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“ANÁLISIS DE LAS SIMETRÍAS DE TEORÍAS MODIFICADAS DE GRAVEDAD: MODIFICACIÓN DE CHERN-SIMONS CUATRIDIMENSIONAL”

TESIS

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Resumen

En este trabajo se realiza el análisis Hamiltoniano de la modificación de Chern-Simons a Relatividad General en cuatro dimensiones. Se tomarán dos caminos: el formalismo Hamiltoniano de Gitman-Lyakhovich-Tyutin y la teoría de Hamilton-Jacobi. Se reporta la estructura completa de las restricciones del modelo. Se construye el Hamiltoniano canónico correspondiente, debido a que la teoría es de alto orden, se investigó si tal Hamiltoniano es estable en el sentido de la inestabilidad Ostrogradsky. Se presenta un conjunto completo de paréntesis generalizados fundamentales no triviales. Se identifican las transformaciones de norma de la teoría y se realiza un conteo de grados de libertad físicos. Además se aplicó la teoría de Hamilton-Jacobi al invariante de Chern-Simons tridimensional.

Abstract

In this work the Hamiltonian analysis of the Chern-Simons modification of General Relativity in four dimensions is performed. Two approaches are taken: the Gitman-Lyakhovich-Tyutin Hamiltonian formalism and the Hamilton-Jacobi theory. The complete structure of constraints of the model is reported. The corresponding canonical Hamiltonian is constructed and, since the theory is of higher-order it is investigated if such Hamiltonian is stable in Ostrogradsky's sense. A full set of non-trivial fundamental generalized brackets is presented. The gauge transformations of the theory are identified and the counting of physical degrees of freedom is carried out. In addition, the Hamilton-Jacobi theory is applied to the 3D Chern-Simons invariant.

Publications

- Alberto Escalante, Aldair Pantoja, Vanessa Castro, *The Hamilton-Jacobi analysis for higher-order Chern-Simons gravity*, Chinese Journal of Physics, doi.org/10.1016/j.cjph.2022.07.014, (2022).
- Alberto Escalante, Aldair Pantoja, *Canonical analysis for Chern-Simons modification of general relativity*, Annals of Physics, doi.org/10.1016/j.aop.2023.169246, (2023).
- Alberto Escalante, Aldair Pantoja, *The Hamilton-Jacobi analysis for higher-order modified gravity*, Chinese Journal of Physics, doi.org/10.1016/j.cjph.2023.05.001 (2023).

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Content

Chapter 1

Introduction

It is well-known that General Relativity (GR) is a successful framework for describing the classical behavior of the gravitational field and its relation with the geometry of space-time [1,6]. From the canonical point of view, GR is a background independent gauge theory with diffeomorphism invariance and the Hamiltonian is a linear combination of first-class constraints and propagates two physical degrees of freedom [7]. From the quantum point of view, the quantization program of gravity remains as a difficult task; in the non-perturbative scheme, the non-linearity of the gravitational field manifested in the constraints obscures the quantization so, making a complete description of a non-perturbative quantum theory of gravity is an open problem [8, 9]. On the other hand, the perturbative point of view of the path integral method leads to the non-renormalizability problem [10, 11] and all tools that have been developed in quantum field theory have not worked successfully. It is common to study modified theories of gravity in order to obtain insights in the classical or quantum regime with the expectation that they will provide new ideas or tools to carry out the quantization program, an example of this being the so-called higher-order theories [12–15]. In fact, higher-order theories are good candidates for fixing the infinities that appear in the renormalization problem of quantum gravity. It is claimed that adding higher-order terms (squared in the curvature) to gravity could help to avoid this problem; since these terms have a dimensionless coupling constant which ensures that the final theory is divergence-free [16, 17]. The study of higher-order theories is a modern topic in physics, these theories are relevant in dark energy physics [18], generalized electrodynamics [19–21] and string theory [22, 23]. A very interesting model in four dimensions can be found in the literature in which the Einstein-Hilbert (EH) action is extended by the addition of a Chern-Simons (CS) 4-current coupled with an auxiliary field, by taking a particular choice of the auxiliary field then the resulting action is a close model to GR [24]. At a Lagrangian level, the extended theory describes the propagation of two degrees of freedom corresponding to the gravitational waves traveling with velocity c, but these waves propagate with different polarization intensities violating spatial reflection symmetry. Moreover, the Schwarzchild metric is a solution of the equations of motion, thus the modified theory and the EH action share the same classical tests. On the other hand, at the Hamiltonian level, the model is a higher-order gauge theory whose Hamiltonian analysis is known not to be easy to perform. The analysis of constrained higher-order systems is usually developed by following the Ostrogradsky-Dirac (OD) [25–27] or the Gitman-Lyakhovich-Tyutin (GLT) [28,29] methods. The OD scheme is based on the extension of the phase space by considering the fields and their velocities as canonical coordinates and then introducing an extension to the canonical momenta, however, the identification of the constraints is not easy to develop; in some cases, the constraints are fixed by hand in order to obtain a consistent algebra [30] and this yields the opportunity to work with alternative methods. The GLT framework is based on the introduction of extra variables, which transform a problem with higher time derivatives to one with only first-order ones, then one can follow the usual Dirac's Hamiltonian formalism for constrained systems. There is an alternative framework for analyzing higher-order theories: the so-called Hamilton-Jacobi method. The HJ formalism for regular field theories [31,32] was developed by Güller and later for singular systems [33, 34]. It is based on the identification of the constraints as Hamiltonians of the theory and on the enforcement of integrability conditions for a collection of partial differential equations for the Hamiltonians (Hamilton-Jacobi equations). The Hamiltonians can be either involutive or non-involutive and they are used for constructing a fundamental differential that codifies all the physical information of the system.

With all of above, the main purpose of this work is to develop a detailed analysis of the CS modification of GR [24] under a particular choice of the auxiliary field in the perturbative approximation and to study its closeness with the canonical Hamiltonian structure of GR. In Chapter 2 and Chapter 3 we have a review of the canonical formalism and of the HJ formalism for constrained systems, respectively. In Chapter 4 we present the main aspects of CS modified gravity. In Chapter 5 we develop a canonical analysis and the HJ analysis of the extended model. Chapter 6 is devoted to conclusions, prospects and remarks.

Chapter 2

Canonical formalism for singular systems

The study of the classical singular systems began its development since the work of Dirac [35, 36]. In his generalization of the Hamiltonian formulation he demonstrated how to identify all the functions $f(q,p) = 0$ constraining the phase-space, this identification can be achieved if one imposes certain consistency conditions. In addition, Dirac's formalism provides other relevant aspects of a singular system such as; counting of degrees of freedom, identification of the gauge symmetries, elimination of non-physical degrees of freedom, etc. Here will be presented the basic aspects of the Dirac Hamiltonian formalism.

2.1 Primary constraints

Let's start with a classical system described by N generalized coordinates through the following action principle

$$S = \int L(\dot{q}_i, q_i, t)dt \tag{2.1.1}$$

where $i = 1, ..., N$ and L is a *singular Lagrangian*, we say that a Lagrangian is a singular one if the determinant of the Hessian matrix H^{ij} vanishes

$$\det\left(H^{ij}\right) = \det\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) = 0. \tag{2.1.2}$$

In order to go to the Hamiltonian formalism we introduce the canonical momenta

$$p^i = \frac{\partial L}{\partial \dot{q}_i}, \tag{2.1.3}$$

since $\det(H^{ij})=0$ only $\mathcal{R}=\text{rank}(H^{ij})$ of the velocities can be expressed in terms of the $q's$ and $p's$, on the other hand, we have $\mathcal{N}=\text{null}(H^{ij})$ non-invertible velocities so we expect $\mathcal{N}$ functions φ^n, $n=1,...,\mathcal{N}$ such that

$$\varphi^n(q_i,p^i)\approx 0, \tag{2.1.4}$$

which we are going to call *primary constraints*, the symbol $\approx$ stands for *weakly zero* and it means that the equation (2.1.4) holds only in the subspace $\mathbb{P}_\varphi\subset\mathbb{P}$ ($\mathbb{P}$ the phase-space) defined by the constraints. Once we introduced the canonical momenta we can define the Hamiltonian in the usual way

$$H_0=\dot{q}_ip^i-L. \tag{2.1.5}$$

The Einstein summation convention $\sum_{i=1}^{N}a_ib^i\rightarrow a_ib^i$ has been taken into account, from now and until said otherwise it will be used in all indicated sums. The Hamiltonian depends only of (q_i,p^i) but these variables are not independent because of (2.1.4) then the Hamiltonian is well defined only in $\mathbb{P}_\varphi$. Now we define the *primary Hamiltonian* as follows

$$H_1=H_0+u_n\varphi^n, \tag{2.1.6}$$

where u_n are Lagrange multipliers enforcing the primary constraints. Variation of H_1 lead us to the Hamilton's equations of motion

$$\dot{q}_i = \frac{\partial H_0}{\partial p^i}+u_n\frac{\partial\varphi^n}{\partial p^i}, \tag{2.1.7}$$

$$-\dot{p}^i = \frac{\partial H_0}{\partial q_i}+u_n\frac{\partial\varphi_n}{\partial q_i}, \tag{2.1.8}$$

in principle u_n are arbitrary functions of (q_i,p^i) so the equations of motion are not determined in a unique way until all multipliers are specified, but this is not always the case, if all multipliers cannot be found then the system has a certain freedom, we will see later that the undetermined multipliers are directly related to the gauge symmetries of the theory.

2.2 Consistency conditions

One expects some *consistency* of the constraints when the system evolves in time i.e. they should not change in time, in other words

$$\dot{\varphi}^n\approx 0, \tag{2.2.1}$$

we can express these conditions in terms of the *Poisson Brackets* (PB) by noticing that the time derivative of any function $F = F(q_i, p^i)$ is

$$\dot{F} = \{F, H_1\}, \tag{2.2.2}$$

it is straightfoward to deduce the equations of motion (2.1.7) and (2.1.8) from (2.2.2). The consistency conditions becomes

$$\{\varphi^n, H_0\} + u_m\{\varphi^n, \varphi^m\} \approx 0, \tag{2.2.3}$$

after developing the equation (2.2.3) for some constraint φ^n one end up with one of the following scenarios:

1. The final equation involves some of the multipliers u_n.
2. A relation between $q's$ and $p's$ emerge, thus we get expressions of the form
$$\varphi^{\mathcal{N}+1}(q_i, p^i) \approx 0. \tag{2.2.4}$$
3. It reduces to $0 \approx 0$.

From the first possibility one may solve for some of the multipliers and then substitute them back in H_1. If the second possibility occurs and if $\varphi^{\mathcal{N}+1}$ is independent of any φ^n then we have a new constraint (*secondary constraint*) and it lead us to a new consistency condition

$$\dot{\varphi}^{\mathcal{N}+1} \approx 0, \tag{2.2.5}$$

we are now in the same situation as before and if the second scenario occurs again then we need to go further and repeat the process until no more constraints appear. In the end we have a collection of secondary, tertiary, etc. constraints that sometimes are called generically as secondary constraints. After repeating this process for all φ^n we end up with a total number of A constraints φ^a, $a = 1, ..., A$.

2.3 First and second-class constraints

It will be worth if we classify all the constraints of the theory in a proper way, let us define a first-class function. Let be a phase-space valued function F, we say that F is *first-class* function if its Poisson bracket with all the constraints are weakly zero

$$\{F, \varphi^a\} \approx 0, \tag{2.3.1}$$

otherwise F is called a *second-class* function. Once we had all the constraints φ^a at hand we can identify all the A_f independent first-class constraints Γ^{a_f} of the theory as

$$\Gamma^{a_f} = \omega_a^{a_f} \varphi^a \tag{2.3.2}$$

where $a_f = 1, ..., A_f$ and $\omega_a^{a_f}$ are the null vectors of the matrix of PB between all constraints φ^a

$$W^{ab} = \{\varphi^a, \varphi^b\}. \tag{2.3.3}$$

On the other hand, the A_s remaining constraints whose PB does not vanish are identified as second-class constraints and we denote them as χ^{a_s}, $a_s = 1, ..., A_s$.

2.4 Total Hamiltonian and Dirac's brackets

We can define the *total Hamiltonian* in a similar way as we did with the primary Hamiltonian but now we take into account all the constraints

$$H_T = H_0 + u_a \varphi^a, \tag{2.4.1}$$

moreover, assuming that we are able to successfully classify all the constraints and to find explicit expressions of the multipliers $u_{a_s}(q_i, p^i)$ associated with the second-class constraints we can make the distinction between first-class and second-class constraints in the total Hamiltonian (2.4.1) renaming it as *extended Hamiltonian*

$$H_E = H_0 + u_{a_s} \chi^{a_s} + u_{a_f} \Gamma^{a_f}, \tag{2.4.2}$$

the dynamical evolution of the system is dictated by H_E

$$\dot{F} = \{F, H_E\}. \tag{2.4.3}$$

Another benefit of the constraint classification is that we can convert the second-class constraints into pure identities, this is $\chi^{a_s}(q_i, p^i) \approx 0 \longrightarrow \chi^{a_s}(q_i, p^i) = 0$ thus removing the non-physical degrees of freedom. We can achieve this by defining a new bracket, for such purpose we take the matrix W^{ab} and rearrange it in the following way

$$W^{ab} = \begin{array}{c c} & \begin{array}{cc} \Gamma^{b_f} & \chi^{b_s} \end{array} \\ \begin{array}{c} \Gamma^{a_f} \\ \chi^{a_s} \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & C^{a_s b_s} \end{pmatrix} \end{array}, \tag{2.4.4}$$

where $C^{a_s b_s}$ stands for the matrix of PB between all the second-class constraints. $C^{a_s b_s}$ is an antisymmetric and invertible matrix. We can define the new bracket (*Dirac's bracket*) as follows

$$\{F, G\}_D = \{F, G\}_D - \{F, \chi^{a_s}\} C^{-1}_{a_s b_s} \{\chi^{b_s}, G\}, \tag{2.4.5}$$

the new bracket satisfies all the good properties of the Poisson bracket and because of this we can use it to describe the dynamics of the system

$$\dot{F} = \{F, H_E\}_D \tag{2.4.6}$$

now that the PB served its purpose of classifying the constraints we can make the substitution $\{\ ,\ \} \longrightarrow \{\ ,\ \}_D$ in all calculations, for example, in the derivation of the gauge transformations.

2.5 Gauge transformations

In (2.4.2) is evident that some multipliers remains unknown (those associated with the first-class constraints) then the equations of motion that H_E generates still allows a free choice of u_{a_f}, this is because the Γ^{a_f} and u_{a_f} are related with transformations that do not affect the physical state of the system, in other words, the *gauge transformations*. Let us consider two states with the same initial conditions at a time t_0 and then let's take its dynamical evolution at time t in a Taylor expansion at first order

$$\begin{aligned} X(t) &= X(t_0) + \dot{X}\delta t & (2.5.1)\\ &= X(t_0) + (\{X, H\}_D + u_{a_f}\{X, \Gamma^{a_f}\}_D)\delta t, & (2.5.2) \end{aligned}$$

if we do this but for a different multiplier u'_{a_f} related to the same constraint Γ^{a_f} we have

$$\begin{aligned} X(t) &= X(t_0) + \dot{X}'\delta t & (2.5.3)\\ &= X(t_0) + (\{X, H\} + u'_{a_f}\{X, \Gamma^{a_f}\})\delta t, & (2.5.4) \end{aligned}$$

the difference is

$$\begin{aligned} \delta X(t) &= X'(t) - X(t) & (2.5.5)\\ &= (u'_{a_f} - u_{a_f})\{X, \Gamma^{a_f}\}\delta t & (2.5.6)\\ &= \{X, \Gamma^{a_f}\}\delta u_{a_f}, & (2.5.7) \end{aligned}$$

where $\delta u_{a_f} \equiv (u'_{a_f} - u_{a_f})\delta t$. If we define $\varepsilon_{a_f} \equiv \delta u_{a_f}$ we establish that: *the change $X' \longrightarrow X + \delta X$ is a transformation with generator $\varepsilon_{a_f}\Gamma^{a_f}$ that does not change the physical state of the system.* This transformation leaves intact the *extended action* given by

$$S_E(q,p,v) = \int (\dot{q}_i p^i - H_E)dt, \tag{2.5.8}$$

such action have all the information of the system; it considers the separation of first-class and second-class constraints, it does not have redundant degrees of freedom because of $\chi^{a_s} = 0$, it makes manifest the gauge symmetries of the system and it gives the equations of motion:

$$\begin{aligned} \dot{F} &= \{F, H_E\}_D, && (2.5.9)\\ \Gamma^{a_f} &\approx 0. && (2.5.10) \end{aligned}$$

2.6 Degrees of freedom

The number of degrees of freedom is the number of physical independent variables nedeed to describe the dynamics of a system. The total number of degrees of freedom is

$$\begin{aligned} DoF &= \frac{1}{2}\Big[\Big(\begin{array}{c}\textit{Total number of}\\ \textit{canonical coordinates}\end{array}\Big) - \Big(\begin{array}{c}\textit{Number of second-class}\\ \textit{constraints}\end{array}\Big)\\ &- 2\times\Big(\begin{array}{c}\textit{Number of first-class}\\ \textit{constraints}\end{array}\Big)\Big] \qquad (2.6.1) \end{aligned}$$

the $\frac{1}{2}$ factor compensates the transition from the configuration space $\mathbb{Q}$ to the phase-space $\mathbb{P}$ and the 2 factor that appears in the third term is due to the double role of the first-class constraints: as restrictions on the $q's$ and $p's$ and as a generators of gauge transformations.

2.7 Higher-order lagrangians

If a Lagrangian contains terms involving derivatives of a coordinate higher than 1, then we are dealing with a *higher-order lagrangian* or *higher-order theory*. These kind of theories can be problematic from a physical point of view, in the subsequent section we will delve into this. The presence of higher-order derivatives complicates the procedure of Hamiltonization and/or quantization of a singular theory, in the literature one can find appropiate developments in order to achieve this goal, for example, one can cite to the Gitman-Lyakhovich-Tyutin (GLT) approach [28, 29]. In the GLT method one looks for a reduction of the order of the Lagrangian by means of auxiliary fields and Lagrange multipliers associated to them. The equations of motion for some auxiliary fields lead to the primary constraints of the theory.

