenta π^i_i, which fixes the previously problematic linearity of these momenta. Thus, the extended Hamiltonian is bounded from below, which means that the two degrees of freedom found above are undoubtedly physical. This is consistent with that reported in [27], but no sufficient proof of this fact is given there, it is only assumed on Lagrangian grounds. In our Hamiltonian analysis we achieve this by including all the second-class constraints, which fixes the Ostrogradski instability. To contrast this, in the Appendix A we introduce the Lagrangian describing only the Einstein-Klein-Gordon (EKG) system (see the first equation in (3.6)). This is also a higher-order system, but the constraint $\underset{Lin}{R} = 0$ is not present. Then, by means of a complete canonical analysis, we found that three degrees of freedom are propagated, and its Hamiltonian is also a linear function of some momenta. However, its second-order constraints are trivial and do not resolve the Ostrogradski instability. In fact, this proposal of the higher-order EKG system is one of the original contributions of our research.

We now continue with the Dirac algebra between the first-class constraints and the extended Hamiltonian. It is given by

$$\begin{aligned} \{\gamma_1, \mathcal{H}_E\}_D &= \frac{1}{2}\gamma_4, \\ \left\{\gamma_2^i, \mathcal{H}_E\right\}_D &= \gamma_3^i, \end{aligned} \tag{3.35}$$

and $\left\{\gamma_3^i, \mathcal{H}_E\right\}_D = \{\gamma_4, \mathcal{H}_E\}_D = 0$. We observe that the algebra is closed and the extended Hamiltonian is of first-class as expected. Furthermore, by using the Dirac brackets and the extended Hamiltonian it is possible to find the following equations of motion

$$\dot{h}_{ij} = \{h_{ij}, \mathcal{H}_E\}_D = 2K_{ij} + \partial_i h_{0j} + \partial_j h_{0i}, \tag{3.36}$$

which is the relation between K_{ij} and h_{ij} given in (2.46), and

$$\dot{K} = \{K, \mathcal{H}_E\}_D = -\frac{1}{2}\nabla^2 h_{00} - \frac{1}{2}R_{ij}{}^{ij}, \tag{3.37}$$

namely

$$2\dot{K} + \nabla^2 h_{00} + R_{ij}{}^{ij} = 0, \tag{3.38}$$

that corresponds to the equation of motion $\underset{Lin}{R} = 0$. This completes our analysis and extends the results found in the literature. Furthermore, in the Appendix B we present the canonical analysis of the purely higher-order sector J (3.2) with the aim of visualizing the consequences of its coupling to GR. This is in analogy to the coupling of the Chern-Simons term and GR in TMG [21]. In TMG both terms are topological, i.e. they do not propagate any degree of freedom, but its coupling origins one massive degree of freedom. On the other hand, J is not topological, since it propagates one ghost degree of freedom as shown in Appendix B. However, its coupling to GR in NMG results in the propagation of two physical degrees of freedom, as we have seen here.

Chapter 4

λR Gravity

In the present chapter, we deal with the Hamiltonian analysis of λR gravity, the low-energy limit of HG. The Hořava's proposal for a higher-order theory of gravity was published in 2009 [29, 30] and has since been the subject of intense study [40].

Unlike higher-order theories based on the Stelle structure [10], HG is completely ghost-free, which removes from the outset one of the main obstacles to reaching the quantization. This is possible because its construction uses a different underlying symmetry. Recall that, like GR, the underlying symmetry of Stelle proposal is that of diffeomorphisms. Its action then involves only higher-order terms that are diffeomorphism-covariant, which can be problematic because they inevitably involve higher order time and thus possibly ghosts. On the other hand, in Hořava gravity the diffeomorphism invariance is sacrificed by establishing a foliation of spacetime and then forcing the symmetries to adapt to the preservation of this foliation; the so-called foliation-preserving diffeomorphisms (FDiff). These are are given by

$$t \to t'(t), \qquad\qquad x^i \to x'^i(x^i, t), \tag{4.1}$$

in coordinates adapted to the foliation. This fixed foliation provides an absolute distinction between time and space similar to the Newtonian one, which allows anisotropy by assuming a different scaling between space and time according to

$$t \to b^{-z}t, \qquad x^i \to b^{-1}x^i, \tag{4.2}$$

where z is the so-called critical exponent. In this way, with the new symmetry (4.1) and the anisotropic scaling (4.2), the Hořava's action is constructed with terms that depend separately on either spatial or temporal derivatives of different orders, depending on the value of the chosen critical exponent. Since ghosts can appear due to higher-order time derivatives, we choose to introduce only quadratic terms in the velocities; however, we can maintain the aspiration to be renormalizable if we still consider terms with higher-order spatial derivatives. In order for the theory to be renormalizable by power counting, it is necessary to consider terms of at least sixth order in spatial derivatives.

Based on these considerations, the realization of HG is easily established by means of the ADM formalism [31], on which the Hamiltonian of GR can be constructed in a non-perturbative way. In contrast to the perturbative approach, where the background metric is used to identify the velocities, the ADM formalism considers a foliation of spacetime to define a dynamical system of geometric quantities defined on a spacelike leaf, and evolving from leaf to leaf. This defines the velocities. However, since the foliation is not fixed, this formulation is still compatible with GR diffeomorphisms. Thus, given the geometric structure established from the foliation, the only thing left to implement the Hořava idea is to fix the foliation. Since the GR action is of second order, we can use a similar kinetic part for the Hořava action, so that

$$S_{\mathrm{HG}} = \int dt d^3x \sqrt{g} N \left(K_{ij} K^{ij} - \lambda K^2 - \mathcal{V} \right), \tag{4.3}$$

where $\mathcal{V}$ is the potential containing the spatial derivative terms. We will not show here the list of possible terms since they will not be relevant to what follows. We invite the interested reader to consult [42].

In the ADM formalism g_{ij} is the metric defined on each spacelike hypersurface, $K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - 2\nabla_{(i} N_{j)}\right)$ is the extrinsic curvature, and N and N_i are the lapse and shift functions respectively. In general g_{ij}, N and N_i are the so-called ADM variables [31]. It is important to mention that in this work we consider only the theory without the projectability condition, where the lapse function has a general dependence on space and time, since this is the scenario that is closer to GR. The choice of a projectable lapse function as a definitory condition leads to a different theory that cannot be smoothly deduced from the non-projectable case. The Hamiltonian analysis for the counterpart of the action (4.3) in the projectable case has been done in [43].

Concerning the kinetic part, a central aspect in setting up the Hořava's action is the introduction of the λ parameter that determines the separate compatibility of the kinetic terms with FDiff. An outstanding feature is that, in the realm of non-projectability, the kinetic part acquires an anisotropic conformal symmetry at $\lambda = \frac{1}{3}$ [29]. The Weyl transformations are anisotropic in the sense that the lapse function scales with a weight different from the one of the spatial metric and the shift vector

$$\tilde{g}_{ij} = \Omega^2 g_{ij}, \quad \tilde{N} = \Omega^3 N, \quad \tilde{N}_i = \Omega^2 N_i, \tag{4.4}$$

where the dependence of $\Omega = \Omega(x,t)$ is consistent only with non-projectability. This conformal symmetry of the kinetic part gives rise to a primary constraint that plays a primordial role in the dynamics of the theory, as we will see bellow. The first or second-class character of this constraint depends on the chosen potential.

For the second-order $\mathcal{V} = -R$, where R is now the spatial Ricci scalar constructed from the spatial metric g_{ij}, the action is known as λR gravity [28, 41]. This lower-order HG version is fundamental to study the theoretical consistency of the full theory, and is the one which we are concerned here. It was initially thought that the very structure of FDiff implied the presence of a strongly coupled degree of freedom additional to the two of GR [44]. Although it has been possible to find useful cosmological applications for this extra mode [71–73], this put into debate the consistency of Hořava theory, since at the IR regime it would differ from GR and its well-tested predictions. It was also believed that the only way to match both theories in this regime would be precisely in the specific case $\lambda = 1$, when the original ADM EH action [44] is restored and thus the extra degree of freedom is suppressed. It was later demonstrated by means of a full Hamiltonian analysis on the λR gravity, that the constraint content of this theory allows to establish an important closeness to GR regardless the λ value [28].

In this understanding, we now present our work on the canonical analysis of this model, but at the perturbative level [41]. Moreover, we have carried out an analysis of an extension of this theory with a term compatible with FDiff, which was not originally considered by Hořava [45]. In general, FDiff-compatible terms are known as BPS [46].

4.1 Perturbative λR Gravity

As mentioned above, the action for λR gravity is

$$S_{\lambda\mathrm{R}} = \int dt d^3x \sqrt{g} N \left(K_{ij} K^{ij} - \lambda K^2 + R \right). \tag{4.5}$$

In the literature we found that this theory has been analyzed at linearized level in [47, 48], where a perturbation around a Minkowski background in the ADM formalism was developed as

$$g_{ij} = \delta_{ij} + \epsilon h_{ij}, \qquad N = 1 + \epsilon n, \qquad N_i = \epsilon n_i; \tag{4.6}$$

and some of the linearized ADM constraints were found. However, a complete identification and classification of the constraints was not developed and Dirac brackets were not reported. In view of this, we perform a complete canonical analysis, but using a different approach. Instead of the perturbation (4.6), we introduce the 3+1 formalism worked above. This is also compatible with the preferred time direction defined by FDiff, and as we have seen in the NMG case, it is helpful to economize the analysis.