Let us consider a higher-order lagrangian L^*, i.e., the coordinate dependence of L^* is

$$L^* = L^*\left(x_1, ..., x_N, \frac{dx_1}{dt}, ..., \frac{dx_N}{dt}, \cdots \frac{d^{\mathcal{O}}x_1}{dt^{\mathcal{O}}}, ... \frac{d^{\mathcal{O}}x_N}{dt^{\mathcal{O}}}, t\right) = L^*\left(x_i^{(l)}, t\right), \tag{2.7.1}$$

where $i = 1, ..., N$, $l = 0, 1, ..., \mathcal{O}$ is the order of the time derivative and $\mathcal{O}$ is the highest order. Now we define a new set of coordinates in the following way

$$q_i^s = x_i^{(s-1)}, \tag{2.7.2}$$

$$v_i = x_i^{(\mathcal{O})}, \tag{2.7.3}$$

where $s = 1, ..., \mathcal{O}$. The superscript s in q_i^s does not denote a derivative of q_i, it only serves to associate the q_i^s to the correspondent $x_i^{(s-1)}$. The definition (2.7.2)-(2.7.3) "extends" the configuration space and the dependence of the Lagrangian becomes

$$L^* = L^*\left(q_i^s, v_i, t\right) = L^*\left(x_i^{(s-1)} = q_i^s, x_i^{(\mathcal{O})} = v_i, t\right), \tag{2.7.4}$$

moreover, from (2.7.2)-(2.7.3) we can infer some restrictions on the $v's$ and the $q's$ and their derivatives

$$\dot{q}_i^{s'} - q_i^{s'+1} = 0, \tag{2.7.5}$$

$$\dot{q}_i^{L} - v_i = 0, \tag{2.7.6}$$

where $s' = 1, ..., \mathcal{O} - 1$. These restrictions are not the primary constraints of the theory, such constraints will arise from the equations of motions for v_i. We need to include the restrictions (2.7.5)-(2.7.6) into the theory, thus we redefine the Lagrangian

$$L = L^* + \lambda_{s'}^i(\dot{q}_i^{s'} - q_i^{s'+1}) + \lambda_{\mathcal{O}}^i(\dot{q}_i^{\mathcal{O}} - v_i). \tag{2.7.7}$$

The Lagrangian L is of order one because L^* has no derivative terms and the presence of $\dot{q}_i^{s'}$ and $\dot{q}_i^{L}$ in the second and fourth terms. Now we are ready to define the phase-space, the canonical momenta are:

$$p_s^i = \frac{\partial L}{\partial \dot{q}_i^s} = \lambda_s^i, \tag{2.7.8}$$

$$\pi^i = \frac{\partial L}{\partial \dot{v}_i} = 0, \tag{2.7.9}$$

$$\Lambda_i^s = \frac{\partial L}{\partial \dot{\lambda}_s^i} = 0, \tag{2.7.10}$$

$$\Lambda_i^{\mathcal{O}} = \frac{\partial L}{\partial \dot{\lambda}_{\mathcal{O}}^i} = 0. \tag{2.7.11}$$

The corresponding Hamiltonian is

$$\begin{aligned} H &= (\dot{q}_i^s p_s^i + \dot{v}_i \pi^i + \dot{\lambda}_s^i \Lambda_i^s + \dot{\lambda}_{\mathcal{O}}^i \Lambda_i^{\mathcal{O}}) - L^* - \lambda_{s'}^i (\dot{q}_i^{s'} - q_i^{s'+1}) - \lambda_L^i (\dot{q}_i^L - v_i) \\ &= \dot{q}_i^s p_s^i - L^* + \lambda_{s'}^i q_i^{s'+1} + \lambda_{\mathcal{O}}^i v_i - (\dot{q}_i^{s'} \lambda_{s'}^i + \dot{q}_i^{\mathcal{O}} \lambda_{\mathcal{O}}^i) \\ &= \dot{q}_i^s p_s^i - L^* + \lambda_{s'}^i q_i^{s'+1} + \lambda_{\mathcal{O}}^i v_i - \dot{q}_i^s \lambda_s^i, \end{aligned}$$

therefore

$$H = p_{s'}^i q_i^{s'+1} + p_{\mathcal{O}}^i v_i - L^*. \tag{2.7.12}$$

In this derivation we use the definition of the momenta, the multipliers λ_s^i correspond directly to the momenta p_s^i, on the other hand, we can express $\lambda_{\mathcal{O}}^i$ in terms of the coordinates q_i^s by looking in the equations of motion for v_i, this is

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \dot{v}_i}\right) - \frac{\partial L^*}{\partial v_i} = 0, \tag{2.7.13}$$

$$\Rightarrow \quad \lambda_{\mathcal{O}}^i - \frac{\partial L^*}{\partial v_i} = 0, \tag{2.7.14}$$

thus we have

$$p_{\mathcal{O}}^i - \frac{\partial L^*}{\partial v_i} = 0. \tag{2.7.15}$$

In the regular case this equation can be used to express all the v_i in terms of the remaining phase-space coordinates. In the singular case, i.e., when $\det(H^{ij}) = 0$ not all v_i can be determined, if we denote the non-invertible sector[1] as v_n we can write the following relations

$$p_{\mathcal{O}}^n - f(q_i, p^j) = 0, \tag{2.7.16}$$

where $f \equiv \frac{\partial L^*}{\partial v_i}$. The dependence $f = f(q_i, p^i)$ comes from the fact that we can express $\mathcal{R}$ of the $v's$ in terms of the $q's$ and $p's$ and then substitute them back in (2.7.15). Finally, we

[1] $n = 1, ..., \mathcal{N}$ where $\mathcal{N}$ is the nullity.

have

$$\varphi^n(q_i, p^i) \approx 0, \tag{2.7.17}$$

these are the primary constraints of the theory. At this point we can follow Dirac's canonical formalism described in the previous sections.

2.8 Ostrogradsky's instability

In the previous section we reviewed a method that allows us to deal with a higher-order Lagrangian. We showed that is possible define a Hamiltonian function and to successfully identify the primary constraints of the theory, but, what physical consequences arise from the presence of higher-order derivatives of a coordinate, in particular, time derivatives? Let us recall a related theorem.

Theorem (of Ostrogradsky's instability): Let a higher-order Lagrangian that involves the $\mathcal{O}$-th order time derivative of the coordinates. If $\mathcal{O} > 2$ and if the Lagrangian is non-degenerate (non-degeneracy means $\frac{\partial^2 L}{\partial q^{(\mathcal{O})}} \neq 0$) with respect to the highest order derivatives, the Hamiltonian of this system depends linearly on the canonical momentum.

In other words, this theorem implies that if there are higher-order terms in the Lagrangian L then the energy of the system is unbounded. Let us consider a simple model that illustrates the Ostrogradsky instability. Let be a point particle system whose dynamical variable $\phi(t)$ is governed by the action

$$S(\phi) = \frac{1}{2}\int(\ddot{\phi}^2 + \alpha\dot{\phi}^2 + \beta\phi^2)dt, \tag{2.8.1}$$

where α and β are constants. We can write down a classically equivalent action if we take $\psi = \dot{\phi}$;

$$S(\phi, \psi) = \frac{1}{2}\int[\dot{\psi}^2 + \alpha\psi^2 + \beta\phi^2 + 2\lambda(\dot{\phi} - \psi)]dt, \tag{2.8.2}$$

by defining the $(p_\psi, p_\phi) = (\frac{\partial L}{\partial \dot{\psi}}, \frac{\partial L}{\partial \dot{\phi}})$ we can construct the Hamiltonian function

$$H = \frac{1}{2}(p_\psi^2 - \alpha\psi^2 - \beta\phi^2) + p_\phi\psi, \tag{2.8.3}$$

as we can see, the Hamiltonian depends linearly on the momentum p_ϕ thus the Hamiltonian is unbounded. In some sense, the theorem of Ostrogradsky "rules out" all Lagrangians of the form $L(q, \dot{q}, \ddot{q}, ..., t)$ if we want a bounded energy, the only assumption is non-degeneracy, but, what happens if one considers a higher-order Lagrangian but one relaxes the non-degeneracy requeriment? This is precisely the case if one considers a constrained system. In the development of the analysis in Chapter 5 we will show how the constraints of a theory can *heal* the Ostrogradsky instability.

2.9 Unfree gauge symmetries

It is known from the literature that if the implementation of consistency conditions (2.2.3) does not end at secondary constraints, then the parameters of the gauge transformations are restricted by differential equations [37]. Let be a collection of $\mathcal{N}$ primary constraints φ_{n_1}, the primary Hamiltonian is given by

$$H_1 = H_0 + u_{n_1} \overset{(1)}{\varphi}{}^{n_1}. \tag{2.9.1}$$

We adopted the notation $n_1 = 1, ..., \mathcal{N}_1 = \mathcal{N}$ and we are explicitly remarking the primary constraints with a label (1). In the general case the consistency conditions may give the following structure

$$\frac{d}{dt} \overset{(1)}{\varphi}{}_{n_1} = \overset{(1)}{V}{}^{m_1}_{n_1} \overset{(1)}{\varphi}{}_{m_1} + \overset{(1)}{O}{}^{m_2}_{n_1} \overset{(2)}{\varphi}{}_{m_2} \approx 0, \tag{2.9.2}$$

where m_2 runs from 1 to the total number $\mathcal{M}_2$ of secondary constraints. The coefficients V and O are differential operators. We can see that the evolution of some primary constraint $\overset{(1)}{\varphi}{}_{n_1}$ is a linear combination of primary constraints themselves and the new secondary constraints $\overset{(2)}{\varphi}{}_{m_2}$. For constraints of the $\mathfrak{g}$th generation we have

$$\frac{d}{dt} \overset{(\mathfrak{g})}{\varphi}{}_{n_\mathfrak{g}} = \sum_{g=1}^{\mathfrak{g}} \overset{(1)}{V}{}^{m_g}_{n_1} \overset{(g)}{\varphi}{}_{m_g} + \overset{(\mathfrak{g})}{O}{}^{m_{\mathfrak{g}+1}}_{n_\mathfrak{g}} \overset{(\mathfrak{g}+1)}{\varphi}{}_{m_{\mathfrak{g}+1}} \approx 0, \tag{2.9.3}$$

where we indicate the explicit summation of all constraints m_g per generation g, this is, the evolution of the $\overset{(\mathfrak{g})}{\varphi}{}_{n_\mathfrak{g}}$ constraint gives a linear combination of all the m_g constraints φ_{m_g} of each generation $g = 1, ..., \mathfrak{g}$, in addition, the $(\mathfrak{g}+1)$th generation of $m_{\mathfrak{g}+1}$ constraints $\varphi_{m_{\mathfrak{g}+1}}$ emerges. For the last generation $\mathcal{G}$ we have

$$\frac{d}{dt} \overset{(\mathcal{G})}{\varphi}{}_{n_\mathcal{G}} = \sum_{g=1}^{\mathcal{G}} \overset{(\mathcal{G})}{V}{}^{m_g}_{n_\mathcal{G}} \overset{(g)}{\varphi}{}_{m_g} \approx 0, \tag{2.9.4}$$

no new constraints appear, the evolution of $\overset{(\mathcal{G})}{\varphi}{}_{n_\mathcal{G}}$ is only a combination of the already known constraints. If there are at least tertiary constraints then the gauge parameters ε

are constrained [37]

$$\left(\delta^{n_{\mathfrak{g}}}_{m_{\mathfrak{g}}}\frac{d}{dt}+\overset{(\mathfrak{g})}{V}{}^{n_{\mathfrak{g}}}_{m_{\mathfrak{g}}}\right)\varepsilon^{m_{\mathfrak{g}}}+\sum_{g=\mathfrak{g}+1}^{\mathcal{G}}\overset{(g)}{V}{}^{n_{\mathfrak{g}}}_{m_g}\varepsilon^{m_g}+\overset{(\mathfrak{g}-1)}{O}{}^{n_{\mathfrak{g}}}_{m_{\mathfrak{g}-1}}\varepsilon^{m_{\mathfrak{g}-1}}=0, \tag{2.9.5}$$

where $\mathfrak{g}=2,...,\mathcal{G}-1$, for the $\mathcal{G}$ generation we have

$$\left(\delta^{n_{\mathcal{G}}}_{m_{\mathcal{G}}}\frac{d}{dt}+\overset{(\mathcal{G})}{V}{}^{n_{\mathcal{G}}}_{m_{\mathcal{G}}}\right)\varepsilon^{m_{\mathcal{G}}}+\overset{(\mathcal{G}-1)}{O}{}^{n_{\mathcal{G}}}_{m_{\mathcal{G}-1}}\varepsilon^{m_{\mathcal{G}-1}}=0. \tag{2.9.6}$$

We must not forget that the distinction here is between primary $\overset{(1)}{\varphi}_{n_1}$, secondary $\overset{(2)}{\varphi}_{n_2}$, tertiary $\overset{(3)}{\varphi}_{n_3}$, etc. and not between first-class φ_{a_f} and second-class φ_{a_s} constraints. In fact, if we consider the particular case where there is only one first-class constraint per generation the evolution of the constraints are reduced to

$$\begin{aligned}
\{\overset{(1)}{\Gamma},H_0\} &= \overset{(1)}{O}\overset{(2)}{\Gamma}, && (2.9.7)\\
\{\overset{(\mathfrak{g})}{\Gamma},H_0\} &= \overset{(\mathfrak{g})}{O}\overset{(\mathfrak{g}+1)}{\Gamma}, && (2.9.8)\\
\{\overset{(\mathcal{G})}{\Gamma},H_0\} &= 0, && (2.9.9)\\
\{\Gamma_{a_f},\Gamma_{b_f}\} &= 0, && (2.9.10)
\end{aligned}$$

where $\mathfrak{g}=2,...,\mathcal{G}-1$ and $a_f=1,...,A$. In this particular case $A=A_f$. The corresponding equations for the gauge parameters are

$$\dot{\varepsilon}^{\mathfrak{g}'+1}+\overset{(\mathfrak{g}')}{O}\varepsilon^{\mathfrak{g}'}=0 \tag{2.9.11}$$

where $\mathfrak{g}'=1,...,\mathcal{G}-1$. In chapter 5 we will see what happens if not one but several first-class constraints per generation are at stake.

2.10 Canonical quantization

From the classical point of view, an observable is a phase-space valued function $\mathbf{O}(q,p)$ that is gauge invariant, in other words, its Poisson or Dirac bracket with the first-class constraints Γ^{a_f} vanishes

$$\{\mathbf{O},\Gamma^{a_f}\}_D\approx 0. \tag{2.10.1}$$

In the simple case of a free particle, the most fundamental observables are the position X and momentum P.

The *canonical quantization* seeks for a representation $|\Psi\rangle = o_q|\psi_q\rangle$ of the observables and of its algebra on a Hilbert space $\mathcal{H}_{Dirac}$. One associates each *important* function (classical observables) on the phase-space to quantum observables that are self-adjoint operators on $\mathcal{H}_{Dirac}$, in the same manner, the fundamental brackets are promoted to commutators between those operators

$$\mathbf{O} \longrightarrow \hat{\mathbf{O}}, \tag{2.10.2}$$
$$\{\mathbf{O}_1, \mathbf{O}_2\}_D \longrightarrow \frac{1}{i\hbar}[\hat{\mathbf{O}}_1, \hat{\mathbf{O}}_2]. \tag{2.10.3}$$

For the free particle we have that

$$X \longrightarrow \hat{X}, \tag{2.10.4}$$
$$P \longrightarrow \hat{P} = -i\hbar\frac{d}{dX}, \tag{2.10.5}$$
$$\{X, P\} = 1 \longrightarrow \frac{1}{i\hbar}[\hat{X}, \hat{P}] = 1. \tag{2.10.6}$$

Now, let's assume a constrained theory whose dynamics is governed by the extended Hamiltonian H_E such that all constraints φ are first-class constraints[2] $\varphi^a = \Gamma^{a_f}$, $a = a_f$, thus we have

$$\{\Gamma^{a_f}, \Gamma^{b_f}\}_D \approx 0 \tag{2.10.7}$$
$$\{\Gamma^{a_f}, H_E\}_D \approx 0. \tag{2.10.8}$$

The canonical quantization procedure produces a space of states that is too large, in the sense that its quantum states are not gauge invariant, but physical states should be. Hence, the space of *physical states* $\mathcal{H}_{phys} \subset \mathcal{H}_{Dirac}$ must be chosen such that

$$\hat{\Gamma}^{a_f}|\Psi\rangle = 0 \tag{2.10.9}$$

for all $|\Psi\rangle \in \mathcal{H}_{phys}$ so that the finite gauge transformations act as

$$\exp^{i\varepsilon_{a_f}\hat{\Gamma}^{a_f}}|\Psi\rangle = |\Psi\rangle \tag{2.10.10}$$

i.e. the physical states are precisely the gauge invariant states. Thus, the space of physical states is the intersection of all kernels of the constraints operators, which is the quantum analogue of the classical constraint surface. It is worth noticing that we did not choose a gauge of any kind here.

[2]This always can be achieved in principle if one eliminates the second-class constraints by introducing the Dirac brackets.

Chapter 3

The Hamilton-Jacobi formalism

The Hamilton-Jacobi's (HJ) formalism for singular systems was developed by Güller [33,34] as a generalization of Carathéodory's method for regular systems [38]. Caratheodory's method is focused on the *equivalent Lagrangians* concept which is a consequence of the invariance of the Euler-Lagrange equations under the transformation

$$L \longrightarrow L' = L - \frac{d\Lambda(q_i, t)}{dt} \tag{3.0.1}$$

where Λ is an arbitrary function. The physics of the Lagrangians L and L' is the same and they also have simultaneous extremal values as functions of $\dot{q}_i$, this is, when the Lagrangian L reach its minimum value $L\mid_{\dot{q}_i=g_i}$ for some functions g_i thus the Lagrangian L' also reach its minimum $L'\mid_{\dot{q}_i=g_i}$ but both values are not necessarily the same, by exploiting this idea it will be possible to establish a set of partial differential equations for the Λ function.

3.1 Hamilton-Jacobi's partial differential equations

Let us consider consider a singular Lagrangian[1] $L(\dot{q}_i, q_i, t)$, that is, the determinant of the Hessian is equal to zero

$$\det\left(H^{ij}\right) = \left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) = 0. \tag{3.1.1}$$

The matrix H^{ij} have rank $\mathcal{R}$ and nullity $\mathcal{N}$. The rank indicate that $\mathcal{R}$ functions ϕ_r exist such that we may express a sector of velocities $\dot{q}_r$ in terms of the remaining variables

$$\dot{q}_r = \phi_r(q_s, q_n, \frac{\partial L}{\partial \dot{q}_s}, t) \tag{3.1.2}$$

[1] $i = 1, ..., N$ and N is the number of generalized coordinates.

where $r, s = 1, ..., \mathcal{R}$ and $n = 1, ..., \mathcal{N}$. In (3.1.2) we make an explicit distinction between the invertible q_r and non-invertible q_n sector. Now we will define an equivalent Lagrangian $L' = L - \frac{d\Lambda}{dt}$ such that it satisfies the following requeriments:

1. The Lagrangian L' as a function of $\dot{q}_i$ will have a minimum value equal to 0 when in the invertible sector it occurs that $\dot{q}_r = \phi_r$, in other words

$$\begin{aligned} L'(q_i, \phi_r, \dot{q}_n, t) &= 0, && (3.1.3) \\ L' &> 0, && \text{in a neighborhood of } \dot{q}_r = \phi_r. \quad (3.1.4) \end{aligned}$$

2. When L' reaches its minimum then the non-invertible sector q_n will satisfy $\mathcal{N}$ relations of the form

$$\frac{\partial L}{\partial \dot{q}^n} + H_n(q_r, q_m, \frac{\partial L}{\partial \dot{q}_r}, t) = 0, \tag{3.1.5}$$

   where $H_n \equiv -\left[\frac{\partial L}{\partial \dot{q}^n}\right]_{\dot{q}_r = \phi_r}$.

It is worth mentioning that when we refer to the minimum of L' we are seeing L' as a functional of $\dot{q}_i$ and not as a function of t explicitly, time evolution of L' (and of L as a consequence) is allowed but, under the requeriments 1 and 2. In fact, because L' and L share the same physics, this is, the action principles $S = \int L$ and $S = \int L'$ are equivalent therefore the existence of the functions Λ, ϕ_r and H_n that defines L' corresponds to the minimun value of the action S

$$S(\dot{q}_i, q_i, t) = \int L dt, \tag{3.1.6}$$

this is, the dynamical evolution corresponding to L. Let's see that it is possible to express the equations (3.1.5) in terms of Λ and of the coordinates q_i by working with equation (3.1.3). We have that

$$L' = 0 \quad \Rightarrow \quad L - \frac{\partial \Lambda}{\partial q_i}\dot{q}_i - \frac{\partial \Lambda}{\partial t} \;=\; 0, \tag{3.1.7}$$

by taking the derivative with respect to $\dot{q}_i$ we obtain the following equation

$$\frac{\partial \Lambda}{\partial q_i} \;=\; \frac{\partial L}{\partial \dot{q}_i}, \tag{3.1.8}$$

thus, the equations (3.1.5) takes the form

$$H_n(q_r, q_m, \frac{\partial \Lambda}{\partial q_r}, t) + \frac{\partial \Lambda}{\partial q^n} = 0. \tag{3.1.9}$$

In (3.1.9) we have $\mathcal{N}$ partial differential equations for the function Λ. We can add another equation to the initial set (3.1.9) by introducing the canonical Hamiltonian

$$H_0 = \dot{q}_i p^i - L, \tag{3.1.10}$$

where p^i are the respective canonical momenta of q_i

$$p^i = \frac{\partial L}{\partial \dot{q}_i}, \tag{3.1.11}$$

by substituting (3.1.7), (3.1.8) and (3.1.11) into H_0 we get

$$H_0(q_r, q_n, \frac{\partial \Lambda}{\partial q_r}, t) + \frac{\partial \Lambda}{\partial t} = 0. \tag{3.1.12}$$

One can obtain this equation by taking an alternative path, that is, by looking for a canonical transformation that changes the coordinates (q, p) into its initial values, the function Λ takes the role of the generating function of such transformation. Now, if we relabel $(t, q_1, ..., q_{\mathcal{N}})$ as $(t_0, t_1, ..., t_{\mathcal{N}})$ we can unify (3.1.9) and (3.1.12) as follows

$$H_n(q_r, \frac{\partial \Lambda}{\partial q_r}, t_m) + \frac{\partial \Lambda}{\partial t^n} = 0, \tag{3.1.13}$$

where now $n = 0, 1, ..., \mathcal{N}$ and $p_0 \equiv \frac{\partial \Lambda}{\partial t}$. We have a set of $\mathcal{N} + 1$ equations, these equations are known as the *Hamilton-Jacobi's partial differential equations* (HJPDE), from now we will call the functions H_n as Hamiltonians. In brief we will see that the variables $t_{n+1} \equiv q_{n+1}$ really have the same status as $t_0 \equiv t$, i.e., they are evolution parameters.

3.2 Characteristic equations

Once we established the system of equations (3.1.13) we can proceed to try to find their characteristic curves, this is, the set of total differential equations such that

$$\begin{aligned} dq_i &= A_i^n dt_n, && (3.2.1)\\ dp^n &= B_n^i dt^n, && (3.2.2)\\ d\Lambda &= C^n dt_n. && (3.2.3) \end{aligned}$$

Now the goal is to find the coefficients A_i^n, B_n^i and C^n, then we will set up certain conditions that ensures the integrability of (3.1.13) and that also guarantees the existence of unique solutions for the characteristics given some initial data. Let's rewrite the HJPDE in terms of the phase-space variables

$$H'_n(q_r, p^r, t_n, p^n) \equiv p_n + H_n(q_r, p^r, t_n) = 0, \tag{3.2.4}$$

in the case when $n = 0$ the function H'_0 can be viewed as the canonical Hamiltonian that emerges from a Legendre transformation that includes t_0 as a coordinate

$$H'_0 \equiv \dot{t}_0 p^0 + \dot{q}_i p^i - L = 0, \tag{3.2.5}$$

this is just the definition of H_0, we will keep referring to H'_n as Hamiltonians. Now let's try to derive the coefficients A^n_i first by taking the partial derivative of H'_0 with respect to p^r

$$\begin{aligned}
\frac{\partial H'_0}{\partial p^r} &= \frac{\partial}{\partial p_r}\left[p^0 + \dot{q}_i p^i - L(q_i, \dot{q}_s, \dot{t}_{n+1})\right] &(3.2.6)\\
&= 0 + \frac{\partial}{\partial p^r}\left[\dot{q}_s p^s + \dot{t}_{n+1} p^{n+1} - L(q_i, \dot{q}_s, \dot{t}_{n+1})\right] &(3.2.7)\\
&= \frac{\partial}{\partial p^i}\left[\phi_r p^r - \dot{t}_{n+1} H^{n+1} - L(q_i, \phi_s, \dot{t}_{n+1})\right], &(3.2.8)
\end{aligned}$$

from the first line to the second line we split up the coordinates $q_i \to (q_r, t_{n+1})$, the $n+1$ labeling comes from the fact that the definition of H_0 does not include t_0 in the Legendre transformation; in the last line we used the equation (3.2.4) and the fact that the velocities $\dot{q}_r$ can be expressed as $\dot{q}_r = \phi_r(q_s, t_{n+1}, p^s, t)$, hence by using the differentiation product rule we get

$$\begin{aligned}
\frac{\partial H'_0}{\partial p^r} &= \dot{q}_r + \frac{\partial \dot{q}_s}{\partial p^r} p^s - \dot{t}_{n+1}\frac{\partial H^{n+1}}{\partial p^r} - \frac{\partial L}{\partial \dot{q}_s}\frac{\partial \dot{q}_s}{\partial p^r} &(3.2.9)\\
&= \dot{q}_r - \frac{\partial H^{n+1}}{\partial p^r}\dot{t}_{n+1}, &(3.2.10)
\end{aligned}$$

therefore

$$\frac{dq_r}{dt} = \frac{\partial H'_0}{\partial p^r} + \frac{\partial H^{n+1}}{\partial p^r}\frac{dt_{n+1}}{dt}, \tag{3.2.11}$$

we now multiply both sides of the equation by dt and by recalling the fact that $\frac{\partial H'_{n+1}}{\partial p^r} = \frac{\partial H_{n+1}}{\partial p^r}$ we obtain a preliminary expression for the curves dq_r

$$\begin{aligned}
dq_r &= \frac{\partial H'_0}{\partial p^r} dt + \frac{\partial H^{n+1}}{\partial p^r} dt_{n+1} &(3.2.12)\\
&= \frac{\partial H'_n}{\partial p^r} dt^n, &(3.2.13)
\end{aligned}$$

we can incorporate to this expression the terms associated to t_n by taking advantage of the following identity

$$dt_n = \delta_n^m dt_m = \frac{\partial p^m}{\partial p^n} dt_m = \frac{\partial H'_m}{\partial p^n} dt^m, \tag{3.2.14}$$

in this manner we obtain the desired expansion of dq_i in terms of the variables t^n

$$dq_i = \frac{\partial H'_n}{\partial p^i} dt^n. \tag{3.2.15}$$

In a similar way, we can find the curves dp^i but this time by starting from the substitution of (3.1.8) in the definition of the momenta (3.1.11)

$$p^i = \frac{\partial \Lambda}{\partial q_i}, \tag{3.2.16}$$

if we take its differential version we have

$$\begin{aligned} dp^i &= \frac{\partial}{\partial q_i}\left(\frac{\partial \Lambda}{\partial q_j}\right) dq_j && (3.2.17) \\ &= \frac{\partial^2 \Lambda}{\partial q_n \partial q_i} dq_r + \frac{\partial^2 \Lambda}{\partial t_n \partial q_i} dt_n, && (3.2.18) \end{aligned}$$

here we can substitute eq. (3.2.15)

$$\begin{aligned} dp^i &= \frac{\partial^2 \Lambda}{\partial q_r \partial q_i} \frac{\partial H'_n}{\partial p^r} dt^n + \frac{\partial^2 \Lambda}{\partial t^n \partial q_i} dt_n && (3.2.19) \\ &= \left(\frac{\partial^2 \Lambda}{\partial q_r \partial q_i} \frac{\partial H'_n}{\partial p^r} + \frac{\partial^2 \Lambda}{\partial t^n \partial q_i}\right) dt^n && (3.2.20) \end{aligned}$$

we are going to leave this result at the moment. On the other hand, by taking the variation of $H'_n(q_i, \frac{\partial \Lambda}{\partial q_i})$ with respect to q_i we have

$$\begin{aligned} & \frac{\partial H'_n}{\partial q_i} + \frac{\partial H'_n}{\partial p^i} \frac{\partial^2 \Lambda}{\partial q_i^2} = 0 && (3.2.21) \\ \Rightarrow \quad & \frac{\partial H'_n}{\partial q_i} + \frac{\partial H'_n}{\partial p^r} \frac{\partial^2 \Lambda}{\partial q_i \partial q_r} + 1 \cdot \frac{\partial^2 \Lambda}{\partial q_i \partial t^n} = 0, && (3.2.22) \end{aligned}$$

by permuting second-order derivatives and by rearranging terms we have

$$\frac{\partial^2 \Lambda}{\partial q_r \partial q_i} \frac{\partial H'_n}{\partial p^r} + \frac{\partial^2 \Lambda}{\partial t_n \partial q_i} = -\frac{\partial H'_n}{\partial q_i}, \tag{3.2.23}$$

the left side of this equation is just the coefficients of eq. (3.2.20), a direct substitution yields

$$dp^i = -\frac{\partial H'_n}{\partial q_i} dt^n. \tag{3.2.24}$$

Finally, for the function Λ we have the following expansion

$$
\begin{aligned}
d\Lambda &= \frac{\partial \Lambda}{\partial q_r} dq_r + \frac{\partial \Lambda}{\partial t_n} dt_n && (3.2.25)\\
&= p^r \frac{\partial H'_n}{\partial p^r} dt^n - H^n dt_n && (3.2.26)\\
&= \left(-H_n + p_r \frac{\partial H'_n}{\partial p^r} \right) dt^n. && (3.2.27)
\end{aligned}
$$

With this last result we have just identified all the correspondent coefficients of eqs. (3.2.1)-(3.2.3). The characteristic equations of the system (3.2.4) are:

$$
\begin{aligned}
dq_i &= \frac{\partial H'_n}{\partial p^i} dt^n, && (3.2.28)\\
dp^i &= -\frac{\partial H'_n}{\partial q_i} dt^n, && (3.2.29)\\
d\Lambda &= \left(-H_n + p_r \frac{\partial H'_n}{\partial p^r} \right) dt^n, && (3.2.30)
\end{aligned}
$$

if equations (3.2.28) and (3.2.29) form an integrable set then its solutions automatically determine the function Λ in a unique way. The next thing to do is to explore what are the conditions that guarantee the integrability of dq_i and dp^i.

3.3 Integrability conditions

In order to specify under which conditions (3.2.29) and (3.2.28) are integrable let us first consider a general system of N total differential equations (TDE)

$$
dx^i = G^i_n(x_r, x_m) dx^n, \qquad (3.3.1)
$$

such system will have attached a set of partial differential equations (PDE) for a potential function F

$$
X_n F = G^i_n \frac{\partial F}{\partial x^i} = 0, \qquad (3.3.2)
$$

where X_n are linear operators. The integrability conditions of (3.3.1) will be posed in terms of the function F: *if the solution F exist then the system* (3.3.1) *will be integrable if and only if*

$$
[X_n, X_m] F = 0, \qquad (3.3.3)
$$

where $[X_n, X_m] = X_n X_m - X_m X_n$.

Example:

Let's take the following equation

$$P(x,y)dx + Q(x,y)dy = 0, \tag{3.3.4}$$

if we suppose that exist a potential function F such that

$$\frac{\partial F}{\partial x} = P, \tag{3.3.5}$$
$$\frac{\partial F}{\partial y} = Q, \tag{3.3.6}$$

this pair of equations are no other than the equations (3.3.2) with coefficients G^i_n;

$$G^x_x = 1, \quad G^x_y = -\frac{Q}{P}, \quad G^y_x = -\frac{P}{Q}, \quad G^y_y = 1, \tag{3.3.7}$$

this is

$$\frac{\partial F}{\partial x} - \frac{P}{Q}\frac{\partial F}{\partial y} = 0. \tag{3.3.8}$$

The corresponding integrability condition is

$$\begin{aligned}
[X_m, X_n]F &= (X_m X_n - X_n X_m)F && (3.3.9)\\
&= \left(G^i_n \frac{\partial}{\partial x^i}\right)\left(G^j_m \frac{\partial F}{\partial x^j}\right) - \left(G^i_m \frac{\partial}{\partial x^i}\right)\left(G^j_n \frac{\partial F}{\partial x^j}\right) && (3.3.10)\\
&= -\frac{Q}{P}\frac{\partial P}{\partial x} + \frac{\partial^2 F}{\partial x \partial y} + \frac{Q}{P}\frac{\partial P}{\partial x} - \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} - \frac{Q}{P}\frac{\partial Q}{\partial y} && (3.3.11)\\
&\quad - \frac{\partial^2 F}{\partial x \partial y} + \frac{Q}{P}\frac{\partial Q}{\partial y} && (3.3.12)\\
&= -\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = 0, && (3.3.13)
\end{aligned}$$

so, the equation (3.3.4) is integrable if it happens that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{\partial^2 F}{\partial x \partial y}. \tag{3.3.14}$$

Now, back to the general case, if (3.3.3) is satisfied there exists some functions ${M_{mn}}^{n'}$ such that

$$[X_n, X_m]F = {M_{nm}}^{n'} X_{n'} F, \tag{3.3.15}$$

if (3.3.15) holds then we say that the system is in involution, any commutator that is not in the form (3.3.15) will be taken as a new operator $X_{\mathcal{N}+1}$ and it will be added to the original set. This process will be repeated until all the operators satisfy the involution relation (3.3.15), when this is achieved we will say that the system is *complete*. We rephrase: *the system of equations* (3.3.1) *is integrable if and only if the system* (3.3.2) *is complete.*

Now let's see what happens if we impose such conditions on the characteristics dq_i and dp^i. An operator X_n acts on a phase-space valued function F in the following way

$$\begin{aligned} X_n F &= G_n^i \frac{\partial F}{\partial x^i} && (3.3.16)\\ &= \frac{\partial F}{\partial q_i}\frac{\partial H'_n}{\partial p^i} - \frac{\partial F}{\partial p^i}\frac{\partial H'_n}{\partial q_i} && (3.3.17)\\ &= \{F, H'_n\}, && (3.3.18) \end{aligned}$$

where $\{\ ,\ \} \equiv \frac{\partial}{\partial q_i}\frac{\partial}{\partial p^i} - \frac{\partial}{\partial p^i}\frac{\partial}{\partial q_i}$ is the PB. We passed from the first line to the second line just by setting $x = q, p$ and by identifying the coefficients G_n^i as (3.2.28) and (3.2.29). The commutator $[X_n, X_m]$ also has a PB structure

$$\begin{aligned} [X_n, X_m]F &= (X_n X_m - X_m X_n)F && (3.3.19)\\ &= X_n\{F, H'_m\} - X_m\{F, H'_n\} && (3.3.20)\\ &= \{\{F, H'_n\}, H'_m\} - \{\{F, H'_n\}, H'_m\} && (3.3.21)\\ &= \{F, \{H'_n, H'_m\}\}. && (3.3.22) \end{aligned}$$

To get the last line the Jacobi's identity has been invoked. The integrability conditions (IC) are reformulated as

$$\{H'_m, H'_n\} = 0, \qquad (3.3.23)$$

moreover, the integrability of the system can be expressed in a more condensed way if we define what we will call as *fundamental differential.* The differential of any phase-space valued function is

$$df = \frac{\partial f}{\partial q^i}dq^i + \frac{\partial f}{\partial p^i}dp^i = \frac{\partial f}{\partial q^r}dq^r + \frac{\partial f}{\partial t^n}dt^n + \frac{\partial f}{\partial p^i}dp^i \qquad (3.3.24)$$

by bringing back eqs. (3.2.28) and (3.2.29) we get that

$$
\begin{aligned}
df &= \left(\frac{\partial f}{\partial q^r}\frac{\partial H'_n}{\partial p_r} + \frac{\partial f}{\partial t^n} - \frac{\partial f}{\partial p^i}\frac{\partial H'_n}{\partial q_i}\right) dt^n &\quad (3.3.25)\\
&= \left(\frac{\partial f}{\partial q^r}\frac{\partial H'_n}{\partial p_r} + \frac{\partial f}{\partial t_m}\frac{\partial H'_n}{\partial p^m} - \frac{\partial f}{\partial p^i}\frac{\partial H'_n}{\partial q_i}\right) dt^n &\quad (3.3.26)\\
&= \left(\frac{\partial f}{\partial q^i}\frac{\partial H'_n}{\partial p_i} - \frac{\partial f}{\partial p^i}\frac{\partial H'_n}{\partial q_i}\right) dt^n &\quad (3.3.27)
\end{aligned}
$$

thus

$$df = \{f, H'_n\}dt^n. \quad (3.3.28)$$

As mentioned before, the fundamental diferential reveals that the Hamiltonians H_{n+1} along with their respective parameters t^{n+1}, dictate the dynamics of the theory in the same way that the canonical Hamiltonian H_0 tell us how the system described by (q, p) evolves in time. IC are reduced to

$$dH'_n = 0, \quad (3.3.29)$$

if (3.3.29) are not satisfied identically then one, but just one of the following scenarios can happen:

1. A new Hamiltonian $H'_{\mathcal{N}+1} = 0$ will emerge which must also satisfy (3.3.29).
2. Relations between the evolution parameters will appear and may be used in the rest of the analysis.

3.4 Hamilton-Jacobi's generalized bracket

At the end we have a number of A Hamiltonians H'_a. It may happen that not all the Hamiltonians are in involution, we will refeer to a Hamiltonian of this type as *non-involutive* and we will denote them as $H_{a_{ni}}$, $a_{ni} = 1, ..., A_{ni}$. The remaining Hamiltonians will be called *involutive* H_{a_i}, $a_i = 1, ..., A_i$. Now we are going to redefine the PB in such a manner that the involution relations (3.3.15) are satisfied and that at the same time the non-involutive Hamiltonians are removed. Let's take the IC for a non-involutive Hamiltonian

$$dH'_{a_{ni}} = \{H'_{a_{ni}}, H'_0\}dt + C_{a_{ni}b_{ni}}dt^{b_{ni}} = 0 \quad (3.4.1)$$

where $C_{a_{ni}b_{ni}} \equiv \{H'_{a_{ni}}, H'_{b_{ni}}\}$, if we resolve for $dt^{b_{ni}}$ we have

$$dt^{b_{ni}} = -C^{-1}_{a_{ni}b_{ni}}\{H'_{a_{ni}}, H'_0\}dt, \quad (3.4.2)$$

now let's substitute this into the fundamental differential

$$df = [\{f, H'_0\} - \{f, H'_{a_{ni}}\}C^{-1}_{a_{ni}b_{ni}}\{H'_{b_{ni}}, H'_0\}]dt + ... + \{f, H'_A\}dt^A. \tag{3.4.3}$$

Definifiton: We define the *Hamilton-Jacobi's generalized bracket* $\{\ ,\ \}^*$ as

$$\{f, g\}^* \equiv \{f, g\} - \{f, H'_{a_{ni}}\}C^{-1}_{a_{ni}b_{ni}}\{H'_{b_{ni}}, g\}, \tag{3.4.4}$$

under this definition the new bracket satisfy the following properties:

$$\begin{aligned}
\{f, g\}^* &= -\{g, f\}^*, && (3.4.5)\\
\{f, gh\}^* &= \{f, g\}^*h + h\{f, h\}^*, && (3.4.6)\\
\{\{f, g\}^*, h\}^* &+ \{\{h, f\}^*, g\}^* + \{\{g, h\}^*, f\}^* = 0, && (3.4.7)\\
\{H'_{a_{ni}}, f\}^* &= 0 \quad \forall f, && (3.4.8)\\
\{f, H'_{a_i}\}^* &= \{f, H'_{a_i}\}. && (3.4.9)
\end{aligned}$$

Equation (3.4.3) and property (3.4.9) allows us to rewrite the fundamental differential

$$\begin{aligned}
df &= \{f, H'_{a-a_{ni}}\}^* dt^{a-a_{ni}}\\
&= \{f, H'_0\}^* dt + \{f, H'_{a_i}\}^* dt^{a_i}. && (3.4.10)
\end{aligned}$$

3.5 Gauge transformations and degrees of freedom

The dynamical evolution described by the parameters t^{a_i} attached to involutive Hamiltonians can be understood as a canonical transformation. Let be a dynamical variable X, its evolution will be

$$dX = \{X, H'_0\}^* dt + \{X, H'_{a_i}\}^* dt^{a_i} \tag{3.5.1}$$

now, if we suppose that the t parameter does not change then the variation of X is

$$\delta X = \{X, H_{a_i}\}^* \delta t^{a_i}. \tag{3.5.2}$$

Because there is no time variation we can infer a transformation that connects states that are physically equivalent with $H'_{a_i}\delta q^{a_i}$ being its generator, such transformation is

$$X \longrightarrow X + \delta X. \tag{3.5.3}$$

Last but not least, the number of degrees of freedom will depend on the number of involutive Hamiltonians and on the number of dynamical variables of the system, that is, the variables that have an equation of motion that does not lead to relations between the phase-space coordinates and the parameters t^a

$$DoF = \frac{1}{2}\Big[\Big(\begin{array}{c} \textit{Total number of} \\ \textit{dynamical variables} \end{array} \Big) - \Big(\begin{array}{c} \textit{Total number of} \\ \textit{involutive Hamiltonians} \end{array} \Big)\Big].$$

Chapter 4

Chern-Simons modified gravity

4.1 Chern-Simons gravity in three dimensions

Originally, the three-dimensional Chern-Simons invariant was first introduced by Deser *et. al.* [39] as a deformation of the Einstein-Hilbert action in three dimensions,

$$S[g_{\mu\nu}] = \int_M R\sqrt{-g}d^3x + \int_M \varepsilon^{\mu\nu\alpha}\left(\frac{1}{2}\Gamma^{\beta}_{\mu\gamma}\partial_\nu\Gamma^{\gamma}_{\alpha\beta} + \frac{1}{3}\Gamma^{\beta}_{\mu\gamma}\Gamma^{m}_{\nu\delta}\Gamma^{\delta}_{\alpha\beta}\right)d^3x, \tag{4.1.1}$$

where $g_{\mu\nu}$ is the 3-metric tensor, M is the space-time manifold, $\varepsilon^{\mu\nu\alpha}$ is the Levi-Civita tensor and $\Gamma^{\alpha}_{\mu\nu}$ are the Christoffel symbols. The second term is recognized as the *Chern-Simons term.* Such deformation produces a gravitational theory (a higher-order theory in the sense of section 2.7) endowed with mass and spin 2. Later it was demonstrated that the resulting theory known as *Topologically massive gravity* (TMG) is renormalizable [40]. A canonical quantization approach was developed in [41], this is, by following the canonical Hamiltonian method described in Chapter 1.

The three-dimensional Chern-Simons invariant on its own can be very helpful as a laboratory in order to gain some insight in the application of the techniques developed in Chapters 1 and 2, in particular, the HJ theory (see Appendix A for details).