In this way, we can easily establish the perturbative λR action by resorting to the GR action (2.47) presented above. It is

$$\mathcal{L} = K_{ij}K^{ij} - K^2 - \frac{1}{2}h^{00}R_{ij}{}^{ij} - \frac{1}{2}h^{ij}\left(R_{ikj}{}^{k} - \frac{1}{2}\delta_{ij}R_{lm}{}^{lm}\right). \tag{4.7}$$

Given that the kinetic part is similar to that of (4.5), we can directly introduce the λ parameter in such a way that the kinetic part now becomes $K_{ij}K^{ij} - \lambda K^2$. Recall that here the perturbed extrinsic curvature is written in terms of h_{ij} and h_{0i} (equation (2.46)). Considering this, we choose to set the action in the following way,

$$\mathcal{L} = G^{ijkl}K_{ij}K_{kl} - \frac{1}{2}h^{00}R_{ij}{}^{ij} - \frac{1}{2}h^{ij}\left(R_{ikj}{}^{k} - \frac{1}{2}\delta_{ij}R_{lm}{}^{lm}\right), \tag{4.8}$$

where

$$G^{ijkl} = \frac{1}{2}\left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right) - \lambda\delta^{ij}\delta^{kl}. \tag{4.9}$$

This latter expression is a linearized version of the generalized De Witt metric; note that with $\lambda = 1$ we recover the Fierz-Pauli Lagrangian for perturbative GR. It is also important to underline that our construction of this linearized Lagrangian is not followed from (4.5). In fact, we propose this as a FDiff-compatible second-order action, but it is based directly on linearized GR. However, as we will see, our proposal corresponds to the dynamics of linearized λR gravity.

Now, with this Lagrangian at hand, we will carry out the canonical analysis, but considering separately the cases $\lambda \neq \frac{1}{3}$ and $\lambda = \frac{1}{3}$ for the reasons already explained above.

4.1.1 Canonical Analysis for $\lambda \neq \frac{1}{3}$

We start by calculating the canonical momenta of the action (4.8), they are given by

$$\pi^{00} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{00}} = 0, \tag{4.10}$$

$$\pi^{0i} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{0i}} = 0, \tag{4.11}$$

$$\pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = G^{ijkl} K_{kl}. \tag{4.12}$$

For constructing the canonical Hamiltonian we need an expression for the velocity $\dot{h}_{ij}$ in terms of the canonical variables. We achieve this by considering (4.12) and its trace

$$K_{ij} = \pi_{ij} + \frac{\lambda}{1-3\lambda}\delta_{ij}\pi = \mathcal{G}_{ijkl}\pi^{kl}. \tag{4.13}$$

Reinserting the definition (2.46) of K_{ij} we find

$$\dot{h}_{ij} = 2\mathcal{G}_{ijkl}\pi^{kl} + \partial_i h_{j0} + \partial_j h_{i0}, \tag{4.14}$$

where $\mathcal{G}_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \frac{\lambda}{1-3\lambda}\delta_{ij}\delta_{kl}$ is the inverse of the generalized De Witt metric: $G^{ijkl}\mathcal{G}_{klpq} = \frac{1}{2}\left(\delta^i_p\delta^j_q + \delta^i_q\delta^j_p\right)$. It should be noted that $\lambda = \frac{1}{3}$ is a singular value of $\mathcal{G}$, therefore the treatment of this case will be discussed separately in the following section.
Since the primary constraints are given by (4.10) and (4.11), we consider

$$\begin{aligned} \phi &: \quad \pi^{00} \approx 0, \\ \phi^i &: \quad \pi^{0i} \approx 0. \end{aligned} \tag{4.15}$$

to establish the primary Hamiltonian as

$$\mathcal{H}_p = \mathcal{G}_{ijkl}\pi^{kl}\pi^{ij} - 2h_{j0}\partial_i\pi^{ij} + \frac{1}{2}h^{00}R_{ij}{}^{ij} + \frac{1}{2}h^{ij}\left(R_{ikj}{}^{k} - \frac{1}{2}\delta_{ij}R_{lm}{}^{lm}\right) + u\phi + u_i\phi^i, \tag{4.16}$$

where u and u_i are the Lagrange multipliers enforcing the primary constraints. Then, by using the fundamental Poisson-bracket relations

$$\left\{h_{ij}(x), \pi^{kl}(y)\right\} = \frac{1}{2}\left(\delta^k_i\delta^l_j + \delta^l_i\delta^k_j\right)\delta^3(x-y), \tag{4.17}$$

we obtain two secondary constraints given by

$$\begin{aligned} \mathcal{H} &: \quad \{\phi, \int d^3x\, \mathcal{H}'\} = R_{ij}{}^{ij} \approx 0, \\ \mathcal{H}^i &: \quad \{\phi^i, \int d^3x\, \mathcal{H}'\} = \partial_j\pi^{ji} \approx 0. \end{aligned} \tag{4.18}$$

We notice here an important difference with GR, since when applying the consistency condition on these constraints, the Hamiltonian constraint leads to a tertiary constraint

$$\theta: \quad \dot{\mathcal{H}} = \left(\frac{\lambda-1}{1-3\lambda}\right)\nabla^2\pi + \partial_i\partial_j\pi^{ij} \approx 0, \tag{4.19}$$

and from the time evolution of the above expression the following constraint arises

$$\gamma: \quad \dot{\theta} = \left(\frac{\lambda-1}{1-3\lambda}\right)\left(\nabla^2\nabla^2 h^{00} + \frac{1}{2}\nabla^2 R_{ij}{}^{ij}\right) \approx 0. \tag{4.20}$$

Of course if we set $\lambda = 1$ these constraints vanish and we recover the constraint structure of GR. Here the generation of constraints ends, and the attempt to obtain more constraints only leads to relations involving Lagrange multipliers u and u_i. The total set of 10 constraints is listed as

$$\begin{aligned}
\phi &: \pi^{00} \approx 0, \\
\phi^i &: \pi^{0i} \approx 0, \\
\mathcal{H} &: R_{ij}{}^{ij} \approx 0, \\
\mathcal{H}^i &: \partial_j \pi^{ji} \approx 0, \\
\theta &: \left(\frac{\lambda-1}{1-3\lambda}\right)\nabla^2\pi \approx 0, \\
\gamma &: \left(\frac{\lambda-1}{1-3\lambda}\right)\left(\nabla^2\nabla^2 h^{00} + \frac{1}{2}\nabla^2 R_{ij}{}^{ij}\right) \approx 0,
\end{aligned} \tag{4.21}$$

which we will now proceed to classify into first-class and second-class constraints. To do this, let us first look at the calculation of the Poisson brackets between the constraints. Those that do not vanish are

$$\begin{aligned}
\{\gamma, \phi\} &= \left(\frac{\lambda-1}{1-3\lambda}\right)\nabla^2\nabla^2\delta^3(x-y) \\
\{\mathcal{H}, \theta\} &:= -2\left(\frac{\lambda-1}{1-3\lambda}\right)\nabla^2\nabla^2\delta^3(x-y), \\
\{\gamma, \theta\} &= -\left(\frac{\lambda-1}{1-3\lambda}\right)^2\nabla^2\nabla^2\nabla^2\delta^3(x-y),
\end{aligned} \tag{4.22}$$

As we know, the constraints whose Poisson brackets vanish with the entire set of constraints are the first-class constraints and generate gauge transformations [15]. Then, we identify the following 6 first-class constraints

$$\begin{aligned}
\gamma_1^i &: \pi^{0i} \approx 0, \\
\gamma_2^i &: \partial_j \pi^{ji} \approx 0.
\end{aligned} \tag{4.23}$$

In the opposite case we obtain second-class constraints. We identify the following 4 constraints of this kind

$$
\begin{aligned}
\chi_1 &: \quad R_{ij}{}^{ij} \approx 0, \\
\chi_2 &: \quad \left(\frac{\lambda-1}{1-3\lambda}\right) \nabla^2 \pi \approx 0, \\
\chi_3 &: \quad \pi^{00} \approx 0, \\
\chi_4 &: \quad \left(\frac{\lambda-1}{1-3\lambda}\right) \nabla^2 \nabla^2 h^{00} \approx 0.
\end{aligned} \tag{4.24}
$$

In this manner, the counting of the degrees of freedom is carried out in the following form

$$
DOF = \frac{1}{2}(20 - 4 - 2 * 6) = 2. \tag{4.25}
$$

This is consistent with that reported in [49]. On the other hand, we now calculate the Dirac brackets with the set (4.24). The inverse of the matrix of Poisson brackets of this set is

$$
C^{\alpha\beta} = \begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{array} \begin{array}{c} \begin{array}{cccc} \chi_1 & \chi_2 & \chi_3 & \chi_4 \end{array} \\ \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \end{array} \frac{1}{\beta \nabla^2 \nabla^2} \delta^3(x-y), \tag{4.26}
$$

where $\beta = \frac{\lambda-1}{1-3\lambda}$. In this way the non vanishing Dirac's brackets are

$$
\left\{ h_{ij}, \pi^{lm} \right\}_D = \frac{1}{2} \left(\delta_i^l \delta_j^m + \delta_i^m \delta_l^k \right) \delta^3(x-y) + \frac{1}{2\nabla^2} \delta_{ij} \left(\partial^l \partial^m - \delta^{lm} \nabla^2 \right) \delta^3(x-y). \tag{4.27}
$$

4.1.2 Canonical Analysis for $\lambda = \frac{1}{3}$

At this value of λ let us consider the particular form of the generalized De Witt metric

$$
\hat{G}^{ijkl} = \frac{1}{2} \left(\delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} \right) - \frac{1}{3} \delta^{ij} \delta^{kl}. \tag{4.28}
$$

Then the Lagrangian (4.8) is written as

$$
\mathcal{L} = \hat{G}^{ijkl} K_{ij} K_{kl} - \frac{1}{2} h^{00} R_{ij}{}^{ij} - \frac{1}{2} h^{ij} \left(R_{ikj}{}^{k} - \frac{1}{2} \delta_{ij} R_{lm}{}^{lm} \right), \tag{4.29}
$$

and now the expressions for the canonical momenta are

$$
\pi^{00} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{00}} = 0, \tag{4.30}
$$

$$\pi^{0i} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{0i}} = 0, \tag{4.31}$$

$$\pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \hat{G}^{ijkl} K_{kl}. \tag{4.32}$$

In this particular case we observe from (4.32) that $\delta_{ij}\pi^{ij} = \pi = 0$, i.e. there is an extra primary constraint. Thus the set of primary constraints are

$$\begin{aligned} \phi &: \pi^{00} \approx 0, \\ \phi^i &: \pi^{0i} \approx 0, \\ \xi &: \pi \approx 0, \end{aligned} \tag{4.33}$$

and the primary Hamiltonian takes the form

$$\mathcal{H}_p = \pi^{ij}\pi_{ij} - 2\partial_i \pi^{ij} h_{0j} + \frac{1}{2} h^{00} R_{ij}{}^{ij} + \frac{1}{2} h^{ij} \left(R_{ikj}{}^{k} - \frac{1}{2}\delta_{ij} R_{lm}{}^{lm} \right) + u\phi + u_i \phi^i + v\xi. \tag{4.34}$$