4.2 Chern-Simons invariant as an anomaly

Although in [39] the CS invariant is defined in three dimensions, some Chern-Simons-type terms in four or more dimensions appeared before in other physical scenarios. For example, it appeared in the fundamental particles physics context as an *anomaly*, that is, a correction

$\mathcal{A}$ to the divergence of a current J^μ that makes such current a non-conserved quantity: $\partial_\mu J^\mu = \mathcal{A} \neq 0$. Some gauge theories with fermions exhibits inherent anomalies as is the case of the electromagnetic field A_μ coupled to a Dirac fermion of mass m [42,43]. To clarify this point let's consider a massless spinorial electrodynamics theory that is described by the following action

$$S[\psi, A_\mu] = \int (\bar{\psi} i\gamma^\mu \partial_\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - ie\bar{\psi}\gamma^\mu A_\mu \psi) d^4x \qquad (4.2.1)$$

where ψ is the Dirac fermion, γ^μ are the Dirac matrices, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the strength tensor and A_μ is the gauge field. This theory is invariant under the following transformation

$$\psi \longrightarrow e^{i\lambda\gamma^5}\psi = \psi + i\lambda\gamma^5\psi + ... \qquad (4.2.2)$$

where λ is a parameter and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is the chiral matrix. Noether's theorem indicates that with this *chiral* symmetry it comes along a conserved quantity, a conserved current given by

$$J^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi, \qquad (4.2.3)$$

by calculating the divergence of J^μ one finds that

$$\partial_\mu J^\mu = -\frac{1}{8\pi^2}\varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}, \qquad (4.2.4)$$

details of this result can be found in [44]. We recognize the Chern-Simons-type invariant in the r.h.s. of equation (4.2.4) in the gauge field language. Similarly, another example of anomalous behaviour can be found in the coupling between a spinor particle of mass m and the gravitational field with metric $g_{\mu\nu}$ [45, 46], the corresponding discrepancy is

$$\nabla_\mu(\sqrt{-g}J^\mu) = \frac{1}{384\pi^2}\varepsilon^{\mu\nu\alpha\beta} R_{\gamma\delta\alpha\beta} R^{\gamma\delta}{}_{\mu\nu} \qquad (4.2.5)$$

where $R_{\alpha\beta\mu\nu}$ is the Riemann curvature tensor. There are more scenarios where Chern-Simons-type terms are considered such as: loop quantum gravity (LQG) [8, 47, 48], string theory [49], supergravity [50, 51], etc [52–56].

4.3 Chern-Simons gravity in four dimensions

For the four-dimensional case we start with a general coupling between the Einstein-Hilbert action and the Chern-Simons-type term

$$S = \frac{1}{16\pi G}\int_M R\sqrt{-g}d^4x + \frac{\mathcal{A}}{4}\int_M \theta^* \mathcal{R}\mathcal{R}\sqrt{-g}d^4x, \qquad (4.3.1)$$

G is the gravitational constant, M is the space-time manifold, R is the scalar curvature, g is the determinant of the metric, $\mathcal{A}$ is a constant, ${}^*\mathcal{R}\mathcal{R}$ is the Pontryagin invariant and θ is known as the coupling field which in general is a function of the space-time. Put in this fashion, the action (4.3.1) describes a *non-dynamical* theory in the sense that there is no kinetic term of θ, of course one can add the corresponding action that involves these kinetic terms

$$S_\theta = \mathcal{B}\int_M [\frac{1}{2}g^{\mu\nu}(\nabla_\mu\theta)(\nabla_\nu\theta) + V(\theta)] \tag{4.3.2}$$

where $\mathcal{B}$ is a constant, ∇_μ is the covariant derivative and V is the potential. When the action S_θ is considered in (4.3.1) the resulting theory is called a *dynamical* one. In the following we will consider $\mathcal{B}=0$ and $\mathcal{A}=(16\pi G)^{-1}$, for further insight on the dynamical case one can see [57].

4.4 Non-dynamical CS gravity in four dimensions

By taking $\mathcal{A}=(16\pi G)^{-1}$ and $\mathcal{B}=0$ action takes the form

$$S[g_{\mu\nu}] = \frac{1}{16\pi G}\int_M \left(R\sqrt{-g} + \frac{1}{4}\theta^* R^\sigma{}_\tau{}^{\mu\nu} R^\tau{}_{\sigma\mu\nu}\right) d^4x \tag{4.4.1}$$

where

$${}^*\mathcal{R}\mathcal{R} \equiv {}^*R^\sigma{}_\tau{}^{\mu\nu} R^\tau{}_{\sigma\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} R^\sigma{}_{\tau\alpha\beta} R^\tau{}_{\sigma\mu\nu}, \tag{4.4.2}$$

$\varepsilon^{\mu\nu\alpha\beta}$ is the Levi-Civita tensor and the Riemann tensor $R^\alpha{}_{\beta\mu\nu}$ is given by

$$R^\alpha{}_{\beta\mu\nu} = \partial\Gamma^\alpha_{\mu\alpha} - \partial_\mu\Gamma^\alpha_{\nu\beta} + \Gamma^\alpha_{\nu\gamma}\Gamma^\gamma_{\mu\beta} - \Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\nu\beta}. \tag{4.4.3}$$

$\Gamma^\alpha_{\mu\nu}$ are the same as in (4.1.1) but now the metric tensor $g_{\mu\nu}$ and all objects that are made of it are four-dimensional entities. The action (4.4.1) was first proposed by Jackiw in [24]. One can rewrite the action by noticing that

$$\frac{1}{2}{}^*R^\sigma{}_\tau{}^{\mu\nu} R^\tau{}_{\sigma\mu\nu} \quad = \quad 2\varepsilon^{\mu\alpha\beta\nu}\partial_\mu\left(\frac{1}{2}\Gamma^\sigma_{\alpha\tau}\partial_\beta\Gamma^\tau_{\nu\sigma} + \frac{1}{3}\Gamma^\sigma_{\alpha\tau}\Gamma^\tau_{\beta\eta}\Gamma^\eta_{\nu\sigma}\right), \tag{4.4.4}$$

then, up to a boundary term the action takes the form

$$S[g_{\mu\nu}] = \int \left(R\sqrt{-g} - \frac{1}{2}\mathfrak{v}_\mu J^\mu\right) d^4x. \tag{4.4.5}$$

where $\mathfrak{v}_\mu \equiv \partial_\mu\theta$ and $J^\mu \equiv 2\varepsilon^{\mu\alpha\beta\nu}\left(\frac{1}{2}\Gamma^\sigma_{\alpha\tau}\partial_\beta\Gamma^\tau_{\nu\sigma} + \frac{1}{3}\Gamma^\sigma_{\alpha\tau}\Gamma^\tau_{\beta\eta}\Gamma^\eta_{\nu\sigma}\right)$. The similarity of the second term with the three-dimensional Chern-Simons theory is more clear now, in fact, there is a direct relation between the Chern-Simons invariant and the Pontryagin class, namely, the exterior derivative of the former is equal to the latter [58]. Variation of the action (4.4.5) with respect to the metric tensor $g_{\mu\nu}$ produces the equations of motions (EoM) of the theory

$$\mathcal{G}_{\mu\nu} + \mathcal{C}_{\mu\nu} = 0, \tag{4.4.6}$$

where $\mathcal{G}_{\mu\nu}$ is the Einstein tensor

$$\mathcal{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{4.4.7}$$

and $\mathcal{C}^{\mu\nu}$ is a Cotton-type tensor

$$\mathcal{C}^{\mu\nu} = -\frac{1}{2\sqrt{-g}}[\mathfrak{v}_\sigma(\epsilon^{\sigma\mu\alpha\beta}\nabla_\alpha R^\nu{}_\beta + \epsilon^{\sigma\nu\alpha\beta}\nabla_\alpha R^\mu{}_\beta) + \mathfrak{v}_{\sigma\tau}({}^*R^{\tau\mu\sigma\nu} + {}^*R^{\tau\nu\sigma\mu})], \tag{4.4.8}$$

where $\mathfrak{v}_{\mu\nu} \equiv \partial_\mu \partial_\nu \theta$. The tensor $\sqrt{-g}\mathcal{C}^{\mu\nu}$ is a conformal invariant tensor in the infinitesimal sense (see Appendix B for details), $\mathcal{C}^{\mu\nu}$ its also symmetric and has zero trace. It is known that the covariant derivative of the Einstein tensor vanishes due to the Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$, on the other hand, if matter is present then its energy-momentum tensor $T_{\mu\nu}$ will appear in the r.h.s. of equation (4.4.6), its divergence is zero $\nabla_\mu T^{\mu\nu} = 0$ so, what happens if we calcualte the derivative of $\mathcal{C}_{\mu\nu}$? The result is

$$\nabla_\mu \mathcal{C}^{\mu\nu} = \frac{1}{8\sqrt{-g}} \mathfrak{v}^\nu \times {}^*\mathcal{R}\mathcal{R}, \tag{4.4.9}$$

therefore, if we want consistency between the l.h.s. and the r.h.s. from the EoM its solutions must satisfy the following requirement

$${}^*\mathcal{R}\mathcal{R} = 0. \tag{4.4.10}$$

Despite this condition one can connect the extended theory with pure GR by choosing a particular form of θ

$$\theta = \frac{t}{\Omega} \longrightarrow \mathfrak{v}_\mu = (1/\Omega, 0, 0, 0), \tag{4.4.11}$$

with this election the Schwarzchild metric remains as a solution of the extended EoM (4.4.6), in this manner, the classical tests of GR are considered. In addition, the Schwarzchild metric also satisfies the requirement (4.4.10). The Schwarzchild metric is not the only solution of the extended theory, the Reissner-Nordstrom and the Friedmann-Robertson-Walker metrics are also solutions of the extended theory [59], in contrast, the Kerr, Kerr-Newman and Kerr-NUT line elements does not satisfy ${}^*\mathcal{R}\mathcal{R} = 0$ [60].

Chapter 5

Analysis of a Chern-Simons modification of general relativity

The content of this chapter is original and it can be found in doi.org/10.1016/j.aop.2023.169246 and in doi.org/10.1016/j.cjph.2023.05.001.

5.1 Canonical analysis

We start with the action composed by the EH action plus the Chern-Simons term

$$S[g_{\mu\nu}] = \int \left(R\sqrt{-g} - \frac{1}{2}\mathfrak{v}_\mu J^\mu \right) d^4x, \tag{5.1.1}$$

we are interested in finding any contribution of the CS term to the well-known canonical Hamiltonian structure of GR, thus, we will make the choice (4.4.11) of the field θ and we will write down a linearized action by considering a perturbation $h_{\mu\nu}$ around the Minkowski background $\eta_{\mu\nu}$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}(x), \tag{5.1.2}$$

hence, by substituting (4.4.11) and (5.1.2) into (5.1.1) and by performing integration by parts we get

$$\begin{aligned} S[h_{\mu\nu}] &= \int \left(\sqrt{-g}R - \frac{1}{2}\mathfrak{v}_\mu J^\mu \right) d^4x \\ &= \int \Bigg[\left(\frac{1}{4}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} - \frac{1}{4}\partial_\lambda h^\mu{}_\mu \partial^\lambda h^\nu{}_\nu + \frac{1}{2}\partial_\lambda h^\lambda{}_\mu \partial^\mu h^\nu{}_\nu - \frac{1}{2}\partial_\lambda h^\lambda{}_\mu \partial_\nu h^{\nu\mu} \right) \\ &\quad - \frac{1}{4\Omega}\epsilon^{0\lambda\mu\nu} \left(\partial_\sigma h_\lambda{}^\rho \partial_\rho \partial_\mu h_\nu{}^\sigma - \partial_\sigma h_\lambda{}^\rho \partial^\sigma \partial_\mu h_{\rho\nu} \right) \Bigg] d^4x, \end{aligned} \tag{5.1.3}$$

in the second line we also neglect $O(h^3)$ terms of $h_{\mu\nu}$ and the signature $\eta_{\mu\nu}=(-1,1,1,1)$ is adopted. Variation of (5.1.3) yields

$$\mathcal{G}^{lin}_{\mu\nu}+\mathcal{C}^{lin}_{\mu\nu}=0, \tag{5.1.4}$$

where $\mathcal{G}^{lin}_{\mu\nu}$ and $\mathcal{C}^{lin}_{\mu\nu}$ are the linearized versions of the Einstein and Cotton tensors respectively

$$\begin{aligned}\mathcal{G}^{lin}_{\mu\nu} &= \frac{1}{2}[\Box h_{\mu\nu}+\partial_\mu\partial_\nu h^\lambda{}_\lambda-\partial_\mu\partial_\lambda h^\lambda{}_\nu-\partial_\nu\partial_\lambda h^\lambda{}_\mu-\eta_{\mu\nu}(\Box h^\lambda{}_\lambda-\partial_\lambda\partial_\gamma h^{\lambda\gamma}]],\\ \mathcal{C}^{lin}_{\mu\nu} &= -\frac{1}{4\Omega}[\epsilon_{0\mu\lambda\gamma}\partial^\lambda(\Box h^\gamma{}_\nu-\partial_\nu\partial_\alpha h^{\alpha\gamma})+\epsilon_{0\nu\lambda\gamma}\partial^\lambda(\Box h^\gamma{}_\mu-\partial_\mu\partial_\alpha h^{\alpha\gamma})],\end{aligned} \tag{5.1.5}$$

$\Box=\partial_\mu\partial^\mu=\nabla^2-\partial_0^2$ is the D'alembertian operator. Furthermore, its quite easy to show that the action is proportional to the sum of the Einstein and Cotton tensors if we rearrange it in a proper way, this is

$$S[h_{\mu\nu}]=-\frac{1}{2}\int h^{\mu\nu}(\mathcal{G}^{lin}_{\mu\nu}+\mathcal{C}^{lin}_{\mu\nu})d^4x. \tag{5.1.6}$$

The action (5.1.6) will be our starting point. The task now is to identify whether or not the action is a higher-order one. We will remove an overall factor of $-\frac{1}{2}$ which of course does not change the dynamics. By performing a $3+1$ decomposition of the action we get

$$\begin{aligned}S &= \int\Big[\frac{1}{2}\dot h_{ij}\dot h^{ij}-\partial_j h_{0i}\partial^j h^{0i}-\frac{1}{2}\partial_k h_{ij}\partial^k h^{ij}-\frac{1}{2}\dot h^i{}_i\dot h^j{}_j+\partial^j h^0{}_0\partial_j h^i{}_i+\frac{1}{2}\partial_k h^i{}_i\partial^k h^j{}_j\\ &\quad -2\partial^i h^0{}_i\dot h^j{}_j-\partial_i h^0{}_0\partial_j h^{ij}-\partial_i h^{ij}\partial_j h^k{}_k+2\partial_j h^0{}_i\dot h^{ij}+\partial_i h^i{}_0\partial_j h^{0j}+\partial_k h^k{}_i\partial_j h^{ij}\\ &\quad +\frac{1}{\mu}\epsilon^{0ijk}(-\ddot h^l{}_i\partial_j h_{lk}+2\dot h^l{}_i\partial_j\partial_l h^0{}_k+\partial_l h^m{}_i\partial_m\partial_j h^l{}_k+\nabla^2 h^0{}_i\partial_j h_{0k}+\nabla^2 h^m{}_i\partial_j h_{mk})\Big]\,d^4x,\end{aligned} \tag{5.1.7}$$

where we have defined $\mu\equiv 2\Omega$. The action is a higher-order theory as we can see in the third line of (5.1.7). One could be tempted to integrate by parts in the higher-order term $\ddot h_i{}^l\partial_j h_{kl}\to-\dot h_i{}^l\partial_j\dot h_{kl}$, this is conceptually correct if there is no boundary. Indeed, by doing such integration one effectively reduce the order of the Lagrangian but, by trying to identify the primary constraints $\varphi(q,p)$ from the definition of the momenta one eventually encounter some issues. Let's write down the canonical momenta $\pi^{\mu\nu}=\frac{\partial\mathcal{L}}{\partial\dot h_{\mu\nu}}$ for such alternative action

$$\pi^{00} = 0, \tag{5.1.8}$$

$$\pi^{0i} = 0, \tag{5.1.9}$$

$$\begin{aligned}\pi^{ij} &= \dot h^{ij}-(\dot h^k{}_k-2\partial^k h_{0k})\eta^{ij}-(\partial^i h_0{}^j+\partial^j h_0{}^i)-\frac{1}{\mu}(\epsilon^{ikl}\partial^j+\epsilon^{jkl}\partial^i)\partial_k h_{0l}\\ &\quad +\frac{1}{\mu}(\epsilon^{ikl}\eta^{jm}+\epsilon^{jkl}\eta^{im})\partial_k\dot h_{lm}.\end{aligned} \tag{5.1.10}$$

From the expression for π^{ij} we can conclude that some velocities $\dot{h}_{lm}$ cannot be solved in terms of the phase-space coordinates, on the other hand, equations (5.1.8)-(5.1.10) are usually taken as the primary constraints of the theory, nevertheless, this is incorrect for (5.1.10) because this is a relation that also involves the velocities of the perturbation in contrast with (2.1.4). In the end, it is not clear how to identify all the primary constraints of the theory. As mentioned before, there are approaches where one extends the definition of the conjugate momenta itself, here we will follow the GLT approach described in Sec. 2.7. Let's begin with the introduction of the following coordinates

$$G_{\mu\nu} \equiv \dot{h}_{\mu\nu}, \tag{5.1.11}$$

$$v_{\mu\nu} \equiv \ddot{h}_{\mu\nu}, \tag{5.1.12}$$

these new coordinates extend the configuration space and the action is redefined

$$S' = \int \mathcal{L}' dx^4 = \int \mathcal{L} dx^4 + \int [\lambda_1^{\mu\nu}(\dot{h}_{\mu\nu} - G_{\mu\nu}) + \lambda_2^{\mu\nu}(\dot{G}_{\mu\nu} - v_{\mu\nu})] d^4x, \tag{5.1.13}$$

where $\mathcal{L}$ is the following Lagrangian

$$\begin{aligned} \mathcal{L} = & \int \Big[\frac{1}{2} G_{ij} G^{ij} - \partial_j h_{0i} \partial^j h^{0i} - \frac{1}{2} \partial_k h_{ij} \partial^k h^{ij} - \frac{1}{2} G^i{}_i G^j{}_j + \partial^j h^0{}_0 \partial_j h^i{}_i + \frac{1}{2} \partial_k h^i{}_i \partial^k h^j{}_j \\ & -2\partial^i h^0{}_i G^j{}_j - \partial_i h^0{}_0 \partial_j h^{ij} - \partial_i h^{ij} \partial_j h^k{}_k + 2\partial_j h^0{}_i G^{ij} + \partial_i h^i{}_0 \partial_j h^{0j} + \partial_k h^k{}_i \partial_j h^{ij} \\ & + \frac{1}{\mu} \epsilon^{0ijk} (-v^l{}_i \partial_j h_{lk} + 2G^l{}_i \partial_j \partial_l h^0{}_k + \partial_l h^m{}_i \partial_m \partial_j h^l{}_k + \nabla^2 h^0{}_i \partial_j h_{0k} + \nabla^2 h^m{}_i \partial_j h_{mk}) \Big] d^3x. \end{aligned} \tag{5.1.14}$$

Now is time to introduce the canonical momenta $(\pi^{\mu\nu}, p^{\mu\nu}, \hat{v}^{\mu\nu}, \Lambda^1_{\mu\nu}, \Lambda^2_{\mu\nu})$ canonically conjugate to $(h_{\mu\nu}, G_{\mu\nu}, v_{\mu\nu}, \lambda_1^{\mu\nu}, \lambda_2^{\mu\nu})$, these are

$$\pi^{\mu\nu} \equiv \frac{\partial \mathcal{L}'}{\partial \dot{h}_{\mu\nu}} = \lambda_1^{\mu\nu}, \tag{5.1.15}$$

$$p^{\mu\nu} \equiv \frac{\partial \mathcal{L}'}{\partial \dot{G}_{\mu\nu}} = \lambda_2^{\mu\nu}, \tag{5.1.16}$$

$$\hat{v}^{\mu\nu} \equiv \frac{\partial \mathcal{L}'}{\partial \dot{v}_{\mu\nu}} = 0, \tag{5.1.17}$$

$$\Lambda^1_{\mu\nu} \equiv \frac{\partial \mathcal{L}'}{\partial \dot{\lambda}_1^{\mu\nu}} = 0, \tag{5.1.18}$$

$$\Lambda^2_{\mu\nu} \equiv \frac{\partial \mathcal{L}'}{\partial \dot{\lambda}_2^{\mu\nu}} = 0. \tag{5.1.19}$$

We can observe that eqs. (5.1.15)-(5.1.16) allows us to identify the Lagrange multipliers $(\lambda_1^{\mu\nu}, \lambda_2^{\mu\nu})$ as canonical variables through the momenta $(\pi^{\mu\nu}, p^{\mu\nu})$ respectively. Additionally, the primary constraints will be given by (2.7.15), this is,

$$\varphi^{\mu\nu} \equiv p^{\mu\nu} - \frac{\partial \mathcal{L}'}{v_{\mu\nu}} \approx 0, \tag{5.1.20}$$

thus we have

$$\begin{aligned} \varphi^{00} &\equiv p^{00} \approx 0, \\ \varphi^{0i} &\equiv p^{0i} \approx 0, \\ \varphi^{ij} &\equiv p^{ij} + \frac{1}{2\mu}(\epsilon^{ikl}\eta^{jm} + \epsilon^{jkl}\eta^{im})\partial_k h_{lm} \approx 0. \end{aligned} \tag{5.1.21}$$