From consistency on the primary constraints we obtain the following five secondary constraints

$$\begin{aligned} \mathcal{H} &: R_{ij}{}^{ij} \approx 0, \\ \mathcal{H}^i &: \partial_j \pi^{ji} \approx 0, \\ \gamma &: \nabla^2 h^{00} + \frac{1}{2} R_{ij}{}^{ij} \approx 0. \end{aligned} \tag{4.35}$$

Since the evolution of these expressions results in the following relations between Lagrange multipliers

$$\begin{aligned} \dot{\mathcal{H}} &= \partial_i \partial_j \pi^{ij} - \nabla^2 \pi - \nabla^2 v \approx 0, \\ \dot{\gamma} &= \nabla^2 u \approx 0, \end{aligned} \tag{4.36}$$

the complete set of constraints is

$$\begin{aligned} \phi &: \pi^{00} \approx 0, \\ \phi^i &: \pi^{0i} \approx 0, \\ \xi &: \pi \approx 0, \\ \mathcal{H} &: R_{ij}{}^{ij} \approx 0, \\ \mathcal{H}^i &: \partial_j \pi^{ji} \approx 0, \\ \gamma &: \nabla^2 h^{00} + \frac{1}{2} R_{ij}{}^{ij} \approx 0, \end{aligned} \tag{4.37}$$

and the nonzero Poisson brackets between them are

$$\begin{aligned}\{\gamma, \phi\} &= \nabla^2 \delta^3(x-y), \\ \{\psi, \xi\} &= -2\nabla^2 \delta^3(x-y), \\ \{\gamma, \xi\} &= -\nabla^2 \delta^3(x-y).\end{aligned} \tag{4.38}$$

With this we can perform the classification of constraints. We obtain 6 first-class constraints given by

$$\begin{aligned}\gamma_1^i \ &: \ \pi^{0i} \approx 0, \\ \gamma_2^i \ &: \ \partial_j \pi^{ji} \approx 0,\end{aligned} \tag{4.39}$$

and the following 4 second-class constraints

$$\begin{aligned}\chi_1 \ &: \ R_{ij}{}^{ij} \approx 0, \\ \chi_2 \ &: \ \pi \approx 0, \\ \chi_3 \ &: \ \pi^{00} \approx 0, \\ \chi_4 \ &: \ \nabla^2 h^{00} \approx 0.\end{aligned} \tag{4.40}$$

Just as in the previous case, the counting of the degrees of freedom yields two. We also construct the Dirac brackets. In this case the matrix between the second-class constraints is given by

$$C_{\alpha\beta} = \begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} \chi_1 & \chi_2 & \chi_3 & \chi_4 \end{array} \\ \begin{pmatrix} 0 & -2\nabla^2 & 0 & 0 \\ 2\nabla^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\nabla^2 \\ 0 & 0 & \nabla^2 & 0 \end{pmatrix} \end{array} \delta^3(x-y), \tag{4.41}$$

and its inverse takes the form

$$C^{\alpha\beta} = \begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} \chi_1 & \chi_2 & \chi_3 & \chi_4 \end{array} \\ \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \end{array} \frac{1}{\nabla^2} \delta^3(x-y). \tag{4.42}$$

In this manner, the nonzero Dirac's brackets are given by

$$\left\{h_{ij}, \pi^{lm}\right\}_D = \frac{1}{2}\left(\delta_i^l \delta_j^m + \delta_i^m \delta_l^k\right) \delta^3(x-y) + \frac{1}{2\nabla^2} \delta_{ij} \left(\partial^l \partial^m - \delta^{lm} \nabla^2\right) \delta^3(x-y), \tag{4.43}$$

which also are the same as in the previous case.

4.1.3 Gauge Fixing

The equivalence between λR gravity and GR in this perturbative approach becomes further evident by fixing the gauge. For this purpose, let us consider the Coulomb gauge $\partial_i h^{ij} \approx 0$ together with $h^{0i} \approx 0$ which agree with the first-class constraints obtained above. In both cases, the set of second-class constraints becomes

$$
\begin{aligned}
\chi_1 &: h^i_i \approx 0, \\
\chi_2 &: \pi \approx 0, \\
\chi_3 &: \pi^{00} \approx 0, \\
\chi_4 &: \nabla^2 h^{00} \approx 0, \\
\chi_5 &: \pi^{0i} \approx 0, \\
\chi_6 &: h^{0i} \approx 0, \\
\chi_7 &: \partial_j \pi^{ji} \approx 0, \\
\chi_8 &: \partial_j h^{ji} \approx 0.
\end{aligned}
\tag{4.44}
$$

After a long algebraic work, we obtain that the non-vanishing Dirac brackets that follow from this set of second-class constraints are

$$
\begin{aligned}
\left\{h_{ij}, \pi^{lm}\right\}_D = {} & \frac{1}{2}\left(\delta^l_i \delta^m_j + \delta^m_i \delta^l_j\right)\delta^3(x-y) + \frac{1}{2\nabla^2}\left(\delta_{ij}\partial^l\partial^m + \delta^{lm}\partial_i\partial_j\right)\delta^3(x-y) \\
& - \frac{1}{2\nabla^2}\left(\delta^m_i \partial_j\partial^l + \delta^l_i\partial_j\partial^m + \delta^m_j\partial_i\partial^l + \delta^l_j\partial_i\partial^m\right)\delta^3(x-y) + \frac{1}{2}\frac{\partial_i\partial_j\partial^l\partial^m}{\nabla^4}\delta^3(x-y) \\
& - \frac{1}{2}\delta_{ij}\delta^{lm}\delta^3(x-y),
\end{aligned}
\tag{4.45}
$$

which are the same as (2.59) for perturbative GR. Indeed, the set of second-class constraints obtained in this way is the same in both models. Furthermore, from our results it is possible to obtain a remarkable result found in the full model (4.3) [28]. It arises from the equation of motion for h_{ij},

$$
\dot{h}_{ij} = \{h_{ij}, \mathcal{H}_p\}_D = 2\pi_{ij} + \partial_i h_{0j} + \partial_j h_{0i},
\tag{4.46}
$$

then by considering its trace $\delta^{ij}\dot{h}_{ij} = \dot{h}^i_i = 2\pi + 2\partial_i h^i_0$, and since $\pi \approx 0$, we obtain the following expression

$$K = \frac{1}{2}\left(\dot{h}^i_i - 2\partial_i h^i_0\right) \approx 0, \tag{4.47}$$

which is of second-class as it can be seen by using (4.44). As a consequence, in the constrained phase space the term λK^2 is not relevant, so that the constant λ no longer promotes a distinction between GR and λR gravity. In this way the λR gravity provides evidence in favor of the consistency of Hořava's theory. This model is equivalent to GR in a particular gauge, the so-called maximal slicing gauge $K = 0$.

4.2 λR Gravity plus the BPS term

In [46], Blas, Pujolás, and Sibiryakov (BPS) found a quantity not originally considered by Hořava and which is also FDiff symmetry-compatible. This is the FDiff-covariant vector $a_i = \partial_i \ln N$. Regarding the λR gravity, we can include the second-order BPS term that can be constructed with a_i. The resulting action is

$$S = \int dt d^3x \sqrt{g} N \left(K_{ij}K^{ij} - \lambda K^2 + R + \alpha a_i a^i\right). \tag{4.48}$$

By including this extension we can generate a full anisotropic conformal theory if the higher-order potential is chosen to be conformal. Conversely, if this potential is not conformal, we obtain a non-conformal gravitational theory, which is called the kinetic-conformal Hořava theory [50]. In both cases, the conformal symmetry of the kinetic part gives rise to a primary constraint that decreases the degrees of freedom of the theory, propagating two as in GR. However, this constraint changes from second-class in the kinetic-conformal case to a gauge symmetry associated with infinitesimal conformal transformations in the anisotropic-conformal case [50]. An excellent analysis of the dynamics of both versions is performed in [51].

In this way, now the question arises whether, even with this extension, Hořava gravity still retains some closeness to GR beyond the number of degrees of freedom. To explore this, we show in the following our perturbative Hamiltonian analysis of this theory [41]. We start from the Lagrangian (4.8) and simply include the linearization of the BPS term. This is

$$\mathcal{L} = G^{ijkl}K_{ij}K_{kl} - \frac{1}{2}h^{00}R - \frac{1}{2}h^{ij}\left(R_{ij} - \frac{1}{2}\delta_{ij}R\right) + \alpha\partial_i h_{00}\partial^i h^{00}. \tag{4.49}$$

4.2.1 Canonical Analysis for $\lambda \neq \frac{1}{3}$

As we have seen above, since the expression (4.49) does not depend on the velocities $\dot{h}_{00}$ and $\dot{h}_{0i}$, its conjugate momenta, π^{00} and π^{0i} respectively, will be primary constraints. On the other hand, the canonical momenta conjugate to h_{ij} are given by

$$\pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = G^{ijkl} K_{kl}. \tag{4.50}$$

In fact, at this point there is still no influence of the BPS term, so we can also directly set the primary Hamiltonian as

$$H = \mathcal{G}_{ijkl}\pi^{kl}\pi^{ij} - 2h_{j0}\partial_i\pi^{ij} + \frac{1}{2}h^{00}R + \frac{1}{2}h^{ij}\left(R_{ij} - \frac{1}{2}\delta_{ij}R\right) - \alpha\partial_i h_{00}\partial^i h^{00} + u\pi^{00} + u_i\pi^{0i}, \tag{4.51}$$

where u and u_i are the Lagrange multipliers enforcing the primary constraints $\pi^{00} \approx 0$ and $\pi^{0i} \approx 0$ respectively. Now, from the consistency of the primary constraints we obtain four secondary constraints

$$\begin{aligned} \mathcal{H} \quad &: \quad \left\{\pi^{00}, \int d^3x\ H\right\} = \frac{1}{2}R + 2\alpha\nabla^2 h_{00} \approx 0, \\ \mathcal{H}^i \quad &: \quad \left\{\pi^{0i}, \int d^3x\ H\right\} = \partial_j\pi^{ji} \approx 0. \end{aligned} \tag{4.52}$$

Since the consistency condition on $\mathcal{H}^i$ is identically satisfied, while for $\mathcal{H}$ leads to an equation involving the multiplier u, the generation of constraints ends. We have obtained a set of 8 constraints, $(\pi^{00}, \pi^{0i}, \mathcal{H}^i, \mathcal{H})$, which, following the scheme, they need to be classified into first-class and second-class constraints. In this case, since

$$\left\{\mathcal{H}, \pi^{00}\right\} \quad := \quad 2\alpha\nabla^2\delta^3(x-y),$$

is the only non-null Poisson bracket, there are 2 second-class constraints

$$\begin{aligned} \chi_1 \quad &: \quad \frac{1}{2}R + 2\alpha\nabla^2 h_{00} \approx 0, \\ \chi_2 \quad &: \quad \pi^{00} \approx 0. \end{aligned} \tag{4.53}$$

which are the vanishing of the momentum conjugated to h_{00} and the analogous to the so-called Hamiltonian constraint in linearized GR [32]. On the other hand, we obtain the following 6 first-class constraints

$$\begin{aligned} \gamma_1^i \quad &: \quad \pi^{0i} \approx 0, \\ \gamma_2^i \quad &: \quad \partial_j\pi^{ji} \approx 0, \end{aligned} \tag{4.54}$$

which are the generators of gauge symmetries. Remember that in the non-extended λR gravity, at $\lambda \neq \frac{1}{3}$, the consistency condition on the Hamiltonian constraint leads to the second-class $\pi = 0$, and the evolution of π yields another second-class constraint. These two additional second-class constraints contribute to obtain two degrees of freedom. However, in this extended model no such constraints are generated, and thus the counting of degrees of freedom yields

$$DOF = \frac{1}{2}(20 - 2 - 2 * 6) = 3, \tag{4.55}$$

one more than linearized λR gravity. Hence, outside the conformal point, the extended λR model, in this sense, is not equivalent to linearized GR. The relevance of adding the BPS extension is related to the behavior of this additional mode, giving it a description that goes from a first-order to a second-order equation. That is, turning it into an useful even mode [46].

On the other hand, we remove the second-class constraints by introducing the Dirac brackets. In this case we get the matrix

$$C_{ab} = \begin{array}{c c} & \begin{array}{cc} \chi_1 & \chi_2 \end{array} \\ \begin{array}{c} \chi_1 \\ \chi_2 \end{array} & \begin{pmatrix} 0 & 2\alpha\nabla^2 \\ -2\alpha\nabla^2 & 0 \end{pmatrix} \end{array} \delta^3(x-y). \tag{4.56}$$

Its inverse is

$$C^{ab} = \begin{array}{c c} & \begin{array}{cc} \chi_1 & \chi_2 \end{array} \\ \begin{array}{c} \chi_1 \\ \chi_2 \end{array} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{array} \frac{1}{2\alpha\nabla^2}\delta^3(x-y). \tag{4.57}$$

Due to the canonical variables involved in the set (4.53), we observe a change only for brackets related to h_{00}. Namely, the fundamental bracket $\left\{h_{00}, \pi^{00}\right\} = \delta^3(x-y)$ changes to

$$\left\{h_{00}, \pi^{00}\right\}_D = 0, \tag{4.58}$$

since $\pi^{00} \approx 0$ is second-class, and the otherwise null Poisson bracket between h_{00} and π^{ij} becomes

$$\left\{h_{00}, \pi^{ij}\right\}_D = \frac{1}{4\alpha\nabla^2}\left(\partial^i\partial^j - \delta^{ij}\nabla^2\right)\delta^3(x-y). \tag{4.59}$$

This bracket is associated with the dynamics of the third degree of freedom and is not present in the non-extended model reported in [41]. Furthermore, we observe that α cannot be zero; this indicates that the field h_{00} is strongly coupled.

On the other side, one of the most important aspects related to the consistency of lowest-order effective Hořava theory is the existence of a solution for the lapse function N (see (4.48)) since it is a dynamical variable which is expected to be fixed by the Hamiltonian constraint. This is, by performing the non-perturbative canonical analysis of (4.48), the following Hamiltonian constraint is obtained

$$\left(4\alpha\nabla^2 - R + \mathcal{G}_{ijkl}\frac{\pi^{ij}\pi^{kl}}{g}\right)\sqrt{N} \approx 0, \tag{4.60}$$

that equation becomes relevant because any source of indetermination on N could either lead to inconsistencies of the theory (4.48) or to reinterpret the Hamiltonian constraint as a condition for another variable as was claimed in [28, 52, 53]. At the non-perturbative level, the Hamiltonian constraint (4.60) is a second-order elliptic PDE for N totally compatible with the standard (flat) asymptotic behavior of all gravitational variables. In fact, if it is taken $\sqrt{N} = 1 + n$, and $g_{ij} = \delta_{ij} + h_{ij}$, (4.60) is reduced to

$$4\alpha\nabla^2 n = R, \tag{4.61}$$

this is the equivalent Hamiltonian constraint found in (4.53) using our approach. In fact, we can identify that h_{00} in our formalism is equivalent to the perturbation n. The equation (4.53) is a Poisson equation that can be solved for h_{00} under appropriate boundary conditions; this ensures that the solution for h_{00} exists, and is unique, at least in the sense of distributions. In this manner our approach completes the results found in the literature.

4.2.2 Canonical Analysis for $\lambda = \frac{1}{3}$

At the kinetic conformal point, the canonical momenta change; thus, in addition to the primary constraints found in the previous section, one more will arise from the definition of the canonical momenta conjugate to h_{ij}, this is

$$\pi^{ij} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{ij}} = K^{ij} - \frac{1}{3}\delta^{ij}K. \tag{4.62}$$

The new constraint is given by $\pi \equiv \delta_{ij}\pi^{ij} = 0$. This constraint must be of second-class because conformal gauge transformations are not a gauge symmetry of the theory. Hence the primary Hamiltonian takes the form

$$\mathcal{H}_p = \pi^{ij}\pi_{ij} - 2\partial_i\pi^{ij}h_{0j} + \frac{1}{2}h^{00}R + \frac{1}{2}h^{ij}\left(R_{ij} - \frac{1}{2}\delta_{ij}R\right) - \alpha\partial_i h_{00}\partial^i h^{00} + u\pi^{00} + u_i\pi^{0i} + v\pi. \tag{4.63}$$

From consistency on the above primary constraints the following secondary constraints arise

$$\begin{aligned} \mathcal{H} &: \frac{1}{2}R + 2\alpha\nabla^2 h_{00} \approx 0, \\ \mathcal{H}^i &: \partial_j \pi^{ji} \approx 0, \\ \gamma &: \nabla^2 h^{00} + \frac{1}{2}R \approx 0, \end{aligned} \tag{4.64}$$

but evolution of these secondary constraints does not generate any new constraints. On the other hand, we notice that the sole difference between $\mathcal{H} \approx 0$ and $\gamma \approx 0$ is given by the α parameter. Thus, if $\alpha \neq \frac{1}{2}$, these constraints are independent and resolvable to produce the following set of independent constraints [1]

$$\begin{aligned} &\pi^{00} \approx 0, \\ &\pi^{0i} \approx 0, \\ &\pi \approx 0, \\ &\mathcal{H}^i \approx 0, \\ &\frac{1}{2}R \approx 0, \\ &2\alpha\nabla^2 h^{00} \approx 0. \end{aligned} \tag{4.65}$$

Given that the nonzero Poisson brackets between them are

$$\begin{aligned} \left\{2\alpha\nabla^2 h^{00}, \pi^{00}\right\} &= 2\alpha\nabla^2\delta^3(x-y), \\ \left\{\frac{1}{2}R, \pi\right\} &= -\nabla^2\delta^3(x-y), \end{aligned} \tag{4.66}$$

we obtain the following six first-class constraints

$$\begin{aligned} \gamma_1^i &: \pi^{0i} \approx 0, \\ \gamma_2^i &: \partial_j \pi^{ji} \approx 0, \end{aligned} \tag{4.67}$$

which are the same as in the previous case, and the following four second-class constraints

$$\begin{aligned} \chi_1 &: R \approx 0, \\ \chi_2 &: \pi \approx 0, \\ \chi_3 &: \pi^{00} \approx 0, \\ \chi_4 &: \nabla^2 h^{00} \approx 0. \end{aligned} \tag{4.68}$$

[1] If $\alpha = \frac{1}{2}$ the set of constraints would be $\pi = \pi^{00} = \mathcal{H} \approx 0$, which is an inconsistent odd second-class set.

The important difference with the case $\lambda \neq \frac{1}{3}$ is the presence of two additional second-class constraints that modify the dynamics of the theory, which now propagates two degrees of freedom just like in linearized GR [32]. In fact, the first and second-class constraints (4.67) and (4.68), correspond to the sets obtained in the non-extended model previously. For this reason, by equally fixing the gauge via the Coulomb gauge $\partial_i h^{ij} \approx 0$, and $h^{0i} \approx 0$, the following non-zero Dirac brackets are obtained

$$\begin{aligned}\left\{h_{ij}, \pi^{lm}\right\}_D =& \frac{1}{2}\left(\delta_i^l \delta_j^m + \delta_i^m \delta_j^l\right)\delta^3(x-y) + \frac{1}{2\nabla^2}\left(\delta_{ij}\partial^l\partial^m + \delta^{lm}\partial_i\partial_j\right)\delta^3(x-y) \\ & - \frac{1}{2\nabla^2}\left(\delta_i^m \partial_j \partial^l + \delta_i^l \partial_j \partial^m + \delta_j^m \partial_i \partial^l + \delta_j^l \partial_i \partial^m\right)\delta^3(x-y) + \frac{1}{2}\frac{\partial_i\partial_j\partial^l\partial^m}{\nabla^4}\delta^3(x-y) \\ & - \frac{1}{2}\delta_{ij}\delta^{lm}\delta^3(x-y).\end{aligned} \tag{4.69}$$

It is worth commenting that these brackets are α independent; thus, the propagators between the fields are well defined. From the propagators, we will see that the theory at the critical point propagates two massless degrees of freedom.