The fundamental PB of the theory are

$$\{h_{\mu\nu}, \pi^{\alpha\beta}\} = \frac{1}{2}(\delta^\alpha_\mu \delta^\beta_\nu + \delta^\beta_\mu \delta^\alpha_\nu)\delta^3(x-y), \tag{5.1.22}$$

$$\{G_{\mu\nu}, p^{\alpha\beta}\} = \frac{1}{2}(\delta^\alpha_\mu \delta^\beta_\nu + \delta^\beta_\mu \delta^\alpha_\nu)\delta^3(x-y). \tag{5.1.23}$$

With all momenta identified we can build the canonical Hamiltonian as follows

$$\begin{aligned} H_{can} = & \int [\pi^{\mu\nu} G_{\mu\nu} + p^{\mu\nu} v_{\mu\nu}] d^3x - \int [\frac{1}{2} G_{ij} G^{ij} - \partial_j h_{0i} \partial^j h^{0i} - \frac{1}{2}\partial_k h_{ij} \partial^k h^{ij} - \frac{1}{2} G^i{}_i G^j{}_j \\ & - \partial^j h_{00} \partial_j h^i{}_i + \frac{1}{2}\partial_k h^i{}_i \partial^k h^j{}_j + 2\partial^i h_{0i} G^j{}_j + \partial_i h_{00} \partial_j h^{ij} - \partial_i h^{ij} \partial_j h^k{}_k - 2\partial_j h_{0i} G^{ij} \\ & - \partial^i h_{0i} \partial^j h_{0j} + \partial_k h^k{}_i \partial_j h^{ij} + \frac{1}{\mu}\epsilon^{ijk}(-v^l{}_i \partial_j h_{lk} - 2G^l{}_i \partial_j \partial_l h_{0k} + \partial_l h^m{}_i \partial_m \partial_j h^l{}_k \\ & - \nabla^2 h_{0i} \partial_j h_{0k} + \nabla^2 h^m{}_i \partial_j h_{mk})] d^3x, \end{aligned} \tag{5.1.24}$$

it is worth mentioning that the canonical Hamiltonian presents linear terms in the momenta, this fact could be associated to Ostrogradski's instabilites, however, we will see later that this apparently instability can be healed by introducing the Dirac brackets. Now, in order to identify constraints of second generation (secondary constraints) we need to calculate the consistency conditions of the primary constraints. For this purpose we introduce the primary Hamiltonian as in (2.1.6)

$$\mathcal{H}_1 = \mathcal{H}_{can} + \int \Delta_{\mu\nu} \varphi^{\mu\nu} d^3x, \tag{5.1.25}$$

where $\Delta_{\mu\nu}$ are Lagrange multipliers enforcing the primary constraints. From consistency conditions we obtain the following secondary constraints

$$\begin{aligned}
\dot{\varphi}^{00} &= \{\varphi^{00}, \mathcal{H}_1\} \approx 0, \\
&\Rightarrow \Phi^{00} \equiv \pi^{00} \approx 0, && (5.1.26) \\
\dot{\varphi}^{0i} &= \{\varphi^{0i}, \mathcal{H}_1\} \approx 0, \\
&\Rightarrow \Phi^{0i} \equiv \pi^{0i} \approx 0, && (5.1.27) \\
\dot{\varphi}^{ij} &= \{\varphi^{ij}, \mathcal{H}_1\} \approx 0, \\
&\Rightarrow \Phi^{ij} \equiv \pi^{ij} + \frac{1}{\mu}(\epsilon^{ikl}\partial^j + \epsilon^{jkl}\partial^i)\partial_k h_{0l} - \frac{1}{2\mu}(\epsilon^{ikl}\eta^{jm} + \epsilon^{jkl}\eta^{im})\partial_k G_{lm} \\
&- G^{ij} + (G^k{}_k - 2\partial^k h_{0k})\eta^{ij} + (\partial^i h_0{}^j + \partial^j h_0{}^i) \approx 0. && (5.1.28)
\end{aligned}$$

From consistency of the secondary constraints we have

$$\begin{aligned}
\dot{\Phi}^{00} &= \{\Phi^{00}, \mathcal{H}_1\} \approx 0, \\
&\Rightarrow \nabla^2 h^i{}_i - \partial_i\partial_j h^{ij} \approx 0, && (5.1.29)
\end{aligned}$$

$$\begin{aligned}
\dot{\Phi}^{0i} &= \{\Phi^{0i}, \mathcal{H}_1\} \approx 0, \\
&\Rightarrow \frac{1}{\mu}\epsilon^{ijk}(\partial_j\partial^l G_{kl} - \nabla^2\partial_j h_{0k}) - \nabla^2 h_0{}^i - \partial^i G^j{}_j + \partial_j G^{ij} + \partial^i\partial^j h_{0j} \approx 0, && (5.1.30)
\end{aligned}$$

$$\begin{aligned}
\dot{\Phi}^{ij} &= \{\Phi^{ij}, \mathcal{H}_1\} \approx 0, \\
&\Rightarrow \frac{1}{\mu}[(\epsilon^{ikl}\partial^j + \epsilon^{jkl}\partial^i)\partial_k G_{0l} + (\epsilon^{ikl}\eta^{jm} + \epsilon^{jkl}\eta^{im})\nabla^2\partial_k h_{lm} - (\epsilon^{ikl}\partial^j + \epsilon^{jkl}\partial^i)\partial^m\partial_k h_{lm} \\
&- (\epsilon^{ikl}\eta^{jm} + \epsilon^{jkl}\eta^{im})\partial_k v_{lm}] + \frac{1}{2\mu}(\epsilon^{ikm}\eta^{jl} + \epsilon^{jkm}\eta^{il} + \epsilon^{ilm}\eta^{jk} + \epsilon^{jlm}\eta^{ik})\partial_m\Delta_{kl} \\
&+ \nabla^2 h^{ij} - \partial^i\partial^j h_{00} + \partial^i\partial^j h^k{}_k - (\partial^i\partial_k h^{jk} + \partial^j\partial_k h^{ik}) + (\nabla^2 h_{00} - \nabla^2 h^k{}_k + \partial_k\partial_l h^{kl})\eta^{ij} \\
&+ (\partial^i G_0{}^j + \partial^j G_0{}^i) - 2\partial^k G_{0k}\eta^{ij} - v^{ij} + v^k{}_k\eta^{ij} - \left[\frac{1}{2}(\eta^{ik}\eta^{jl} + \eta^{jk}\eta^{il}) - \eta^{ij}\eta^{kl}\right]\Delta_{kl} \approx 0, \\
& && (5.1.31)
\end{aligned}$$

then, we identify the following tertiary constraints

$$\begin{aligned}
\Psi^{00} &\equiv \nabla^2 h^i{}_i - \partial_i\partial_j h^{ij}, && (5.1.32) \\
\Psi^{0i} &\equiv \frac{1}{\mu}\epsilon^{ijk}(\partial_j\partial^l G_{kl} - \nabla^2\partial_j h_{0k}) - \nabla^2 h_0{}^i + \partial^i\partial^j h_{0j} - \partial^i G^j{}_j + \partial_j G^{ij} && (5.1.33)
\end{aligned}$$

and some relations for the multipliers Δ_{ij}

$$\frac{1}{\mu}[(\epsilon^{ikl}\partial^j+\epsilon^{jkl}\partial^i)\partial_k G_{0l}+(\epsilon^{ikl}\eta^{jm}+\epsilon^{jkl}\eta^{im})\nabla^2\partial_k h_{lm}-(\epsilon^{ikl}\partial^j+\epsilon^{jkl}\partial^i)\partial^m\partial_k h_{lm}$$
$$-(\epsilon^{ikl}\eta^{jm}+\epsilon^{jkl}\eta^{im})\partial_k v_{lm}]+\frac{1}{2\mu}(\epsilon^{ikm}\eta^{jl}+\epsilon^{jkm}\eta^{il}+\epsilon^{ilm}\eta^{jk}+\epsilon^{jlm}\eta^{ik})\partial_m\Delta_{kl}$$
$$+\nabla^2 h^{ij}-\partial^i\partial^j h_{00}+\partial^i\partial^j h^k{}_k-(\partial^i\partial_k h^{jk}+\partial^j\partial_k h^{ik})+(\nabla^2 h_{00}-\nabla^2 h^k{}_k+\partial_k\partial_l h^{kl})\eta^{ij}$$
$$+(\partial^i G_0{}^j+\partial^j G_0{}^i)-2\partial^k G_{0k}\eta^{ij}-v^{ij}+v^k{}_k\eta^{ij}-\left[\frac{1}{2}(\eta^{ik}\eta^{jl}+\eta^{jk}\eta^{il})-\eta^{ij}\eta^{kl}\right]\Delta_{kl}=0. \tag{5.1.34}$$

If we go further and calculate consistency of the tertiary constraints we find that

$$\dot{\Psi}^{00} = \{\Psi^{00},\mathcal{H}_1\}\approx 0,$$
$$\Rightarrow \nabla^2 G^i{}_i-\partial^i\partial^j G_{ij}=\partial_i\Psi^{0i}=0, \tag{5.1.35}$$
$$\dot{\Psi}^{0i} = \{\Psi^{0i},\mathcal{H}_1\}\approx 0,$$
$$\Rightarrow \frac{1}{\mu}\epsilon^{ijk}(\partial_j\partial^l v_{kl}-\nabla^2\partial_j G_{0k})+\frac{1}{\mu}\epsilon^{ijk}\partial_j\partial^l\Delta_{kl}-\nabla^2 G_0{}^i+\partial^i\partial^j G_{0j}$$
$$-\partial^i v^j{}_j+\partial_j v^{ij}+[\eta^{kl}\partial^i-\frac{1}{2}(\eta^{ik}\partial^l+\eta^{il}\partial^k)]\Delta_{kl}\approx 0, \tag{5.1.36}$$

therefore no new constraints arise from consistency conditions of tertiary constraints because (5.1.35) is a consequence of (5.1.33) and (5.1.36) are relations between the multipliers Δ_{ij} that also can be obtained from ∂_i(5.1.34). The process of identifying further constraints is finished. The full set of constraints is

$$\varphi^{00} \equiv p^{00}, \tag{5.1.37}$$
$$\varphi^{0i} \equiv p^{0i}, \tag{5.1.38}$$
$$\varphi^{ij} \equiv p^{ij}+\frac{1}{2\mu}(\epsilon^{ikl}\eta^{jm}+\epsilon^{jkl}\eta^{im})\partial_k h_{lm}, \tag{5.1.39}$$
$$\Phi^{00} \equiv \pi^{00}, \tag{5.1.40}$$
$$\Phi^{0i} \equiv \pi^{0i}, \tag{5.1.41}$$
$$\Phi^{ij} \equiv \pi^{ij}+\frac{1}{\mu}(\epsilon^{ikl}\partial^j+\epsilon^{jkl}\partial^i)\partial_k h_{0l}-\frac{1}{2\mu}(\epsilon^{ikl}\eta^{jm}+\epsilon^{jkl}\eta^{im})\partial_k G_{lm}$$
$$-G^{ij}+(G^k{}_k-2\partial^k h_{0k})\eta^{ij}+(\partial^i h_0{}^j+\partial^j h_0{}^i), \tag{5.1.42}$$
$$\Psi^{00} \equiv \nabla^2 h^i{}_i-\partial_i\partial_j h^{ij}, \tag{5.1.43}$$
$$\Psi^{0i} \equiv \frac{1}{\mu}\epsilon^{ijk}(\partial_j\partial^l G_{kl}-\nabla^2\partial_j h_{0k})-\nabla^2 h_0{}^i+\partial^i\partial^j h_{0j}-\partial^i G^j{}_j+\partial_j G^{ij}. \tag{5.1.44}$$

Now that we have all the constraints of the theory we can classify them into first and second class. For this aim we construct the 24×24 matrix W^{IJ} whose entries are the PB between all constraints, this is

$$
W^{IJ} = \begin{array}{c} \\ \varphi^{00} \\ \varphi^{0i} \\ \varphi^{ij} \\ \Phi^{00} \\ \Phi^{0i} \\ \Phi^{ij} \\ \Psi^{00} \\ \Psi^{0i} \end{array}
\begin{array}{c}
\begin{array}{cccccccc} \varphi^{00} & \varphi^{0k} & \varphi^{kl} & \Phi^{00} & \Phi^{0k} & \Phi^{kl} & \Psi^{00} & \Psi^{0k} \end{array} \\
\left(\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \{\varphi^{ij}, \Phi^{kl}\} & 0 & \{\varphi^{ij}, \Psi^{0k}\} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \{\Phi^{0i}, \Phi^{kl}\} & 0 & \{\Phi^{0i}, \Psi^{0k}\} \\
0 & 0 & \{\Phi^{ij}, \varphi^{kl}\} & 0 & \{\Phi^{ij}, \Phi^{0k}\} & 0 & \{\Phi^{ij}, \Psi^{00}\} & 0 \\
0 & 0 & 0 & 0 & 0 & \{\Psi^{00}, \Phi^{kl}\} & 0 & 0 \\
0 & 0 & \{\Psi^{0i}, \varphi^{kl}\} & 0 & \{\Psi^{0i}, \Phi^{0k}\} & 0 & 0 & 0
\end{array}\right)
\end{array} \tag{5.1.45}
$$

The non-zero PB in W^{IJ} are given by

$$
\begin{aligned}
\{\varphi^{ij}, \Phi^{kl}\} &= \left[\frac{1}{2\mu}(\epsilon^{ikm}\eta^{jl} + \epsilon^{jkm}\eta^{il} + \epsilon^{ilm}\eta^{jk} + \epsilon^{jlm}\eta^{ik})\partial_m + \frac{1}{2}(\eta^{ik}\eta^{jl} + \eta^{jk}\eta^{il}) - \eta^{ij}\eta^{kl}\right]\delta^3(x-y), \\
\{\Phi^{0i}, \Phi^{kl}\} &= \left[-\frac{1}{2\mu}(\epsilon^{ikm}\partial^l + \epsilon^{ilm}\partial^k)\partial_m + \eta^{kl}\partial^i - \frac{1}{2}(\eta^{ik}\partial^l + \eta^{il}\partial^k)\right]\delta^3(x-y), \\
\{\varphi^{ij}, \Psi^{0k}\} &= \left[-\frac{1}{2\mu}(\epsilon^{ikl}\partial^j + \epsilon^{jkl}\partial^i)\partial_l + \eta^{ij}\partial^k - \frac{1}{2}(\eta^{ik}\partial^j + \eta^{jk}\partial^i)\right]\delta^3(x-y), \\
\{\Phi^{0i}, \Psi^{0k}\} &= \left[\frac{1}{2\mu}\epsilon^{ikl}\partial_l\nabla^2 + \frac{1}{2}\eta^{ik}\nabla^2 - \frac{1}{2}\partial^i\partial^k\right]\delta^3(x-y), \\
\{\Phi^{ij}, \Psi^{00}\} &= -\eta^{ij}\nabla^2\delta^3(x-y) + \partial^i\partial^j\delta^3(x-y).
\end{aligned} \tag{5.1.46}
$$

Hence, after some algebraic work we find that the matrix W^{IJ} has 12 null vectors, this implies that we can identify 12 independent first-class constraints, such constraints are given by:

$$\Gamma_1 \equiv p^{00} \approx 0, \tag{5.1.47}$$

$$\Gamma_2 \equiv \pi^{00} \approx 0, \tag{5.1.48}$$

$$\Gamma_3^i \equiv \pi^{0i} - \partial_j p^{ij} - \frac{1}{2\mu}\epsilon^{ijk}\partial_j\partial^l h_{kl} \approx 0, \tag{5.1.49}$$

$$\Gamma_4^i \equiv \partial_j\pi^{ij} + \frac{1}{2\mu}\epsilon^{ijk}\partial_j\partial^l G_{kl} \approx 0, \tag{5.1.50}$$

$$\Gamma_5^i \equiv p^{0i} \approx 0, \tag{5.1.51}$$

$$\Gamma_6 \equiv \nabla^2 h^i{}_i - \partial_i\partial_j h^{ij} + \partial_i\partial_j p^{ij} \approx 0, \tag{5.1.52}$$

on the other hand, since the matrix W^{IJ} has rank 12 therefore we can identify 12 second-class constraints

$$\chi_1^{ij} \equiv p^{ij} + \frac{1}{2\mu}(\epsilon^{ikl}\eta^{jm} + \epsilon^{jkl}\eta^{im})\partial_k h_{lm} \approx 0, \tag{5.1.53}$$

$$\begin{aligned}\chi_2^{ij} \equiv\ & \pi^{ij} + \frac{1}{\mu}(\epsilon^{ikl}\partial^j + \epsilon^{jkl}\partial^i)\partial_k h_{0l} - \frac{1}{2\mu}(\epsilon^{ikl}\eta^{jm} + \epsilon^{jkl}\eta^{im})\partial_k G_{lm} \\ & - G^{ij} + (G^k{}_k - 2\partial^k h_{0k})\eta^{ij} + (\partial^i h_0{}^j + \partial^j h_0{}^i) \approx 0.\end{aligned} \tag{5.1.54}$$

The PB between all constraints are:

$$\begin{aligned}\{\Gamma^a, \Gamma^b\} &= 0, \\ \{\chi_1^{ij}, \chi_1^{kl}\} &= 0, \\ \{\chi_1^{ij}, \chi_2^{kl}\} &= \left[\frac{1}{2\mu}(\epsilon^{ikm}\eta^{jl} + \epsilon^{jkm}\eta^{il} + \epsilon^{ilm}\eta^{jk} + \epsilon^{jlm}\eta^{ik})\partial_m + \frac{1}{2}(\eta^{ik}\eta^{jl} + \eta^{jk}\eta^{il}) - \eta^{ij}\eta^{kl}\right]\delta^3(x-y), \\ \{\chi_2^{ij}, \chi_2^{kl}\} &= 0.\end{aligned} \tag{5.1.55}$$

With all constraints classified into first and second class we can perform the counting of physical degrees of freedom as follows: there are 40 canonical variables $(h_{\mu\nu}, G_{\mu\nu})$, the number of first-class constraints Γ^a is 12 and there are 12 second-class constraints χ^a, therefore, the number of physical degrees of freedom is $DoF = \frac{1}{2}[(40) - 2(12) - 12] = 2$, just like the EH action. These 2 degrees of freedom are associated to two linearly independent polarizations of gravitational waves [24]. We can remove all the second-class constraints by introducing the Dirac brackets as in (2.4.5)

$$\{F(x), G(y)\}_D = \{F(x), G(y)\} - \int \{F(x), \chi^a(u)\}(C_{ab})^{-1}\{\chi^b(v), G(y)\} du dv, \tag{5.1.56}$$

where $(C_{ab})^{-1}$ is the inverse of the 12×12 matrix C_{ab} whose entries are the PB between the second-class constraints, this is

$$C_{ab} = \begin{array}{c c} & \begin{array}{cc} \chi_1^{kl} & \chi_2^{kl} \end{array} \\ \begin{array}{c} \chi_1^{ij} \\ \\ \chi_2^{ij} \end{array} & \begin{pmatrix} 0 & \{\chi_1^{ij}, \chi_2^{kl}\} \\ & \\ \{\chi_2^{ij}, \chi_1^{kl}\} & 0 \end{pmatrix} \end{array}, \tag{5.1.57}$$

where $\{\chi_1^{ij},\chi_2^{kl}\}$ is given in (5.1.55). The explicit form of C_{ab} is

$$
\frac{1}{\mu}
\begin{array}{c}
\\ \chi_1^{11}\\ \chi_1^{12}\\ \chi_1^{13}\\ \chi_1^{22}\\ \chi_1^{23}\\ \chi_1^{33}\\ \chi_2^{11}\\ \chi_2^{12}\\ \chi_2^{13}\\ \chi_2^{22}\\ \chi_2^{23}\\ \chi_2^{33}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccc}
\chi_1^{11} & \chi_1^{12} & \chi_1^{13} & \chi_1^{22} & \chi_1^{23} & \chi_1^{33} & \chi_2^{11} & \chi_2^{12} & \chi_2^{13} & \chi_2^{22} & \chi_2^{23} & \chi_2^{33}
\end{array}\\
\left(\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_3 & -\partial_2 & -\mu & 0 & -\mu\\
0 & 0 & 0 & 0 & 0 & 0 & -\partial_3 & \frac{\mu}{2} & \frac{1}{2}\partial_1 & \partial_3 & -\frac{1}{2}\partial_2 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \partial_2 & -\frac{1}{2}\partial_1 & \frac{\mu}{2} & 0 & \frac{1}{2}\partial_3 & -\partial_2\\
0 & 0 & 0 & 0 & 0 & 0 & -\mu & -\partial_3 & 0 & 0 & \partial_1 & -\mu\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\partial_2 & -\frac{1}{2}\partial_3 & -\partial_1 & \frac{\mu}{2} & \partial_1\\
0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & \partial_2 & -\mu & -\partial_1 & 0\\
0 & -\partial_3 & \partial_2 & \mu & 0 & \mu & 0 & 0 & 0 & 0 & 0 & 0\\
\partial_3 & -\frac{\mu}{2} & -\frac{1}{2}\partial_1 & -\partial_3 & \frac{1}{2}\partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-\partial_2 & \frac{1}{2}\partial_1 & -\frac{\mu}{2} & 0 & -\frac{1}{2}\partial_3 & \partial_2 & 0 & 0 & 0 & 0 & 0 & 0\\
\mu & \partial_3 & 0 & 0 & -\partial_1 & \mu & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -\frac{1}{2}\partial_2 & \frac{1}{2}\partial_3 & \partial_1 & -\frac{\mu}{2} & -\partial_1 & 0 & 0 & 0 & 0 & 0 & 0\\
\mu & 0 & -\partial_2 & \mu & \partial_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\end{array}
\delta^3(x-y). \tag{5.1.58}
$$

The matrix C_{ab} is regular, if we write it as

$$
C^{ab} = \frac{1}{\mu}\begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}\delta^3(x-y) \tag{5.1.59}
$$

then, its inverse has the form

$$
C^{-1}_{ab} = \frac{\mu}{\Xi}\begin{pmatrix} 0 & -A^{-1} \\ A^{-1} & 0 \end{pmatrix}\delta^3(x-y), \tag{5.1.60}
$$

where the submatrix A^{-1} is built of an arrangement of six 1×6 columns;

$$
A^{-1}_{\mathfrak{a}1} = \begin{pmatrix}
-\frac{\mu}{2}\partial_1^4 + \mu\partial_1^2\partial_2^2 + \mu\partial_1^2\partial_3^2 - \frac{5\mu^3}{8}\partial_1^2 - \frac{\mu^3}{8}\partial_2^2 - \frac{\mu^3}{8}\partial_3^2 - \frac{\mu^5}{8}\\
-2\mu\partial_1^3\partial_2 + \mu\partial_1\partial_2^3 + \mu\partial_1\partial_2\partial_3^2 - 2\mu^2\partial_1^2\partial_3 - \frac{\mu^2}{2}\partial_2^2\partial_3 - \frac{\mu^2}{2}\partial_3^3 - \frac{\mu^3}{2}\partial_1\partial_2 - \frac{\mu^4}{2}\partial_3\\
-2\mu\partial_1^3\partial_3 + \mu\partial_1\partial_2^2\partial_3 + \mu\partial_1\partial_3^3 + 2\mu^2\partial_1^2\partial_2 + \frac{\mu^2}{2}\partial_2^3 + \frac{\mu^2}{2}\partial_2\partial_3^2 - \frac{\mu^3}{2}\partial_1\partial_3 + \frac{\mu^4}{2}\partial_2\\
\frac{\mu}{2}\partial_1^4 - \frac{\mu}{2}\partial_1^2\partial_2^2 + \frac{\mu}{2}\partial_2^4 + \frac{\mu}{2}\partial_1^2\partial_3^2 + \frac{\mu}{2}\partial_2^2\partial_3^2 - \frac{3\mu^2}{2}\partial_1\partial_2\partial_3 + \frac{5\mu^3}{8}\partial_1^2 + \frac{5\mu^3}{8}\partial_2^2 + \frac{\mu^3}{8}\partial_3^2 + \frac{\mu^5}{8}\\
-2\mu\partial_1^2\partial_2\partial_3 + \mu\partial_2^3\partial_3 + \mu\partial_2\partial_3^3 + \frac{3\mu^2}{2}\partial_1\partial_2^2 - \frac{3\mu^2}{2}\partial_1\partial_3^2 + \mu^3\partial_2\partial_3\\
\frac{\mu}{2}\partial_1^4 + \frac{\mu}{2}\partial_1^2\partial_2^2 - \frac{\mu}{2}\partial_1^2\partial_3^2 + \frac{\mu}{2}\partial_2^2\partial_3^2 + \frac{\mu}{2}\partial_3^4 + \frac{3\mu^2}{2}\partial_1\partial_2\partial_3 + \frac{5\mu^3}{8}\partial_1^2 + \frac{\mu^3}{8}\partial_2^2 + \frac{5\mu^3}{8}\partial_3^2 + \frac{\mu^5}{8}
\end{pmatrix},
$$

$$A_{\mathfrak{a}2}^{-1} = \begin{pmatrix} -2\mu\partial_1^3\partial_2 + \mu\partial_1\partial_2^3 + \mu\partial_1\partial_2\partial_3^2 + 2\mu^2\partial_1^2\partial_3 + \frac{\mu^2}{2}\partial_2^2\partial_3 + \frac{\mu^2}{2}\partial_3^3 - \frac{\mu^3}{2}\partial_1\partial_2 + \frac{\mu^4}{2}\partial_3 \\ -6\mu\partial_1^2\partial_2^2 - 2\mu^3\partial_1^2 - 2\mu^3\partial_2^2 - \frac{\mu^3}{2}\partial_3^2 - \frac{\mu^5}{2} \\ -6\mu\partial_1^2\partial_2\partial_3 - 2\mu^2\partial_1^3 + \mu^2\partial_1\partial_2^2 + \mu^2\partial_1\partial_3^2 - \frac{3\mu^3}{2}\partial_2\partial_3 - \frac{\mu^4}{2}\partial_1 \\ \mu\partial_1^3\partial_2 - 2\mu\partial_1\partial_2^3 + \mu\partial_1\partial_2\partial_3^2 - \frac{\mu^2}{2}\partial_1^2\partial_3 - 2\mu^2\partial_2^2\partial_3 - \frac{\mu^2}{2}\partial_3^3 - \frac{\mu^3}{2}\partial_1\partial_2 - \frac{\mu^4}{2}\partial_3 \\ -6\mu\partial_1\partial_2^2\partial_3 - \mu^2\partial_1^2\partial_2 + 2\mu^2\partial_2^3 - \mu^2\partial_2\partial_3^2 - \frac{3\mu^3}{2}\partial_1\partial_3 + \frac{\mu^4}{2}\partial_2 \\ \mu\partial_1^3\partial_2 + \mu\partial_1\partial_2^3 - 2\mu\partial_1\partial_2\partial_3^2 - \frac{3\mu^2}{2}\partial_1^2\partial_3 + \frac{3\mu^2}{2}\partial_2^2\partial_3 + \mu^3\partial_1\partial_2 \end{pmatrix},$$

$$A_{\mathfrak{a}3}^{-1} = \begin{pmatrix} -2\mu\partial_1^3\partial_3 + \mu\partial_1\partial_2^2\partial_3 + \mu\partial_1\partial_3^3 - 2\mu^2\partial_1^2\partial_2 - \frac{\mu^2}{2}\partial_2^3 - \frac{\mu^2}{2}\partial_2\partial_3^2 - \frac{\mu^3}{2}\partial_1\partial_3 - \frac{\mu^4}{2}\partial_2 \\ -6\mu\partial_1^2\partial_2\partial_3 + 2\mu^2\partial_1^3 - \mu^2\partial_1\partial_2^2 - \mu^2\partial_1\partial_3^2 - \frac{3\mu^3}{2}\partial_2\partial_3 + \frac{\mu^4}{2}\partial_1 \\ -6\mu\partial_1^2\partial_3^2 - 2\mu^3\partial_1^2 - \frac{\mu^3}{2}\partial_2^2 - 2\mu^3\partial_3^2 - \frac{\mu^5}{2} \\ \mu\partial_1^3\partial_3 - 2\mu\partial_1\partial_2^2\partial_3 + \mu\partial_1\partial_3^3 + \frac{3\mu^2}{2}\partial_1^2\partial_2 - \frac{3\mu^2}{2}\partial_2\partial_3^2 + \mu^3\partial_1\partial_3 \\ -6\mu\partial_1\partial_2\partial_3^2 + \mu^2\partial_1^2\partial_3 + \mu^2\partial_2^2\partial_3 - 2\mu^2\partial_3^3 - \frac{3\mu^3}{2}\partial_1\partial_2 - \frac{\mu^4}{2}\partial_3 \\ \mu\partial_1^3\partial_3 + \mu\partial_1\partial_2^2\partial_3 - 2\mu\partial_1\partial_3^3 + \frac{\mu^2}{2}\partial_1^2\partial_2 + \frac{\mu^2}{2}\partial_2^3 + 2\mu^2\partial_2\partial_3^2 - \frac{\mu^3}{2}\partial_1\partial_3 + \frac{\mu^4}{2}\partial_2 \end{pmatrix},$$

$$A_{\mathfrak{a}4}^{-1} = \begin{pmatrix} \frac{\mu}{2}\partial_1^4 - \frac{\mu}{2}\partial_1^2\partial_2^2 + \frac{\mu}{2}\partial_2^4 + \frac{\mu}{2}\partial_1^2\partial_3^2 + \frac{\mu}{2}\partial_2^2\partial_3^2 + \frac{3\mu^2}{2}\partial_1\partial_2\partial_3 + \frac{5\mu^3}{8}\partial_1^2 + \frac{5\mu^3}{8}\partial_2^2 + \frac{\mu^3}{8}\partial_3^2 + \frac{\mu^5}{8} \\ \mu\partial_1^3\partial_2 - 2\mu\partial_1\partial_2^3 + \mu\partial_1\partial_2\partial_3^2 + \frac{\mu^2}{2}\partial_1^2\partial_3 + 2\mu^2\partial_2^2\partial_3 + \frac{\mu^2}{2}\partial_3^3 - \frac{\mu^3}{2}\partial_1\partial_2 + \frac{\mu^4}{2}\partial_3 \\ \mu\partial_1^3\partial_3 - 2\mu\partial_1\partial_2^2\partial_3 + \mu\partial_1\partial_3^3 - \frac{3\mu^2}{2}\partial_1^2\partial_2 + \frac{3\mu^2}{2}\partial_2\partial_3^2 + \mu^3\partial_1\partial_3 \\ \mu\partial_1^2\partial_2^2 - \frac{\mu}{2}\partial_2^4 + \mu\partial_2^2\partial_3^2 - \frac{\mu^3}{8}\partial_1^2 - \frac{5\mu^3}{8}\partial_2^2 - \frac{\mu^3}{8}\partial_3^2 - \frac{\mu^5}{8} \\ \mu\partial_1^2\partial_2\partial_3 - 2\mu\partial_2^3\partial_3 + \mu\partial_2\partial_3^3 - \frac{\mu^2}{2}\partial_1^3 - 2\mu^2\partial_1\partial_2^2 - \frac{\mu^2}{2}\partial_1\partial_3^2 - \frac{\mu^3}{2}\partial_2\partial_3 - \frac{\mu^4}{2}\partial_1 \\ \frac{\mu}{2}\partial_1^2\partial_2^2 + \frac{\mu}{2}\partial_2^4 + \frac{\mu}{2}\partial_1^2\partial_3^2 - \frac{\mu}{2}\partial_2^2\partial_3^2 + \frac{\mu}{2}\partial_3^4 - \frac{3\mu^2}{2}\partial_1\partial_2\partial_3 + \frac{\mu^3}{8}\partial_1^2 + \frac{5\mu^3}{8}\partial_2^2 + \frac{5\mu^3}{8}\partial_3^2 + \frac{\mu^5}{8} \end{pmatrix},$$

$$A_{\mathfrak{a}5}^{-1} = \begin{pmatrix} -2\mu\partial_1^2\partial_2\partial_3 + \mu\partial_2^3\partial_3 + \mu\partial_2\partial_3^3 - \frac{3\mu^2}{2}\partial_1\partial_2^2 + \frac{3\mu^2}{2}\partial_1\partial_3^2 + \mu^3\partial_2\partial_3 \\ -6\mu\partial_1\partial_2^2\partial_3 + \mu^2\partial_1^2\partial_2 - 2\mu^2\partial_2^3 + \mu^2\partial_2\partial_3^2 - \frac{3\mu^3}{2}\partial_1\partial_3 - \frac{\mu^4}{2}\partial_2 \\ -6\mu\partial_1\partial_2\partial_3^2 - \mu^2\partial_1^2\partial_3 - \mu^2\partial_2^2\partial_3 + 2\mu^2\partial_3^3 - \frac{3\mu^3}{2}\partial_1\partial_2 + \frac{\mu^4}{2}\partial_3 \\ \mu\partial_1^2\partial_2\partial_3 - 2\mu\partial_2^3\partial_3 + \mu\partial_2\partial_3^3 + \frac{\mu^2}{2}\partial_1^3 + 2\mu^2\partial_1\partial_2^2 + \frac{\mu^2}{2}\partial_1\partial_3^2 - \frac{\mu^3}{2}\partial_2\partial_3 + \frac{\mu^4}{2}\partial_1 \\ -6\mu\partial_2^2\partial_3^2 - \frac{\mu^3}{2}\partial_1^2 - 2\mu^3\partial_2^2 - 2\mu^3\partial_3^2 - \frac{\mu^5}{2} \\ \mu\partial_1^2\partial_2\partial_3 + \mu\partial_2^3\partial_3 - 2\mu\partial_2\partial_3^3 - \frac{\mu^2}{2}\partial_1^3 - \frac{\mu^2}{2}\partial_1\partial_2^2 - 2\mu^2\partial_1\partial_3^2 - \frac{\mu^3}{2}\partial_2\partial_3 - \frac{\mu^4}{2}\partial_1 \end{pmatrix},$$

$$A^{-1}_{\mathfrak{a}6} = \begin{pmatrix} \frac{\mu}{2}\partial_1^4 + \frac{\mu}{2}\partial_1^2\partial_2^2 - \frac{\mu}{2}\partial_1^2\partial_3^2 + \frac{\mu}{2}\partial_2^2\partial_3^2 + \frac{\mu}{2}\partial_3^4 - \frac{3\mu^2}{2}\partial_1\partial_2\partial_3 + \frac{5\mu^3}{8}\partial_1^2 + \frac{\mu^3}{8}\partial_2^2 + \frac{5\mu^3}{8}\partial_3^2 + \frac{\mu^5}{8} \\ \mu\partial_1^3\partial_2 + \mu\partial_1\partial_2^3 - 2\mu\partial_1\partial_2\partial_3^2 + \frac{3\mu^2}{2}\partial_1^2\partial_3 - \frac{3\mu^2}{2}\partial_2^2\partial_3 + \mu^3\partial_1\partial_2 \\ \mu\partial_1^3\partial_3 + \mu\partial_1\partial_2^2\partial_3 - 2\mu\partial_1\partial_3^3 - \frac{\mu^2}{2}\partial_1^2\partial_2 - \frac{\mu^2}{2}\partial_2^3 - 2\mu^2\partial_2\partial_3^2 - \frac{\mu^3}{2}\partial_1\partial_3 - \frac{\mu^4}{2}\partial_2 \\ \frac{\mu}{2}\partial_1^2\partial_2^2 + \frac{\mu}{2}\partial_2^4 + \frac{\mu}{2}\partial_1^2\partial_3^2 - \frac{\mu}{2}\partial_2^2\partial_3^2 + \frac{\mu}{2}\partial_3^4 + \frac{3\mu^2}{2}\partial_1\partial_2\partial_3 + \frac{\mu^3}{8}\partial_1^2 + \frac{5\mu^3}{8}\partial_2^2 + \frac{5\mu^3}{8}\partial_3^2 + \frac{\mu^5}{8} \\ \mu\partial_1^2\partial_2\partial_3 + \mu\partial_2^3\partial_3 - 2\mu\partial_2\partial_3^3 + \frac{\mu^2}{2}\partial_1^3 + \frac{\mu^2}{2}\partial_1\partial_2^2 + 2\mu^2\partial_1\partial_3^2 - \frac{\mu^3}{2}\partial_2\partial_3 + \frac{\mu^4}{2}\partial_1 \\ \mu\partial_1^2\partial_3^2 + \mu\partial_2^2\partial_3^2 - \frac{\mu}{2}\partial_3^4 - \frac{\mu^3}{8}\partial_1^2 - \frac{\mu^3}{8}\partial_2^2 - \frac{5\mu^3}{8}\partial_3^2 - \frac{\mu^5}{8} \end{pmatrix},$$

where $\Xi \equiv -\mu^2(\nabla^2+\mu^2)(\nabla^2+\frac{\mu^2}{4})$ and $\mathfrak{a} = 1,...,6$. It is not immediate, albeit straightforward to show that $(C\times C^{-1})_{ab} = \mathbb{I}$. Let's take for example, the product $A^{4\mathfrak{a}}A^{-1}_{\mathfrak{a}4}$ which correspond to the entries $(C\times C^{-1})_{44}$ and $(C\times C^{-1})_{1010}$

$$\begin{aligned} A^{4\mathfrak{a}}A^{-1}_{\mathfrak{a}4} &= -\mu[\frac{\mu}{2}\partial_1^4 - \frac{\mu}{2}\partial_1^2\partial_2^2 + \frac{\mu}{2}\partial_2^4 + \frac{\mu}{2}\partial_1^2\partial_3^2 + \frac{\mu}{2}\partial_2^2\partial_3^2 + \frac{3\mu^2}{2}\partial_1\partial_2\partial_3 + \frac{5\mu^3}{8}\partial_1^2 + \frac{5\mu^3}{8}\partial_2^2 \\ &+\frac{\mu^3}{8}\partial_3^2 + \frac{\mu^5}{8}] - \partial_3[\mu\partial_1^3\partial_2 - 2\mu\partial_1\partial_2^3 + \mu\partial_1\partial_2\partial_3^2 + \frac{\mu^2}{2}\partial_1^2\partial_3 + 2\mu^2\partial_2^2\partial_3 + \frac{\mu^2}{2}\partial_3^3 \\ &-\frac{\mu^3}{2}\partial_1\partial_2 + \frac{\mu^4}{2}\partial_3] + \partial_1[\mu\partial_1^2\partial_2\partial_3 - 2\mu\partial_2^3\partial_3 + \mu\partial_2\partial_3^3 - \frac{\mu^2}{2}\partial_1^3 - 2\mu^2\partial_1\partial_2^2 \\ &-\frac{\mu^2}{2}\partial_1\partial_3^2 - \frac{\mu^3}{2}\partial_2\partial_3 - \frac{\mu^4}{2}\partial_1] - \mu[\frac{\mu}{2}\partial_1^2\partial_2^2 + \frac{\mu}{2}\partial_2^4 + \frac{\mu}{2}\partial_1^2\partial_3^2 - \frac{\mu}{2}\partial_2^2\partial_3^2 + \frac{\mu}{2}\partial_3^4 \\ &-\frac{3\mu^2}{2}\partial_1\partial_2\partial_3 + \frac{\mu^3}{8}\partial_1^2 + \frac{5\mu^3}{8}\partial_2^2 + \frac{5\mu^3}{8}\partial_3^2 + \frac{\mu^5}{8}] \\ &= -\frac{5\mu^4}{4}\nabla^2 - \mu^2(\partial_1^4+\partial_2^4+\partial_3^4) - 2\mu^2(\partial_1^2\partial_2^2 + \partial_1^2\partial_3^2 + \partial_2^2\partial_3^3) - \frac{\mu^6}{4} \\ &= \Xi, \end{aligned}$$

thus we have that

$$(C\times C^{-1})_{44} = (\frac{1}{\mu}A^{4\mathfrak{a}}) \times (\frac{\mu}{\Xi}A^{-1}_{\mathfrak{a}4}) = \frac{1}{\mu}\frac{\mu}{\Xi}\Xi\delta^3(x-y) = \delta^3(x-y). \tag{5.1.61}$$