4.3 Extended λR Gravity plus a Cotton-Square term

Now we consider a Hořava theory with soft breaking of conformal symmetry [50]. This is composed by the previous extended λR model that is not conformal and a square term of the Cotton tensor that is conformally invariant. The Cotton term that we shall add is $C_{ij}C^{ij}$, where

$$C^{ij} = \epsilon^{ikl}\nabla_k\left(R_l^j - \frac{1}{4}R\delta_l^j\right). \tag{4.70}$$

On a Minkowski background, the linearized Lagrangian now is written as

$$\begin{aligned}\mathcal{L} =& G^{ijkl}K_{ij}K_{kl} - \frac{1}{2}h^{00}R - \frac{1}{2}h^{ij}\left(R_{ij} - \frac{1}{2}\delta_{ij}R\right) + \alpha\partial_i h_{00}\partial^i h^{00} - w\partial_i R_k^j \partial^i R_j^k \\ & + w\partial^i R_i^j \partial_k R_j^k + \frac{3w}{8}\partial_i R \partial^i R + \frac{w}{2}\partial_i R_j^i \partial^j R,\end{aligned} \tag{4.71}$$

where w is an arbitrary constant. In this extension the equations of motion for h^{00}, h^{0i} and h^{ij} are respectively

$$4\alpha\nabla^2 h^{00} + R = 0, \tag{4.72}$$

$$\partial_i \dot{h}^{ij} - \lambda\partial^j \dot{h}_i^i + (2\lambda - 1)\partial^j\partial_i h_0^i - \nabla^2 h_0^j = 0, \tag{4.73}$$

and

$$\dot{K}^{ij} - \lambda \dot{K}\delta^{ij} + R^{ij} - \frac{1}{2}\delta^{ij}R + \frac{1}{2}(\partial^i\partial^j - \delta^{ij}\nabla^2)h^{00} + w\nabla^2\left(\partial^i\partial^j R - \partial^l\partial^i R^j_l - \partial^l\partial^j R^i_l\right)$$
$$+ \frac{w}{2}\left(3\partial^i\partial^j - \delta^{ij}\nabla^2\right)\partial_l\partial_m R^{lm} + \frac{3w}{4}\nabla^4\left(R^{ij} - \delta^{ij}R\right) = 0. \tag{4.74}$$

Moreover, by choosing the gauges $h^{0i} = 0$ and $\partial_i h^{ij} = 0$, we simplify the above equations.

$$4\alpha\nabla^2 h^{00} - \nabla^2 h^i_i = 0, \tag{4.75}$$

$$\partial_i \dot{h}^{ij} - \lambda\partial^j \dot{h}^i_i = 0, \tag{4.76}$$

$$\frac{1}{2}\Box\left(h^{ij} - \lambda\delta^{ij}h^k_k\right) - \frac{1}{2}\left(\partial^i\partial^j + \delta^{ij}(\lambda-1)\nabla^2\right)h^k_k - \frac{w}{4}\left(3\partial^i\partial^j - \delta^{ij}\nabla^2\right)\nabla^4 h^k_k$$
$$+ \frac{1}{2}(\partial^i\partial^j - \delta^{ij}\nabla^2)h^{00} - \frac{3w}{4}\nabla^4\left(\nabla^2 h^{ij} + \partial^i\partial^j h^k_k + 2\delta^{ij}\nabla^2 h^k_k\right) = 0. \tag{4.77}$$

We identify equation (4.76) as the so-called Hořava gauge. This gauge has been proposed in several works [47, 48] to remove unphysical degrees of freedom; however, in our framework, this gauge emerges naturally. On the other hand, we see that in the UV limit α plays a relevant role; it is involved in the dynamics of h_{ij} through the inescapable presence of both h^k_k and h^{00} in (4.77), and its α-dependent relation (4.75). This fact is also responsible for the presence of a third degree of freedom for $\lambda \neq \frac{1}{3}$. However, at $\lambda = \frac{1}{3}$ we have

$$\partial_i \dot{h}^{ij} - \frac{1}{3}\partial^j \dot{h}^i_i = 0, \tag{4.78}$$

$$\frac{1}{2}\Box\left(h^{ij} - \frac{1}{3}\delta^{ij}h^k_k\right) - \frac{1}{2}\left(\partial^i\partial^j - \frac{2}{3}\delta^{ij}\nabla^2\right)h^k_k + \frac{1}{2}(\partial^i\partial^j - \delta^{ij}\nabla^2)h^{00}$$
$$- \frac{w}{4}\left(3\partial^i\partial^j - \delta^{ij}\nabla^2\right)\nabla^4 h^k_k - \frac{3w}{4}\nabla^4\left(\nabla^2 h^{ij} + \partial^i\partial^j h^k_k + 2\delta^{ij}\nabla^2 h^k_k\right) = 0. \tag{4.79}$$

Then, by taking the trace of (4.79) we obtain

$$\frac{1}{2}\nabla^2 h^i_i - \nabla^2 h^{00} - 6w\nabla^6 h^i_i = 0. \tag{4.80}$$

In conjunction with (4.75), this result allows to decoupling the dependence between h^{00} and h^i_i, and thus from (4.79). Furthermore, in the IR limit $w \to 0$ this is

$$\frac{1}{2}\nabla^2 h^i_i - \nabla^2 h^{00} = 0. \tag{4.81}$$

Considering this expression together with (4.75), (4.79) and $\alpha \neq \frac{1}{2}$ we get

$$\nabla^2 h^{00} = 0, \qquad \nabla^2 h^i_i = 0 \quad \text{and} \quad \Box h_{ij} = 0, \tag{4.82}$$

which are the standard wave equations of linearized gravity. As we saw previously, this limit does not depend on the value of α beyond $\alpha \neq \frac{1}{2}$. Now we proceed to perform the canonical analysis.

4.3.1 Canonical Analysis for $\lambda \neq \frac{1}{3}$

Since the primary constraints depend only on the kinetic part, these are the same as the respective previous case. Thus, the primary Hamiltonian takes the form

$$\begin{aligned}\mathcal{H}_p =& \mathcal{G}_{ijkl}\pi^{kl}\pi^{ij} - 2h_{j0}\partial_i\pi^{ij} + \frac{1}{2}h^{00}R + \frac{1}{2}h^{ij}\left(R_{ij} - \frac{1}{2}\delta_{ij}R\right) - \alpha\partial_i h_{00}\partial^i h^{00} \\ &+ w\partial_i R^j_k\partial^i R^k_j - w\partial^i R^j_i\partial_k R^k_j - \frac{3w}{8}\partial_i R\partial^i R - \frac{w}{2}\partial_i R^i_j\partial^j R + u\pi^{00} + u_i\pi^{0i}.\end{aligned} \tag{4.83}$$

Similarly, since the high-order potential does not involve h_{00} or h_{0i}, the consistency of primary constraints results in the following secondary constraints

$$\begin{aligned}\mathcal{H} &: \frac{1}{2}R + 2\alpha\nabla^2 h_{00} \approx 0, \\ \mathcal{H}^i &: \partial_j\pi^{ji} \approx 0.\end{aligned} \tag{4.84}$$

The preservation in time of these constraints does not lead us to more constraints. Note that, at the perturbative level, the added potential does not affect the sets of first and second-class constraints. Thus, there are 3 degrees of freedom and the Dirac brackets are those found in (4.58) and (4.59). The same would be true for any higher-order potential that does not involve terms that depend on h_{00}, such as the BPS terms.

4.3.2 Canonical Analysis for $\lambda = \frac{1}{3}$

Following the same above consideration about the primary constraints, now the primary Hamiltonian takes the form

$$\begin{aligned}H =& \pi^{ij}\pi_{ij} - 2\partial_i\pi^{ij}h_{0j} + \frac{1}{2}h^{00}R + \frac{1}{2}h^{ij}\left(R_{ij} - \frac{1}{2}\delta_{ij}R\right) - \alpha\partial_i h_{00}\partial^i h^{00} \\ &+ w\partial_i R^j_k\partial^i R^k_j - w\partial^i R^j_i\partial_k R^k_j - \frac{3w}{8}\partial_i R\partial^i R - \frac{w}{2}\partial_i R^i_j\partial^j R + u\pi^{00} + u_i\pi^{0i} + v\pi.\end{aligned} \tag{4.85}$$

In addition to the set (4.84), the presence of π adds a secondary constraint whose structure is determined by the dependence on h_{ij} of the higher-order terms, so that it differs from its counterpart in (4.64). Thus, from consistency of π we obtain the following constraint

$$\gamma : \nabla^2 h^{00} + \frac{1}{2}R + 2w\nabla^2\partial_i\partial_j R^{ij} + \frac{1}{2}w\nabla^2\nabla^2 R \approx 0. \tag{4.86}$$

The presence of $\nabla^2 h^{00}$ in $\mathcal{H}$ and γ leads to expressions containing the Lagrange multiplier u when the consistency condition is applied, and as in the previous cases, the preservation of $\mathcal{H}^i$ is identically satisfied. Thus we have obtained the complete set of constraints. To make the separation into first and second-class, let us note that the non-zero Poisson brackets between them are

$$\begin{aligned} \left\{\gamma, \pi^{00}\right\} &= \nabla^2 \delta^3(x-y), \\ \{\pi, \gamma\} &= (\nabla^2 + w\nabla^2\nabla^2\nabla^2)\delta^3(x-y), \\ \{\mathcal{H}, \pi\} &= -\nabla^2 \delta^3(x-y), \\ \left\{\mathcal{H}, \pi^{00}\right\} &= 2\alpha\nabla^2\delta^3(x-y). \end{aligned} \tag{4.87}$$