With the inverse $(C_{ab})^{-1}$ calculated we can compute (5.1.56) for the canonical variables, for the pair $\{h_{11}, \pi^{11}\}_D$ for example, we have

$$
\begin{aligned}
\{h_{11}, \pi^{11}\}_D \;&=\; \{h_{11}, \pi^{11}\} - \int (\{h_{11}, \chi_2^{11}\} C_{72}^{-1} \{\chi_1^{12}, \pi^{11}\} + \{h_{11}, \chi_2^{11}\} C_{73}^{-1} \{\chi_1^{13}, \pi^{11}\}) du dv \\
&=\; \delta^3(x-y) - \frac{1}{2\mu} \int [\delta^3(x-u) C_{72}^{-1} \partial_3^v \delta^3(v-y) - \delta^3(x-u) C_{73}^{-1} \partial_2^v \delta^3(v-y)] du dv \\
&=\; \delta^3(x-y) - \frac{1}{2\Xi} \int [\delta^3(x-u)(-2\mu\partial_1^3\partial_2 + \mu\partial_1\partial_2^3 + \mu\partial_1\partial_2\partial_3^2 + 2\mu^2\partial_1^2\partial_3 + \frac{\mu^2}{2}\partial_2^2\partial_3 \\
&\quad + \frac{\mu^2}{2}\partial_3^3 - \frac{\mu^3}{2}\partial_1\partial_2 + \frac{\mu^4}{2}\partial_3)^v \delta^3(u-v) \partial_3^v \delta^3(v-y) - \delta^3(x-u)(-2\mu\partial_1^3\partial_3 + \mu\partial_1\partial_2^2\partial_3 + \mu\partial_1\partial_3^3 \\
&\quad - 2\mu^2\partial_1^2\partial_2 - \frac{\mu^2}{2}\partial_2^3 - \frac{\mu^2}{2}\partial_2\partial_3^2 - \frac{\mu^3}{2}\partial_1\partial_3 - \frac{\mu^4}{2}\partial_2)^v \delta^3(u-v) \partial_2^v \delta^3(v-y)] du dv \\
&=\; \delta^3(x-y) - \frac{1}{2\Xi} \int [(-2\mu\partial_1^3\partial_2 + \mu\partial_1\partial_2^3 + \mu\partial_1\partial_2\partial_3^2 + 2\mu^2\partial_1^2\partial_3 + \frac{\mu^2}{2}\partial_2^2\partial_3 \\
&\quad + \frac{\mu^2}{2}\partial_3^3 - \frac{\mu^3}{2}\partial_1\partial_2 + \frac{\mu^4}{2}\partial_3)^v \delta^3(x-v) \partial_3^v \delta^3(v-y) - (-2\mu\partial_1^3\partial_3 + \mu\partial_1\partial_2^2\partial_3 + \mu\partial_1\partial_3^3 \\
&\quad - 2\mu^2\partial_1^2\partial_2 - \frac{\mu^2}{2}\partial_2^3 - \frac{\mu^2}{2}\partial_2\partial_3^2 - \frac{\mu^3}{2}\partial_1\partial_3 - \frac{\mu^4}{2}\partial_2)^v \delta^3(x-v) \partial_2^v \delta^3(v-y)] dv \\
&=\; \delta^3(x-y) + \frac{1}{2\Xi} \int [(-2\mu\partial_1^3\partial_2 + \mu\partial_1\partial_2^3 + \mu\partial_1\partial_2\partial_3^2 + 2\mu^2\partial_1^2\partial_3 + \frac{\mu^2}{2}\partial_2^2\partial_3 \\
&\quad + \frac{\mu^2}{2}\partial_3^3 - \frac{\mu^3}{2}\partial_1\partial_2 + \frac{\mu^4}{2}\partial_3)^v \partial_3^v \delta^3(x-v) \delta^3(v-y) - (-2\mu\partial_1^3\partial_3 + \mu\partial_1\partial_2^2\partial_3 + \mu\partial_1\partial_3^3 \\
&\quad - 2\mu^2\partial_1^2\partial_2 - \frac{\mu^2}{2}\partial_2^3 - \frac{\mu^2}{2}\partial_2\partial_3^2 - \frac{\mu^3}{2}\partial_1\partial_3 - \frac{\mu^4}{2}\partial_2)^v \partial_2^v \delta^3(x-v) \delta^3(v-y)] dv \\
&=\; \delta^3(x-y) + \frac{1}{2\Xi} [(-2\mu\partial_1^3\partial_2 + \mu\partial_1\partial_2^3 + \mu\partial_1\partial_2\partial_3^2 + 2\mu^2\partial_1^2\partial_3 + \frac{\mu^2}{2}\partial_2^2\partial_3 \\
&\quad + \frac{\mu^2}{2}\partial_3^3 - \frac{\mu^3}{2}\partial_1\partial_2 + \frac{\mu^4}{2}\partial_3)\partial_3 - (-2\mu\partial_1^3\partial_3 + \mu\partial_1\partial_2^2\partial_3 + \mu\partial_1\partial_3^3 \\
&\quad - 2\mu^2\partial_1^2\partial_2 - \frac{\mu^2}{2}\partial_2^3 - \frac{\mu^2}{2}\partial_2\partial_3^2 - \frac{\mu^3}{2}\partial_1\partial_3 - \frac{\mu^4}{2}\partial_2)\partial_2] \delta^3(x-y) \\
&=\; \delta^3(x-y) + \frac{\mu^2}{2\Xi} [\frac{\mu^2}{2}(\partial_2^2 + \partial_3^2) + 2(\partial_1^2\partial_2^2 + \partial_1^2\partial_3^2 + \frac{1}{2}\partial_2^2\partial_3^2) + \frac{1}{2}(\partial_2^4 + \partial_3^4)] \delta^3(x-y),
\end{aligned}
$$

after doing this calculation for all possible combinations of $\{h_{ij}, \pi^{kl}\}_D$ and by writing the results with appropriate space indices we obtain the following Dirac brackets between the perturbation field and its canonical momenta

$$\begin{aligned}\{h_{ij},\pi^{kl}\}_D &= \frac{1}{2}(\delta_i^k\delta_j^l+\delta_i^l\delta_j^k)\delta^3(x-y)+\frac{\mu^2}{4\Xi}\Big[[(\delta_i^k\delta_j^l+\delta_i^l\delta_j^k-\eta_{ij}\eta^{kl})\nabla^2+(\eta_{ij}\partial^k\partial^l+\eta^{kl}\partial_i\partial_j)](\nabla^2+\mu^2)\\ &-3\partial_i\partial_j\partial^k\partial^l-\frac{3\mu^2}{4}(\delta_i^k\partial_j\partial^l+\delta_i^l\partial_j\partial^k+\delta_j^k\partial_i\partial^l+\delta_j^l\partial_i\partial^k)+\frac{\mu}{4}\Big[(\epsilon_i{}^{km}\delta_j^l+\epsilon_j{}^{km}\delta_i^l\\ &+\epsilon_i{}^{lm}\delta_j^k+\epsilon_j{}^{lm}\delta_i^k)(\nabla^2+\mu^2)+3(\epsilon_i{}^{km}\partial_j\partial^l+\epsilon_j{}^{km}\partial_i\partial^l+\epsilon_i{}^{lm}\partial_j\partial^k+\epsilon_j{}^{lm}\partial_i\partial^k)]\partial_m\Big]\delta^3(x-y).\end{aligned} \tag{5.1.62}$$

The remaining non-trivial brackets are:

$$\begin{aligned}\{G_{ij},p^{kl}\}_D &= -\frac{\mu^2}{4\Xi}\Big[[(\delta_i^k\delta_j^l+\delta_i^l\delta_j^k-\eta_{ij}\eta^{kl})\nabla^2+(\eta_{ij}\partial^k\partial^l+\eta^{kl}\partial_i\partial_j)](\nabla^2+\mu^2)-3\partial_i\partial_j\partial^k\partial^l\\ &-\frac{3\mu^2}{4}(\delta_i^k\partial_j\partial^l+\delta_i^l\partial_j\partial^k+\delta_j^k\partial_i\partial^l+\delta_j^l\partial_i\partial^k)+\frac{\mu}{4}[(\epsilon_i{}^{km}\delta_j^l+\epsilon_j{}^{km}\delta_i^l+\epsilon_i{}^{lm}\delta_j^k+\epsilon_j{}^{lm}\delta_i^k)(\nabla^2+\mu^2)\\ &+3(\epsilon_i{}^{km}\partial_j\partial^l+\epsilon_j{}^{km}\partial_i\partial^l+\epsilon_i{}^{lm}\partial_j\partial^k+\epsilon_j{}^{lm}\partial_i\partial^k)]\partial_m\Big]\delta^3(x-y),\end{aligned} \tag{5.1.63}$$

$$\begin{aligned}\{\pi^{ij},p^{kl}\}_D &= \frac{1}{8\mu}(\epsilon^{ikm}\eta^{jl}+\epsilon^{jkm}\eta^{il}+\epsilon^{ilm}\eta^{jk}+\epsilon^{jlm}\eta^{ik})\partial_m\delta^3(x-y)-\frac{\mu^2}{8\Xi}\Big[[(\eta^{ik}\eta^{jl}+\eta^{il}\eta^{jk}-\eta^{ij}\eta^{kl})\nabla^2\\ &+(\eta^{ij}\partial^k\partial^l+\eta^{kl}\partial^i\partial^j)\Big](\nabla^2+\mu^2)-3\partial^i\partial^j\partial^k\partial^l-\frac{3\mu^2}{4}(\eta^{ik}\partial^j\partial^l+\eta^{il}\partial^j\partial^k+\eta^{jk}\partial^i\partial^l+\eta^{jl}\partial^i\partial^k)\\ &+\frac{\mu}{4}[(\epsilon^{ikm}\eta^{jl}+\epsilon^{jkm}\eta^{il}+\epsilon^{ilm}\eta^{jk}+\epsilon^{jlm}\eta^{ik})(\nabla^2+\mu^2)+3(\epsilon^{ikm}\partial^j\partial^l+\epsilon^{jkm}\partial^i\partial^l\\ &+\epsilon^{ilm}\partial^j\partial^k+\epsilon^{jlm}\partial^i\partial^k)]\partial_m\Big]\delta^3(x-y),\end{aligned} \tag{5.1.64}$$

$$\begin{aligned}\{h_{ij},G_{kl}\}_D &= \frac{1}{2}(\eta_{ik}\eta_{jl}+\eta_{il}\eta_{jk}-\eta_{ij}\eta_{kl})\delta^3(x-y)+\frac{\mu^2}{2\Xi}\left[[(\eta_{ik}\eta_{jl}+\eta_{il}\eta_{jk}-\eta_{ij}\eta_{kl})\nabla^2+(\eta_{ij}\partial_k\partial_l\right.\\ &+\eta_{kl}\partial_i\partial_j)](\nabla^2+\mu^2)-3\partial_i\partial_j\partial_k\partial_l-\frac{3\mu^2}{4}(\eta_{ik}\partial_j\partial_l+\eta_{il}\partial_j\partial_k+\eta_{jk}\partial_i\partial_l+\eta_{jl}\partial_i\partial_k)\\ &+\frac{\mu}{4}[(\epsilon_{ik}{}^m\eta_{jl}+\epsilon_{jk}{}^m\eta_{il}+\epsilon_{il}{}^m\eta_{jk}+\epsilon_{jl}{}^m\eta_{ik})(\nabla^2+\mu^2)+3(\epsilon_{ik}{}^m\partial_j\partial_l+\epsilon_{jk}{}^m\partial_i\partial_l\\ &\left.+\epsilon_{il}{}^m\partial_j\partial_k+\epsilon_{jl}{}^m\partial_i\partial_k)]\partial_m\right]\delta^3(x-y),\end{aligned} \tag{5.1.65}$$

$$\{G_{ij},\pi^{0k}\}_D = -\frac{1}{2}(\delta_i^k\partial_j+\delta_j^k\partial_i)\delta^3(x-y), \tag{5.1.66}$$

$$\{\pi^{ij},\pi^{0k}\}_D = \frac{1}{4\mu}(\epsilon^{ikl}\partial^j+\epsilon^{jkl}\partial^i)\partial_l\delta^3(x-y). \tag{5.1.67}$$

We can see a direct contribution due to the CS term which could be important in the quantization analysis of the theory. The new algebra between the first and second-class constraints is

$$\begin{aligned}\{\Gamma^a, \Gamma^b\}_D &= 0, && (5.1.68)\\ \{\Gamma^a, \chi^b\}_D &= 0. && (5.1.69)\end{aligned}$$

Now, in our analysis the canonical Hamiltonian dependes linearly on $\pi^{\mu\nu}$ and $p^{\mu\nu}$, this means that it has no local minimum and its energy will be unbounded from below apparently, however, it is known that this instability can be removed if there are constraints present [61]. Having the Dirac brackets, then the second-class constraints can be considered strongly zero $\chi^a = 0$ and can be used to rewrite the canonical Hamiltonian, the final result is

$$\begin{aligned}H'_{can} &= \int [\frac{1}{2}\pi^{ij}\pi_{ij} - \frac{1}{4}\pi^i{}_i\pi^j{}_j + 2\partial^i h_{0i}\partial^j h_{0j} - 2\partial^i h_0{}^j\partial_i h_{0j} + 2\partial_i h_{0j}G^{ij} - 2\partial^i h_{0i}G^j{}_j \\ &+\frac{1}{2}\partial^k h^{ij}\partial_k h_{ij} - \frac{1}{2}\partial_k h^i{}_i\partial^k h^j{}_j + \partial_i h^{ij}\partial_j h^k{}_k - \partial_k h^k{}_i\partial_j h^{ij} - \frac{1}{\mu}\epsilon^{ijk}[\partial^l h_{im}\partial^m\partial_j h_{kl} \\ &-3\nabla^2 h_{0i}\partial_j h_{0k} + \nabla^2 h_i{}^l\partial_j h_{kl} + 3\partial_i h_{0j}\partial^l G_{kl}] + \frac{1}{\mu^2}[(\partial^i\partial^j h_{0j} - \nabla^2 h_0{}^i)\nabla^2 h_{0i} \\ &-(\partial^i\partial^j h_{0j} - \nabla^2 h_0{}^i)\partial^k G_{ik}] + \frac{1}{4\mu^2}[2\partial^i G_{ij}\partial^j G^k{}_k - \partial^k G^i{}_k\partial^j G_{ij} - \partial_k G^i{}_i\partial^k G^j{}_j]]d^3x.\end{aligned} \tag{5.1.70}$$

Now there are no linear terms on the momenta anymore, in this manner, we got rid of any Ostrogradsky's instability by using the constraints of the theory. Moreover, it is straightfoward to show that (5.1.70) reproduces the correct equations of motion, for example

$$\begin{aligned}\dot{\pi}_{0i} &= \{\pi_{0i}, H'_{can}\}_D \\ &= \nabla^2 h_{0i} - \partial_i\partial^j h_{0j} + \partial_i G^j{}_j - \partial^j G_{ij} - \frac{1}{\mu}\epsilon_i{}^{jk}(\partial_j\partial^l G_{kl} - \nabla^2\partial_j h_{0k}) \\ &= \{\pi_{0i}, H_{can}\}.\end{aligned} \tag{5.1.71}$$

Thus the EoM are in agreement with the evolution of the Dirac and/or Poisson brackets. On the other hand, we can proceed to identify the symmetries of the theory, i.e., the gauge transformations. The local gauge transformations are given by

$$\delta X(x) = \int \{X(x), \omega_a\Gamma^a(y)\}_D d^3y, \tag{5.1.72}$$

where ω_a are the gauge parameters. Calculation for the perturbation $h_{\mu\nu}$ and its velocity

$G_{\mu\nu}$ throws the following infinitesimal variations:

$$\delta h_{00} = \omega_2, \tag{5.1.73}$$
$$\delta h_{0i} = \frac{1}{2}\omega_{3i}, \tag{5.1.74}$$
$$\delta h_{ij} = -\frac{1}{2}\left(\partial_i\omega_{4j} + \partial_j\omega_{4i}\right) \tag{5.1.75}$$
$$\delta G_{00} = \omega_1 \tag{5.1.76}$$
$$\delta G_{0i} = \frac{1}{2}\omega_{5i} \tag{5.1.77}$$
$$\delta G_{ij} = \frac{1}{2}(\partial_i\omega_{3j} + \partial_j\omega_{3i}) + \partial_i\partial_j\omega_6. \tag{5.1.78}$$

Now, since there are third generation constraints the gauge parameters ω_a are restricted. As it is mentioned in Sec. 2.8, the involution relations of all first-class constraints provide equations for unfree gauge transformations, say, the gauge parameters obey the system of partial differential equations (2.9.5) and (2.9.6) or (2.9.11) in the case of one first-class per generation. In this case the situation goes as follows: we have 4 first-class constraints of first generation Γ_1 and Γ_5^i, their time derivative essentially generates 4 more constraints: Γ_2 and Γ_3^i, in the same manner the latter generate the last generation of constraints: Γ_6 and Γ_4^i, all this can be directly verified by looking at the involution relations. The involution relations between the constraints and the Hamiltonian reads:

$$\{\Gamma_1, H_{can}\}_D = -\pi^{00} = -\Gamma_2, \tag{5.1.79}$$
$$\{\Gamma_5^i, H_{can}\}_D = -\pi^{0i} = \partial_j\chi_1^{ij} - \Gamma_3^i = -\Gamma_3^i, \tag{5.1.80}$$
$$\{\Gamma_2, H_{can}\}_D = \nabla^2 h^i{}_i - \partial_i\partial_j h^{ij} = \Gamma_6 - \partial_i\partial_j\chi_1^{ij} = \Gamma_6, \tag{5.1.81}$$
$$\{\Gamma_3^i, H_{can}\}_D = \partial_j\pi^{ij} + \frac{1}{2\mu}\epsilon^{ijk}\partial_j\partial^l G_{kl} = \Gamma_4^i, \tag{5.1.82}$$
$$\{\Gamma_4^i, H_{can}\}_D = 0, \tag{5.1.83}$$
$$\{\Gamma_6, H_{can}\}_D = -\partial_i\partial_j\pi^{ij} = -\partial_i\Gamma_4^i, \tag{5.1.84}$$
$$\{\chi_1^{ij}, H_{can}\}_D = 0, \tag{5.1.85}$$
$$\{\chi_2^{ij}, H_{can}\}_D = 0, \tag{5.1.86}$$

these brackets are no other than the equations (2.9.7)-(2.9.10) but with more constraints per generation, in other words:

$$\begin{aligned}
\{\overset{(1)}{\Gamma}_a, H_0\} &= \overset{(1)}{O}{}_a^b \overset{(2)}{\Gamma}{}_b, && (5.1.87)\\
\{\overset{(\mathfrak{g})}{\Gamma}{}_a, H_0\} &= \overset{(\mathfrak{g})}{O}{}_a^b \overset{(\mathfrak{g}+1)}{\Gamma}{}_b, && (5.1.88)\\
\{\overset{(\mathcal{G})}{\Gamma}{}_a, H_0\} &= 0, && (5.1.89)\\
\{\Gamma_a, \Gamma_b\} &= 0, && (5.1.90)
\end{aligned}$$

where

$$\begin{aligned}
O_{\Gamma_1}^{\Gamma_2} &= -1, && (5.1.91)\\
O_{\Gamma_5^i}^{\Gamma_3^i} &= -1, && (5.1.92)\\
O_{\Gamma_2}^{\Gamma_6} &= +1, && (5.1.93)\\
O_{\Gamma_3^i}^{\Gamma_4^i} &= +1, && (5.1.94)\\
O_{\Gamma_6}^{\Gamma_4^i} &= -\partial_i. && (5.1.95)
\end{aligned}$$