Thus, we obtain six first-class constraints given by

$$\begin{aligned} \gamma_1^i &: \ \pi^{0i} \approx 0, \\ \gamma_2^i &: \ \partial_j \pi^{ji} \approx 0, \end{aligned} \tag{4.88}$$

and the following four second-class constraints

$$\begin{aligned} \chi_1 &: \ \frac{1}{2}R + 2\alpha\nabla^2 h_{00} \approx 0, \\ \chi_2 &: \ \pi \approx 0, \\ \chi_3 &: \ \pi^{00} \approx 0, \\ \chi_4 &: \ \nabla^2 h^{00} + \frac{1}{2}R + 2w\nabla^2\partial_i\partial_j R^{ij} + \frac{1}{2}w\nabla^2\nabla^2 R \approx 0. \end{aligned} \tag{4.89}$$

Although there is an obvious modification in the second-class constraints compared to (4.40), the gauge symmetries associated with the first-class constraints prevail, as well as the propagation of two degrees of freedom. We will now calculate the Dirac brackets that arise from the set (4.89). The matrix of the Poisson brackets between the second-class constraints is

$$C_{ab} = \begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} \quad\chi_1 \quad & \quad\quad \chi_2 \quad\quad & \quad \chi_3 \quad & \quad\quad \chi_4 \quad\quad \end{array} \\ \begin{pmatrix} 0 & -\nabla^2 & 2\alpha\nabla^2 & 0 \\ \nabla^2 & 0 & 0 & \nabla^2 + w\nabla^2\nabla^2\nabla^2 \\ -2\alpha\nabla^2 & 0 & 0 & -\nabla^2 \\ 0 & -\nabla^2 - w\nabla^2\nabla^2\nabla^2 & \nabla^2 & 0 \end{pmatrix} \end{array} \delta^3(x-y), \tag{4.90}$$

and its inverse is given by

$$C^{ab}=\begin{array}{c c} & \begin{array}{cccc}\chi_1 & \chi_2 & \chi_3 & \chi_4\end{array} \\ \begin{array}{c}\chi_1\\ \chi_2\\ \chi_3\\ \chi_4\end{array} & \begin{pmatrix} 0 & 1 & 1+w\nabla^4 & 0 \\ -1 & 0 & 0 & 2\alpha \\ -1-w\nabla^4 & 0 & 0 & 1 \\ 0 & -2\alpha & -1 & 0 \end{pmatrix}\end{array}\frac{1}{(1-2\alpha(1+w\nabla^4))\nabla^2}\delta^3(x-y). \quad (4.91)$$

The Dirac brackets that can be built with this matrix are

$$\left\{h_{00},\pi^{00}\right\}_D=0, \quad (4.92)$$

$$\left\{h_{ij},\pi^{00}\right\}_D=0, \quad (4.93)$$

and

$$\begin{aligned}\left\{h_{ij},\pi^{lm}\right\}_D=&\frac{1}{2}\left(\delta_i^l\delta_j^m+\delta_i^m\delta_j^l\right)\delta^3(x-y)+\frac{\delta_{ij}}{2\Xi}\left(\partial^l\partial^m-\delta^{lm}\nabla^2\right)\delta^3(x-y)\\&-\frac{\alpha\delta_{ij}}{\Xi}\left(\partial^l\partial^m-\delta^{lm}\nabla^2\right)\left(1+3w\nabla^4\right),\end{aligned} \quad (4.94)$$

where $\Xi=(1-2\alpha(1+w\nabla^4))\nabla^2$. Now, for observing the IR effective action we take $w\to 0$ and the Dirac brackets are reduced to

$$\left\{h_{ij},\pi^{lm}\right\}_D=\frac{1}{2}\left(\delta_i^l\delta_j^m+\delta_i^m\delta_j^l-\delta_{ij}\delta^{lm}\right)\delta^3(x-y)+\frac{\delta_{ij}\partial^l\partial^m}{2\nabla^2}\delta^3(x-y), \quad (4.95)$$

however, the first-class constraints remain and we can fix the gauge. In fact, by fixing the gauge the following constraints arrive

$$\begin{aligned}\chi_1 &: \frac{1}{2}R+2\alpha\nabla^2h_{00}\approx 0,\\ \chi_2 &: \pi\approx 0,\\ \chi_3 &: \pi^{00}\approx 0,\\ \chi_4 &: \nabla^2h^{00}+\frac{1}{2}R+2w\nabla^2\partial_i\partial_jR^{ij}+\frac{1}{2}w\nabla^2\nabla^2R\approx 0,\\ \chi_5 &: \pi^{0i}\approx 0,\\ \chi_6 &: h^{0i}\approx 0,\\ \chi_7 &: \partial_j\pi^{ji}\approx 0,\\ \chi_8 &: \partial_jh^{ji}\approx 0.\end{aligned} \quad (4.96)$$

The matrix whose entries are the Poisson brackets between these constraints is given by

$$
C_{\alpha\beta} = \begin{array}{cc}
 & \begin{matrix} \chi_1 & \chi_2 & \chi_3 & \chi_4 & \chi_5^1 & \chi_5^2 & \chi_5^3 & \chi_6^1 & \chi_6^2 & \chi_6^3 & \chi_7^1 & \chi_7^2 & \chi_7^3 & \chi_8^1 & \chi_8^2 & \chi_8^3 \end{matrix} \\
\begin{matrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5^1 \\ \chi_5^2 \\ \chi_5^3 \\ \chi_6^1 \\ \chi_6^2 \\ \chi_6^3 \\ \chi_7^1 \\ \chi_7^2 \\ \chi_7^3 \\ \chi_8^1 \\ \chi_8^2 \\ \chi_8^3 \end{matrix} &
\begin{pmatrix}
0 & -\nabla^2 & 2\alpha\nabla^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\nabla^2 & 0 & 0 & \nabla^2 + w\nabla^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial^1 & -\partial^2 & -\partial^3 \\
-2\alpha\nabla^2 & 0 & 0 & -\nabla^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\nabla^2 - w\nabla^6 & \nabla^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\left(\nabla^2 + \partial^1\partial^1\right) & \frac{1}{2}\partial^1\partial^2 & \frac{1}{2}\partial^1\partial^3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\partial^2\partial^1 & \frac{1}{2}\left(\nabla^2 + \partial^2\partial^2\right) & \frac{1}{2}\partial^2\partial^3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\partial^3\partial^1 & \frac{1}{2}\partial^3\partial^2 & \frac{1}{2}\left(\nabla^2 + \partial^3\partial^3\right) \\
0 & \partial^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\left(\nabla^2 + \partial^1\partial^1\right) & -\frac{1}{2}\partial^2\partial^1 & -\frac{1}{2}\partial^3\partial^1 & 0 & 0 & 0 \\
0 & \partial^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\partial^1\partial^2 & -\frac{1}{2}\left(\nabla^2 + \partial^2\partial^2\right) & -\frac{1}{2}\partial^3\partial^2 & 0 & 0 & 0 \\
0 & \partial^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\partial^1\partial^3 & -\frac{1}{2}\partial^2\partial^3 & -\frac{1}{2}\left(\nabla^2 + \partial^3\partial^3\right) & 0 & 0 & 0
\end{pmatrix}
\end{array}
\delta^3(x-y), \tag{4.97}
$$

and after long calculations, its inverse takes the form

$$C^{\alpha\beta} = \begin{array}{cc} & \begin{array}{cccccccccccccccc} \chi_1 & \chi_2 & \chi_3 & \chi_4 & \chi_5^1 & \chi_5^2 & \chi_5^3 & \chi_6^1 & \chi_6^2 & \chi_6^3 & \chi_7^1 & \chi_7^2 & \chi_7^3 & \chi_8^1 & \chi_8^2 & \chi_8^3 \end{array} \\ \begin{array}{c} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5^1 \\ \chi_5^2 \\ \chi_5^3 \\ \chi_6^1 \\ \chi_6^2 \\ \chi_6^3 \\ \chi_7^1 \\ \chi_7^2 \\ \chi_7^3 \\ \chi_8^1 \\ \chi_8^2 \\ \chi_8^3 \end{array} & \begin{pmatrix} 0 & -\nabla^2 & -\nabla^2 - w\nabla^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial^1 & 0 & 0 & 0 & 0 & 0 \\ \nabla^2 & 0 & 0 & -2\alpha\nabla^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial^1 & -\partial^2 & -\partial^3 \\ \nabla^2 + w\nabla^6 & 0 & 0 & -\nabla^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\alpha\nabla^2 & \nabla^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\alpha\partial^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\zeta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\zeta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\zeta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\zeta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\zeta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2\zeta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \partial^1 & 0 & 0 & -2\alpha\partial^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\left(\nabla^2 + \partial^1\partial^1\right) & \frac{1}{2}\partial^1\partial^2 & \frac{1}{2}\partial^1\partial^3 \\ \partial^2 & 0 & 0 & -2\alpha\partial^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\partial^2\partial^1 & \frac{1}{2}\left(\nabla^2 + \partial^2\partial^2\right) & \frac{1}{2}\partial^2\partial^3 \\ \partial^3 & 0 & 0 & -2\alpha\partial^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\partial^3\partial^1 & \frac{1}{2}\partial^3\partial^2 & \frac{1}{2}\left(\nabla^2 + \partial^3\partial^3\right) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta(2\nabla^2 - \partial^1\partial^1) & -\frac{1}{2}\partial^2\partial^1 & -\frac{1}{2}\partial^3\partial^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta\partial^1\partial^2 & -\frac{1}{2}\left(\nabla^2 + \partial^2\partial^2\right) & -\frac{1}{2}\partial^3\partial^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta\partial^1\partial^3 & -\frac{1}{2}\partial^2\partial^3 & -\frac{1}{2}\left(\nabla^2 + \partial^3\partial^3\right) & 0 & 0 & 0 \end{pmatrix} \end{array} \frac{1}{\zeta}\delta^3(x-y), \tag{4.98}$$

where $\zeta = \left(-1 + 2\alpha\left(1 + w\nabla^4\right)\right)\nabla^4$ and $\beta = -1 + 2\alpha\left(1 + w\nabla^4\right)$. Thus, the final Dirac's brackets are given by

$$\begin{aligned}\left\{h_{ij}, \pi^{lm}\right\}_D =& \frac{1}{2}\left(\delta_i^l\delta_j^m + \delta_i^m\delta_j^l\right)\delta^3(x-y) \\ &+ \frac{1}{2\zeta}\left(\partial_i\partial_j - \delta_{ij}\nabla^2\right)\left(\partial^l\partial^m - \delta^{lm}\nabla^2\right)\left(1 - 2\alpha\left(1 + 3w\nabla^4\right)\right)\delta^3(x-y) \\ &- \frac{1}{2\nabla^4}\left(\left(\partial_i\partial^l\delta_j^m + \partial_i\partial^m\delta_j^l + \partial_j\partial^l\delta_i^m + \partial_j\partial^m\delta_i^l\right)\nabla^2 - 2\partial_i\partial_j\partial^l\partial^m\right)\delta^3(x-y),\end{aligned} \tag{4.99}$$

In the IR limit, say $w \to 0$, these brackets are reduced to those of linearized GR (4.69). Also, if $w \neq 0$ but $\alpha = \frac{1}{2}$, then these brackets are w independent. Otherwise w contributes and will be present in the propagators.

Chapter 5

Conclusions

The goal of a quantum theory of gravity has led us down various paths in our search for the ultimate structure of reality. For example, in modified theories of gravity, particularly higher-order theories, we have even explored the idea of an anisotropic space-time as in the case of Hořava gravity. Identifying the physical implications of these possible scenarios is an important issue that must be handled with care. For this, the best tool at our disposal is the Hamiltonian constraint analysis, which allows to fully characterize dynamical aspects that can be naively misidentified only on Lagrangian grounds.

This technique is also relevant for many applications in gravity theories, such as the cosmology of homogeneous models, perturbations around them, and collapse models of matter distribution into black holes. For the observational aspects of cosmology for instance, canonical methods provide systematic tools for finding gauge-invariant observables and their evolution. Also, for the study of slow-roll inflation[1] we can work with perturbations around the Friedmann-Robertson-Walker (FRW) metric, calculate the canonical equations of motion from the Hamiltonian, and then impose the slow-roll conditions to analyze inflation. The slow-roll approximations reduce the set of equations of motion, the so-called Einstein-Klein-Gordon (EKG) equations, in such a way that one can quickly obtain the solution to the scale factor in this approximation [58]. In addition, expressions for basic observables such as the scalar spectral index and the tensor-to-scalar ratio can be extracted. It is also important to mention that the canonical formalism allows for an easier derivation of many inflation results without such an approximation. For example, we can obtain the exact solution of the complete set of EKG equations. In this regard, our perturbative work will

[1] For this it is necessary to add a scalar and matter fields to the action.

be useful in forthcoming works.

Considering the above, our main objective has been the perturbative Hamiltonian analysis of higher-order gravity theories but on a flat (Minkowskian) spacetime. However, it is well known that there are often complications in the calculations to obtain the complete constraint structure when dealing with higher-order theories in general. For this reason, we have worked on an approach that involves an extrinsic curvature-type variable, which allows us to significantly economize this task, as observed especially in the analysis of NMG. For this 2+1 theory, after achieving the identification of first and second-class constraints, we were able to fully demonstrate that it does not contain ghost degrees of freedom by means of the extended Hamiltonian, which becomes a bounded function. This is a good indicator of unitarity at the quantum level.

In this respect, there are early works on the unitarity and renormalizability beyond power counting in NMG [59, 60]. However, a remarkable feature of these works is that the proofs are based on an equivalence between linearized NMG and the Pauli-Fierz massive gravity with only the spin ± 2 massive graviton modes without the scalar mode. This approach was motivated by the fact that the direct inclusion of gauge fixing conditions to derive the propagator for the gravitational field in NMG was not possible. This is because, in addition to the fact that the identification of the first-class constraints had not been done, the usual gauge fixing condition, the de Donder's gauge, does not work well in this theory. Moreover, a central aspect in the proof of renormalizability and unitarity is the discarding of a possible additional scalar mode coming from the higher derivative part of the Lagrangian[2]. It was assumed that the scalar mode is canceled out only by the fact that the trace part of the stress-energy tensor associated with the higher derivative terms is proportional to the original higher derivative Lagrangian. In contrast, in our work we rigorously determine the degrees of freedom by first identifying the total constraint structure as dictated by the canonical formalism. We found two modes that are physical. Similarly, we can work over the Dirac brackets we have constructed to establish the commutators and then calculate the propagators. Thus, our perturbative work will be useful in future work.

On the other hand, with the above results, NMG can be considered a promising candidate for a fully consistent theory of quantum gravity, albeit in three dimensions and with massive gravitons. In this respect, the most promising tool for this task is the AdS/CFT

[2]The canonical analysis of the higher-order sector of NMG is given in Appendix B. There, it is shown that it uniquely propagates one degree of freedom, which is a ghost.

correspondence [61], which reduces the problem to the construction of a two-dimensional boundary conformal field theory for the AdS bulk of the given 3D theory. The state of affairs with NMG is that it suffers from the so-called "bulk-boundary unitarity clash" [26]: it is unitary only in the bulk or in the boundary, which, in a nutshell, prevents the construction of a quantum version according to the AdS/CFT correspondence. In fact, it has been proven that no theory with the same particle content as NMG can be both bulk and boundary unitary at the same time [62]. This is a strong theorem which also rules out any f (Ricci)-type higher curvature extensions of the NMG such as the cubic and quartic theories obtained by demanding the existence of a holographic c function in [63, 64] and the infinite order Born-Infeld extension [65]. Alternative theories have been proposed in 3D such as Minimal Massive Gravity that try to avoid this bulk-boundary unitarity clash [20]. However, the methodology we have developed can provide a safe path for the analysis of similar higher-order theories.

On the other hand, we have worked on the λR model and some of its extensions to both second and higher-order. We have identified the total set of constraints, and in each case, established the Dirac brackets. Our results support the theoretical consistency at low energies of the Hořava's proposal. In the non-extended version, the compatibility with GR is established regardless the λ value. The same first and second-class sets are obtained for any value of λ, except for $\lambda = 1$, where only first-class constraints are obtained, but at this value the identification with GR is direct. On the other hand, for the BPS-extended model, just at the kinetic conformal point $\lambda = \frac{1}{3}$ the constraint content suppresses the extra degree of freedom, so it contains only two physical modes. This is true even when we add higher-order terms, since the structure of the constraints is very similar.

Since these results support the consistency of HG at the classical level, and given that it contains no ghosts, this theory is a serious candidate for a complete quantization program. In fact, the renormalizability of the projectable version has been proven in [66], whereas the quantization of the non-projectable case, which as we have seen is closer to GR, has been hampered due to the difficulty posed by the presence of second-class constraints [67–69]. A proof has been presented in [70], where the quantization is performed through the Batalin-Fradkin-Vilkovisky (BFV) formalism, and the renormalization is achieved by using the approach of Barvinsky et al. based on the background field formalism.

It is also important to mention that, due to the remarkable property of anisotropy, it has been interesting to explore several cosmological scenarios within the framework of HG. For example, on a flat background, the tensor perturbations can lead to potential problems such as fine-tuning and super-luminal propagation [35]. This has been reexamined by introducing an additional scalar field, leading to an effective phase of dark energy [75]. In addressing late-time acceleration of the Universe, various cosmological quantities have been examined for Tsallis, Rényi, and Sharma-Mittal holographic dark-energy models, alongside modified field equations for logarithmic and power law versions of entropy corrected models [76]. Moreover, the phenomenon of matter–antimatter asymmetry has been scrutinized in the background of HG via gravitational baryogenesis [77]. In [78], a non-perturbative quantum correction in the black hole entropy has been considered and studied the consequence of thermodynamics of the Hořava black hole at quantum scales. While these theoretical models may offer alternatives to the standard ΛCDM model, their viability hinges on their alignment with observational data sets such as TORNY and the Gold sample data set [79, 80]. In this sense, numerous investigations have focused on constraining HG, utilizing different approaches including Big Bang Nucleosynthesis bounds [81], and observational data from sources like CMB [82, 83], baryon acoustic oscillations [82, 83], galaxy power spectrum, and Type Ia supernovae measurements [84–86]. This was further extended in [87, 88] to incorporate observations from GW170817 and GRB170817A, while also addressing degeneracy with massive neutrinos.

We also should mention an interesting fact regarding the two models we worked on. From the conjecture of the quantum inheritance principle, the construction of a four-dimensional renormalizable gravity can be intimately related to a three-dimensional renormalizable relativistic theory. In this sense, the potential for the $z = 3$ HG can be constructed from NMG by the so called "detailed balance condition", thus possibly obtaining a renormalizable HG since the NMG is. This has been explored in [89]. Also, based on our analysis of both theories, a Hamiltonian construction of this extension is one of our pending works.

For all these reasons, HG is undoubtedly a very interesting area of study from both classical and quantum perspectives. Pending further advances in the quantization program, canonical analyses of possible extensions with terms of different orders are of interest from a cosmological point of view.

Appendix A

Klein-Gordon Theory

In this appendix we perform the canonical study of the system whose equations of motion are given by

$$(\Box + m^2)\underset{Lin}{G_{\mu\nu}} = 0. \tag{A.1}$$

This is a system with certain closeness with NMG, however the constrain $\underset{Lin}{R} = 0$ is discarded. It is straightforward to see that these equations can be obtained from the action

$$S[g_{\mu\nu}] = \int d^3x\sqrt{-g}\left(R + \frac{1}{m^2}Z\right), \tag{A.2}$$

where

$$Z = R_{\mu\nu}R^{\mu\nu} - \frac{1}{2}R^2. \tag{A.3}$$

The equations of motion that emerge from the variation of the action (A.2) are

$$\frac{3}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + 2RR_{\mu\nu} - \frac{3}{4}g_{\mu\nu}R^2 - 4R^{\alpha}_{\mu}R_{\nu\alpha} + \left(\Box + m^2\right)G_{\mu\nu} = 0, \tag{A.4}$$

hence, by considering the perturbation of the metric around the Minkowski background into (A.4) the equations (A.1) are obtained. Furthermore, the linearization of the action (A.2) yields

$$\begin{aligned}\mathcal{L} = \frac{1}{2}\partial_\alpha\partial^\alpha h_{\mu\nu}\partial_\alpha\partial^\alpha h^{\mu\nu} + \partial_\mu\partial_\nu h\partial_\alpha\partial^\alpha h^{\mu\nu} - \partial_\mu\partial_\alpha h^{\alpha}_{\nu}\partial_\alpha\partial^\alpha h^{\mu\nu} - \frac{1}{2}\partial_\alpha\partial^\alpha h\partial_\alpha\partial^\alpha h \\ + m^2\left(\frac{1}{2}\partial_\mu h\partial^\mu h + \partial_\mu h^{\mu\rho}\partial_\alpha h^{\alpha}_{\rho} - \frac{1}{2}\partial_\lambda h^{\rho\mu}\partial^\lambda h_{\mu\rho} - \partial_\mu h^{\mu\rho}\partial_\rho h\right).\end{aligned} \tag{A.5}$$

With the introduction of the extrinsic curvature (2.46) once the 2+1 decomposition of this Lagrangian is performed, we obtain

$$\begin{aligned}\mathcal{L} \;=\;& \dot{K}^2 - \dot{K}_{ij}\dot{K}^{ij} + m^2\left(K^2 - K_{ij}K^{ij}\right) - 2\partial_i K\partial^i K - 2\partial_i K^{ij}\partial_k K^k_j + 2\partial_k K_{ij}\partial^k K^{ij} \\ & + 2\partial_l K\partial_m K^{lm} - \partial_i\partial_j h_{00}\dot{K}^{ij} + \nabla^2 h_{00}\dot{K} + \frac{1}{2}\left(R_{ikj}{}^{k} - \frac{1}{2}\delta_{ij}R_{lm}{}^{lm}\right)\left(\nabla^2 + m^2\right)h^{ij} \\ & + \frac{1}{2}R_{ij}{}^{ij}\left(\nabla^2 + m^2\right)h_{00} + \alpha^{ij}\left(\dot{h}_{ij} - 2\partial_i h_{0j} - 2K_{ij}\right). \qquad \text{(A.6)}\end{aligned}$$

The canonical variables of the system are given by $h_{\mu\nu}$, α_{ij} and K_{ij}, and its corresponding canonical momenta $\pi^{\mu\nu}$, τ^{ij} and P^{ij}. Hence, by performing the canonical analysis, the principal results are the following: the canonical Hamiltonian is given by

$$\begin{aligned}\mathcal{H}_c =\;& \frac{1}{4}P^2 - \frac{1}{4}P_{ij}P^{ij} - \frac{1}{2}\partial_i\partial_j h_{00}P^{ij} - m^2\left(K^2 - K_{ij}K^{ij}\right) + 2\partial_i K\partial^i K + 2\partial_i K^{ij}\partial_k K^k_j \\ & - 2\partial_k K_{ij}\partial^k K^{ij} - 2\partial_l K\partial_m K^{lm} + 2\pi^{ij}K_{ij} - 2\partial_i\pi^{ij}h_{0j} - \frac{1}{2}R_{ij}{}^{ij}\left(\nabla^2 + m^2\right)h_{00} \\ & - \frac{1}{2}\left(R_{ikj}{}^{k} - \frac{1}{2}\delta_{ij}R_{lm}{}^{lm}\right)\left(\nabla^2 + m^2\right)h^{ij}, \qquad \text{(A.7)}\end{aligned}$$

where we can observe that there is only one term in the canonical momenta π^{ij} and is linear. This fact is associated to the presence of ghosts degrees of freedom. On the other hand, the complete set of constraints is given by the following six first-class constraints

$$\begin{aligned}\gamma_1 &: \pi^{00} \approx 0, \\ \gamma_2^i &: \pi^{0i} \approx 0, \\ \gamma_3^i &: \partial_j\pi^{ij} \approx 0, \\ \gamma_4 &: \partial_i\partial_j P^{ij} + \left(\nabla^2 + m^2\right)R_{ij}{}^{ij} - \left(\nabla^2 + m^2\right)\left(\partial^i\partial^j - \delta^{ij}\nabla^2\right)\tau^{ij} \approx 0, \qquad \text{(A.8)}\end{aligned}$$

and the following six second-class constraints

$$\begin{aligned}\chi_1^{ij} &: \pi^{ij} - \alpha^{ij} \approx 0, \\ \chi_2^{ij} &: \tau^{ij} \approx 0. \qquad \text{(A.9)}\end{aligned}$$

We can observe that the second-class constraints have a trivial structure, and it is easy to observe that the Dirac and Poisson brackets coincide with each other, thus we expect that the instability of the Hamiltonian (A.7) will be present yet. In fact, we can use the Dirac brackets and the second-class constraints to calculate the extended Hamiltonian, we obtain

$$\begin{aligned}
\mathcal{H}_E &= \frac{1}{4}P^2 - \frac{1}{4}P_{ij}P^{ij} - m^2\left(K^2 - K_{ij}K^{ij}\right) + 2\partial_i K\partial^i K + 2\partial_i K^{ij}\partial_k K_j^k \\
&- 2\partial_k K_{ij}\partial^k K^{ij} - 2\partial_l K\partial_m K^{lm} + 2\pi^{ij}K_{ij} + 2\pi^{ij}\partial_i h_{0j} \\
&- \frac{1}{2}\left(R_{ikj}{}^{k} - \frac{1}{2}\delta_{ij}R_{lm}{}^{lm}\right)\left(\nabla^2 + m^2\right)h^{ij} + \left(R^i{}_k{}^{jk} - \frac{1}{2}\delta^{ij}R_{lm}{}^{lm}\right)\left(\nabla^2 + m^2\right)\tau_{ij} \\
&- \frac{1}{2}h_{00}\left(\partial_i\partial_j P^{ij} + \left(\nabla^2 + m^2\right)R_{ij}{}^{ij} - \left(\nabla^2 + m^2\right)\left(\partial^i\partial^j - \delta^{ij}\nabla^2\right)\tau^{ij}\right). \qquad \text{(A.10)}
\end{aligned}$$

In this case we see that the second-class constraints do not cure instability and there will be ghosts. In fact, the counting of physical degrees of freedom is performed as follows: there are 24 canonical variables, six first-class and six second-class constraints, so

$$DOF = \frac{1}{2}\left(24 - 6 - 2 * 6\right) = 3,$$

now there are two modes of helicity ± 2 and one mode of zero helicity being a ghost.

Appendix B

Purely Higher-Order Theory

In this appendix we resume the canonical analysis of the higher-order term given in the action (3.1), this term is expressed by

$$S[g_{\mu\nu}] = \int d^3x \, \sqrt{-g} \left(R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2 \right). \tag{B.1}$$

By performing the linearization and the change of variables K_{ij}, we find the following Lagrangian,

$$\begin{aligned} \mathcal{L}_J \;\; = \;\; & \dot{K}^{ij} \dot{K}_{ij} - \frac{1}{2} \dot{K}^2 - \frac{1}{2} \dot{K} \nabla^2 h_{00} + \dot{K}^{ij} \partial_i \partial_j h_{00} + 2 \dot{K}_{ij} R^{i}{}_{l}{}^{jl} - \frac{3}{2} \dot{K} R_{ij}{}^{ij} - 2 \partial^l K_{il} \partial^j K^i_j \\ & + 4 \partial^j K_{ij} \partial^i K - 2 \partial_i K \partial^i K + R_{ikj}{}^{k} R^{i}{}_{l}{}^{jl} + \partial_i \partial_j h_{00} R^{i}{}_{k}{}^{jk} - \frac{3}{8} R_{ij}{}^{ij} R_{lm}{}^{lm} + \frac{1}{8} (\nabla^2 h_{00})^2 \\ & - \frac{3}{4} \nabla^2 h_{00} R_{ij}{}^{ij} + \alpha^{ij} (\dot{h}_{ij} - 2 \partial_i h_{0j} - 2 K_{ij}). \end{aligned} \tag{B.2}$$

The canonical Hamiltonian is given by

$$\begin{aligned} \mathcal{H}_J \;\; = \;\; & \frac{1}{4} P^{ij} P_{ij} + \frac{1}{4} P \nabla^2 h_{00} - \frac{1}{2} P^{ij} \partial_i \partial_j h_{00} - P^{ij} R_{ilj}{}^{l} + \frac{3}{4} P R_{lm}{}^{lm} + 2 \partial^l K_{il} \partial^j K^i_j \\ & - 4 \partial^j K_{ij} \partial^i K + 2 \partial_i K \partial^i K + \frac{1}{4} \nabla^2 h_{00} R_{ij}{}^{ij} + 2 \pi^{ij} K_{ij} - 2 \partial_i \pi^{ij} h_{0j}. \end{aligned} \tag{B.3}$$

where we observe again the linear term in the π^{ij} momenta. Furthermore, the complete set of the constraints is given by the following eight first-class constraints

$$
\begin{aligned}
\gamma_1 &: \quad \pi^{00} \approx 0, \\
\gamma_2^i &: \quad \pi^{0i} \approx 0, \\
\gamma_3^i &: \quad \partial_j \pi^{ij} \approx 0, \\
\gamma_4 &: \quad \partial_i \partial_j P^{ij} \approx 0, \\
\gamma_5 &: \quad \partial_i \partial_j K^{ij} - \nabla^2 K + \pi_i^i \approx 0, \\
\gamma_6 &: \quad P + R_{lm}{}^{lm} - \left(\partial_i \partial_j - \delta_{ij} \nabla^2\right) \tau^{ij} \approx 0,
\end{aligned} \tag{B.4}
$$

and the following six second-class constraints

$$
\begin{aligned}
\chi_1^{ij} &: \quad \pi^{ij} - \alpha^{ij} \approx 0, \\
\chi_2^{ij} &: \quad \tau^{ij} \approx 0.
\end{aligned} \tag{B.5}
$$

It is worth commenting that the second-class constraints have a trivial form just like the system analyzed in Appendix A; then the Dirac brackets will be trivial. In this manner, with the results obtained in the previous sections, we expect that the system (B.1) will present the Ostrogradski sickness. In this respect, we can carry out the counting of physical degrees of freedom as follows: there are 24 canonical variables, eight first-class constraints and six second-class constraints, thus

$$
DOF = \frac{1}{2}\left(24 - 6 - 2 * 8\right) = 1.
$$

This degree of freedom corresponds to a ghost.

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