On the other hand, the restrictions on the gauge parameters are

$$\dot{\varepsilon}^{a+1} + O_b^a \varepsilon^b = 0, \tag{5.1.96}$$

identification of the structure coefficients O_a^b allows us to calculate (5.1.96) for the $\omega' s$ parameters

$$\begin{aligned}
& \dot{\varepsilon}^{\Gamma_2} + O_{\Gamma_1}^{\Gamma_2}\omega^1 = 0 \\
\Rightarrow\ & \int [\partial_0\omega_2\delta^3(x-y) - \delta^3(x-y)\omega_1] d^3y = 0 && (5.1.97)\\
& \dot{\varepsilon}^{\Gamma_3^i} + O_{\Gamma_5^i}^{\Gamma_3^i}\varepsilon^{\Gamma_5^i} = 0 \\
\Rightarrow\ & \int [\partial_0\omega_{3i}\delta^3(x-y) - \omega_{5i}\delta^3(x-y)] d^3y = 0 && (5.1.98)\\
& \dot{\varepsilon}^{\Gamma^6} + O_{\Gamma_2}^{\Gamma_6}\varepsilon^{\Gamma_2} = 0 \\
\Rightarrow\ & \int [\partial_0\omega_6\delta^3(x-y) + \omega_2\delta^3(x-y)] d^3y = 0 && (5.1.99)\\
& \dot{\varepsilon}^{\Gamma_4^i} + O_{\Gamma_3^i}^{\Gamma_4^i}\varepsilon^{\Gamma_3^i} + O_{\Gamma_6}^{\Gamma_4^i}\varepsilon^{\Gamma_6} = 0 \\
\Rightarrow\ & \int [\partial_0\omega_i^4\delta^3(x-y) + \omega_{3i}\delta^3(x-y) - \partial_i\delta^3(x-y)\omega_6] d^3y = 0, && (5.1.100)
\end{aligned}$$

thus, we obtain an explicit set of relations between the gauge parameters:

$$\partial_0\omega_2 - \omega_1 = 0, \tag{5.1.101}$$

$$\partial_0\omega_{3i} - \omega_{5i} = 0, \tag{5.1.102}$$

$$\partial_0\omega_6 + \omega_2 = 0, \tag{5.1.103}$$

$$\partial_0\omega_{4i} + \omega_{3i} + \partial_i\omega_6 = 0. \tag{5.1.104}$$

Afterwards, by taking advantage of these relations we can rewrite the gauge transformations for $h_{\mu\nu}$ and $G_{\mu\nu}$ only in terms of the parameters ω_4^i and ω_6:

$$\begin{aligned}
\delta h_{00} &= -\partial_0\omega_6, &(5.1.105)\\
\delta h_{0i} &= -\frac{1}{2}(\partial_0\omega_{4i} + \partial_i\omega_6), &(5.1.106)\\
\delta h_{ij} &= -\frac{1}{2}(\partial_i\omega_{4j} + \partial_j\omega_{4i}), &(5.1.107)\\
\delta G_{00} &= -\partial_0\partial_0\omega_6, &(5.1.108)\\
\delta G_{0i} &= -\frac{1}{2}\partial_0(\partial_0\omega_{4i} + \partial_i\omega_6), &(5.1.109)\\
\delta G_{ij} &= -\frac{1}{2}(\partial_i\partial_0\omega_{4j} + \partial_j\partial_0\omega_{4i}), &(5.1.110)
\end{aligned}$$

moreover, we can express the gauge transformations in a covariant way by introducing the 4-vector Λ_μ such that $\Lambda_0 \equiv -\frac{1}{2}\omega_6$ and $\Lambda_i \equiv -\frac{1}{2}\omega_{4i}$, the gauge transformation reads

$$\begin{aligned}
\delta h_{\mu\nu} &= \partial_\mu\Lambda_\nu + \partial_\nu\Lambda_\mu, &(5.1.111)\\
\delta G_{\mu\nu} &= \partial_0(\partial_\mu\Lambda_\nu + \partial_\nu\Lambda_\mu). &(5.1.112)
\end{aligned}$$

5.2 Hamilton-Jacobi analysis

The main goal here is to identify a complete set of HJPDE and to construct a fundamental diferential df that codifies all the physical information of the system. For such purpose let us start over again with the Lagrangian that defines the action (5.1.1), this is

$$\begin{aligned}
\mathcal{L} = \int \Bigg[&\frac{1}{2}\dot{h}_{ij}\dot{h}^{ij} - \partial_j h_{0i}\partial^j h^{0i} - \frac{1}{2}\partial_k h_{ij}\partial^k h^{ij} - \frac{1}{2}\dot{h}^i{}_i\dot{h}^j{}_j + \partial^j h^0{}_0\partial_j h^i{}_i + \frac{1}{2}\partial_k h^i{}_i\partial^k h^j{}_j \\
&-2\partial^i h^0{}_i\dot{h}^j{}_j - \partial_i h^0{}_0\partial_j h^{ij} - \partial_i h^{ij}\partial_j h^k{}_k + 2\partial_j h^0{}_i\dot{h}^{ij} + \partial_i h^i{}_0\partial_j h^{0j} + \partial_k h^k{}_i\partial_j h^{ij} \\
&+\frac{1}{\mu}\epsilon^{ijk}(-\ddot{h}^l{}_i\partial_j h_{lk} + 2\dot{h}^l{}_i\partial_j\partial_l h^0{}_k + \partial_l h^m{}_i\partial_m\partial_j h^l{}_k + \nabla^2 h^0{}_i\partial_j h_{0k} + \nabla^2 h^m{}_i\partial_j h_{mk})\Bigg] d^3.
\end{aligned} \tag{5.2.1}$$

We will reduce the order of the time derivatives of the Lagrangian by extending the configuration space in a different way than we did in the previous section, this is done by introducing the following change of variables

$$K_{ij} = \frac{1}{2}(\dot{h}_{ij} - \partial_i h_{0j} - \partial_j h_{0i}), \tag{5.2.2}$$

K_{ij} is related with the so-called extrinsic curvature. This change of variables redefine the Lagrangian

$$\begin{aligned} \mathcal{L} \; = \; & \int \Big[2K_{ij}K^{ij} - 2K^i{}_iK^j{}_j - h_{00}R_{ij}{}^{ij} - h_{ij}R^{ij} + \frac{1}{2}h^i{}_iR_{ij}{}^{ij} + \frac{1}{\mu}\epsilon^{ijk}(4K_i{}^l\partial_jK_{kl} \\ & + \partial^m h_{im}\partial_j\partial^l h_{kl} + \nabla^2 h_i{}^m\partial_j h_{km}) + \psi^{ij}(\dot{h}_{ij} - \partial_i h_{0j} - \partial_j h_{0i} - 2K_{ij})\Big] \, d^3x, \end{aligned} \tag{5.2.3}$$

where we have added the Lagrange multipliers ψ^{ij} enforcing the relation (5.2.2) and the expressions $R_{ij}{}^{ij}$ and R_{ij} are defined in the following way

$$R_{ij}{}^{ij} \equiv \partial^i\partial^j h_{ij} - \nabla^2 h^i{}_i, \tag{5.2.4}$$

$$R_{ij} \equiv \frac{1}{2}(\partial_i\partial^k h_{jk} + \partial_j\partial^k h_{ik} - \partial^i\partial^j h^k{}_k - \nabla^2 h_{ij}). \tag{5.2.5}$$

Now we calculate the canonical momenta associated with the canonical variables

$$\pi^{00} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{00}} = 0, \tag{5.2.6}$$

$$\pi^{0i} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{0i}} = 0, \tag{5.2.7}$$

$$\pi^{ij} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{ij}} = \psi^{ij}, \tag{5.2.8}$$

$$P^{ij} = \frac{\partial\mathcal{L}}{\partial\dot{K}_{ij}} = 0, \tag{5.2.9}$$

$$\Lambda^{ij} = \frac{\partial\mathcal{L}}{\partial\dot{\psi}_{ij}} = 0, \tag{5.2.10}$$

from this definition we identify the Hamiltonians of the theory

$$
\begin{aligned}
H' &\equiv H_0 + \Pi = 0, && (5.2.11)\\
H_1^{00} &\equiv \pi^{00} = 0, && (5.2.12)\\
H_2^{0i} &\equiv \pi^{0i} = 0, && (5.2.13)\\
H_3^{ij} &\equiv \pi^{ij} - \psi^{ij} = 0, && (5.2.14)\\
H_4^{ij} &\equiv P^{ij} = 0, && (5.2.15)\\
H_5^{ij} &\equiv \Lambda^{ij} = 0, && (5.2.16)
\end{aligned}
$$

where H_0 is the canonical Hamiltonian defined as usual $H_0 = \dot{h}_{\mu\nu}\pi^{\mu\nu} + \dot{K}_{ij}P^{ij} + \dot{\psi}_{ij}\Lambda^{ij} - \mathcal{L}$ and $\Pi = \partial_0 S$. The corresponding PB between the canonical variables are given by

$$
\begin{aligned}
\{h_{\mu\nu}, \pi^{\alpha\beta}\} &= \frac{1}{2}(\delta^\alpha_\mu\delta^\beta_\nu + \delta^\alpha_\nu\delta^\beta_\mu)\delta^3(x-y), && (5.2.17)\\
\{K_{ij}, \pi^{kl}\} &= \frac{1}{2}(\delta^k_i\delta^l_j + \delta^k_j\delta^l_i)\delta^3(x-y), && (5.2.18)\\
\{\psi^{ij}, \Lambda_{kl}\} &= \frac{1}{2}(\delta^i_k\delta^j_l + \delta^j_k\delta^i_l)\delta^3(x-y). && (5.2.19)
\end{aligned}
$$

Once we have a preliminary set of Hamiltonians we can construct a preliminary differential fundamental $df = \{f, H_n\}dt^n$ but, we can remove some of the non-involutive Hamiltonians in favour of a preliminary generalized bracket

$$
\{f, g\}^* = \{f, g\} - \{f, H_{a'}\}(C_{a'b'})^{-1}\{H_{b'}, g\}, \qquad (5.2.20)
$$

where $C_{a'b'}$ is the matrix of PB between all non-involutive Hamiltonians. The non-involutive Hamiltonians are H_3^{ij} and H_5^{ij} because its PB does not vanish

$$
\{H_3^{ij}, H_5^{ij}\} = -\frac{1}{2}(\eta^{ik}\eta^{jl} + \eta^{il}\eta^{kj})\delta^3(x-y), \qquad (5.2.21)
$$

therefore, the matrix $C_{a'b'}$ is given by

$$
C_{a'b'} = \begin{array}{c c} & \begin{array}{cc} \quad H_3^{kl} \quad & \quad H_5^{kl} \quad \end{array} \\ \begin{array}{c} H_3^{ij} \\ H_4^{ij} \end{array} & \begin{pmatrix} 0 & -\frac{1}{2}(\eta^{ik}\eta^{jl} + \eta^{il}\eta^{kj}) \\ \frac{1}{2}(\eta^{ik}\eta^{jl} + \eta^{il}\eta^{kj}) & 0 \end{pmatrix} \end{array} \delta^3(x-y). \qquad (5.2.22)
$$

Its inverse is

$$
C^{-1}_{a'b'} = \begin{pmatrix} 0 & \frac{1}{2}(\eta^{ik}\eta^{jl} + \eta^{il}\eta^{kj}) \\ -\frac{1}{2}(\eta^{ik}\eta^{jl} + \eta^{il}\eta^{kj}) & 0 \end{pmatrix} \delta^3(x-y). \qquad (5.2.23)
$$

Its easy to calculate the new brackets between the canonical fields:

$$\{h_{\mu\nu}, \pi^{\alpha\beta}\}^* = \frac{1}{2}(\delta^\alpha_\mu \delta^\beta_\nu + \delta^\beta_\mu \delta^\alpha_\nu)\delta^3(x-y), \tag{5.2.24}$$

$$\{K_{ij}, P^{kl}\}^* = \frac{1}{2}(\delta^k_i \delta^l_j + \delta^l_i \delta^k_j)\delta^3(x-y), \tag{5.2.25}$$

$$\{h_{\mu\nu}, \psi^{\alpha\beta}\}^* = \frac{1}{2}(\delta^\alpha_\mu \delta^\beta_\nu + \delta^\beta_\mu \delta^\alpha_\nu)\delta^3(x-y), \tag{5.2.26}$$

$$\{\psi_{ij}, \Lambda^{kl}\}^* = 0, \tag{5.2.27}$$

we can observe from (5.2.27) that the variables ψ^{ij} and Λ^{ij} can be removed which implies that we can perform the substitution $\psi^{ij} = \pi^{ij}$ and $\Lambda^{ij} = 0$. The resulting canonical Hamiltonian is

$$\begin{aligned}\mathcal{H}_0 = & \int [2K^i{}_i K^j{}_j - 2K_{ij}K^{ij} + h_{00}R_{ij}{}^{ij} + h_{ij}R^{ij} - \frac{1}{2}h^i{}_i R_{ij}{}^{ij} - \frac{1}{\mu}\epsilon^{ijk}(4K_i{}^l \partial_j K_{kl} \\ & + \partial^m h_{im}\partial_j \partial^l h_{kl} + \nabla^2 h_i{}^m \partial_j h_{km}) - 2h_{0j}\partial_i \pi^{ij} + 2K_{ij}\pi^{ij}]d^3x. \end{aligned} \tag{5.2.28}$$

The canonical Hamiltonian has linear terms in the momenta such as in the pure canonical analysis, nevertheless, it is known that those instabilities could be healed by means of the correct identification of the Hamiltonians. With the final generalized brackets and the identification of all the Hamiltonians we can remove the linear canonical momenta terms. The fundamental differential is

$$df = \int [\{f, H'\}^* dt + \{f, H_1^{00}\}^* d\omega^1_{00} + \{f, H_2^{0i}\}^* d\omega^2_{0i} + \{f, H_4^{ij}\}^* d\omega^4_{ij}]d^3y, \tag{5.2.29}$$

we will require integrability conditions on the Hamiltonians H_1^{00}, H_2^{0i} and H_4^{ij}, in other words

$$dH_{ai} = 0. \tag{5.2.30}$$

From (5.2.30) we obtain 10 new Hamiltonians:

$$H_6^{00} \equiv \nabla^2 h^i{}_i - \partial^i \partial^j h_{ij} = 0, \tag{5.2.31}$$

$$H_7^{0i} \equiv \partial_j \pi^{ij} = 0, \tag{5.2.32}$$

$$H_8^{ij} \equiv \pi^{ij} - 2K^{ij} + 2\eta^{ij}K^k{}_k - \frac{2}{\mu}(\epsilon^{ikl}\eta^{jm} + \epsilon^{jkl}\eta^{im})\partial_k K_{lm} = 0. \tag{5.2.33}$$

Now we have to incorporate these new Hamiltonians to the prevous set of Hamiltonians, this is:

$$
\begin{aligned}
H' &\equiv \mathcal{H}_0 + \Pi = 0, && (5.2.34)\\
H_1^{00} &\equiv \pi^{00} = 0, && (5.2.35)\\
H_2^{0i} &\equiv \pi^{0i} = 0, && (5.2.36)\\
H_4^{ij} &\equiv P^{ij} = 0, && (5.2.37)\\
H_6^{00} &\equiv \nabla^2 h^i{}_i - \partial^i\partial^j h_{ij} = 0, && (5.2.38)\\
H_7^{0i} &\equiv \partial_j \pi^{ij} = 0, && (5.2.39)\\
H_8^{ij} &\equiv \pi^{ij} - 2K^{ij} + 2\eta^{ij}K^k{}_k - \frac{2}{\mu}(\epsilon^{ikl}\eta^{jm} + \epsilon^{jkl}\eta^{im})\partial_k K_{lm} = 0, && (5.2.40)
\end{aligned}
$$

in addition, we need to recalculate the algebra between Hamiltomians. The 20×20 matrix made of the brackets $\{\ ,\ \}^*$ between all the Hamiltonians is

$$
W^{IJ} = \begin{array}{c} \\ H_1^{00} \\ H_2^{0i} \\ H_4^{ij} \\ H_6^{00} \\ H_7^{0i} \\ H_8^{ij} \end{array}
\begin{array}{c}
\begin{array}{cccccc} H_1^{00} & H_2^{0k} & H_4^{kl} & H_6^{00} & H_7^{0k} & H_8^{kl} \end{array}\\
\left(\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \{H_4^{ij}, H_8^{kl}\}^*\\
0 & 0 & 0 & 0 & 0 & \{H_6^{00}, H_8^{kl}\}^*\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \{H_8^{ij}, H_4^{kl}\}^* & \{H_8^{ij}, H_6^{00}\}^* & 0 & 0
\end{array}\right)
\end{array}, \qquad (5.2.41)
$$

where

$$
\begin{aligned}
\{H_4^{ij}, H_8^{ij}\}^* &= 2[\frac{1}{2\mu}(\epsilon^{ikm}\eta^{jl} + \epsilon^{jkm}\eta^{il} + \epsilon^{ilm}\eta^{jk} + \epsilon^{ilm}\eta^{ik})\partial_m + \frac{1}{2}(\eta^{ik}\eta^{jl} + \eta^{jk}\eta^{il}) - \eta^{ij}\eta^{kl}]\delta^3(x-y),\\
\{H_6^{00}, H_8^{ij}\}^* &= [\eta^{ij}\nabla^2 - \partial^i\partial^j]\delta^3(x-y). \qquad (5.2.42)
\end{aligned}
$$

The null vectors $v = (\frac{1}{2}\partial_i\partial_j, \delta_{ij}, 0)$ immediatly indicate us that an independent Hamiltonian can be infered

$$
H_9 = \nabla^2 h^i{}_i - \partial^i\partial^j h_{ij} + \frac{1}{2}\partial_i\partial_j P^{ij}, \qquad (5.2.43)
$$

the matrix W^{IJ} now has the form

$$W^{IJ} = \begin{array}{c} \\ H_1^{00} \\ H_2^{0i} \\ H_7^{0i} \\ H_9 \\ H_4^{ij} \\ H_8^{ij} \end{array} \begin{array}{c} \begin{array}{cccccc} H_1^{00} & H_2^{0k} & H_7^{0k} & H_9 & H_4^{kl} & H_8^{kl} \end{array} \\ \left(\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \{H_4^{ij}, H_8^{kl}\}^* \\ 0 & 0 & 0 & 0 & \{H_8^{ij}, H_4^{kl}\}^* & 0 \end{array} \right) \end{array}, \qquad (5.2.44)$$

where we arrange the matrix such that in its lower right corner it appears the submatrix $C_{a_{ni}b_{ni}}$ necessary to calculate the generalized brackets $\{f,g\}^{**} = \{f,g\} - \{f, H_{a_{ni}}\}C^{-1}_{a_{ni}b_{ni}}\{H_{b_{ni}}, g\}$,

$$C_{a_{ni}b_{ni}} = \begin{array}{c} \\ H_4^{ij} \\ H_8^{ij} \end{array} \begin{array}{c} \begin{array}{cc} H_4^{kl} & H_8^{kl} \end{array} \\ \left(\begin{array}{cc} 0 & \{H_4^{ij}, H_8^{kl}\} \\ \{H_8^{ij}, H_4^{kl}\} & 0 \end{array} \right) \end{array}. \qquad (5.2.45)$$

In the end, the non-trivial HJ generalized brackets are given by

$$\{h_{ij}, \pi^{kl}\}^{**} = \frac{1}{2}(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k)\delta^3(x-y), \qquad (5.2.46)$$

$$\{K_{ij}, P^{kl}\}^{**} = 0, \qquad (5.2.47)$$

$$\begin{aligned} \{h_{ij}, K_{kl}\}^{**} = {} & \frac{1}{4}(\eta_{ik}\eta_{jl} + \eta_{il}\eta_{jk} - \eta_{ij}\eta_{kl})\delta^3(x-y) + \frac{\mu^2}{4\Xi}[[(\eta_{ik}\eta_{jl} + \eta_{il}\eta_{jk} - \eta_{ij}\eta_{kl})\nabla^2 + (\eta_{ij}\partial_k\partial_l \\ & + \eta_{kl}\partial_i\partial_j)](\nabla^2 + \mu^2) - 3\partial_i\partial_j\partial_k\partial_l - \frac{3\mu^2}{4}(\eta_{ik}\partial_j\partial_l + \eta_{il}\partial_j\partial_k + \eta_{jk}\partial_i\partial_l + \eta_{jl}\partial_i\partial_k) \\ & + \frac{\mu}{4}[(\epsilon_{ik}{}^m\eta_{jl} + \epsilon_{jk}{}^m\eta_{il} + \epsilon_{il}{}^m\eta_{jk} + \epsilon_{jl}{}^m\eta_{ik})(\nabla^2 + \mu^2) + 3(\epsilon_{ik}{}^m\partial_j\partial_l + \epsilon_{jk}{}^m\partial_i\partial_l \\ & + \epsilon_{il}{}^m\partial_j\partial_k + \epsilon_{jl}{}^m\partial_i\partial_k)]\partial_m]\delta^3(x-y), \end{aligned} \qquad (5.2.48)$$

where $\Xi = -\mu(\nabla^2 + \mu^2)(\nabla^2 + \frac{\mu^2}{4})$. The HJ generalized brackets allow us redefine the fundamental differential once more time

$$\begin{aligned} df = {} & \int [\{f, H'(y)\}^{**}dt + \{f, H_1^{00}(y)\}^{**}d\omega_{00}^1 + \{f, H_2^{0i}(y)\}^{**}d\omega_{0i}^2 + \{f, H_7^{0i}(y)\}^{**}d\omega_{0i}^7 \\ & + \{f, H_9(y)\}^{**}d\omega^9]d^3y, \end{aligned} \qquad (5.2.49)$$

where

$$
\begin{aligned}
H_1^{00} &= \pi^{00}, && (5.2.50)\\
H_2^{0i} &= \pi^{0i}, && (5.2.51)\\
H_7^{0i} &= \partial_j \pi^{ij}, && (5.2.52)\\
H_9 &= \nabla^2 h^i{}_i - \partial^i\partial^j h_{ij}, && (5.2.53)
\end{aligned}
$$

we can note that in (5.2.53) there is no P^{ij} term, this is because now we can safely use $H_4^{ij} = 0$ as an identity, in fact, if we use $H_4^{ij} = 0$ and $H_8^{ij} = 0$ in the canonical Hamiltonian then the Ostrogradski's instabilities can be healed in the same manner we did in the previous section. The resulting Hamiltonian is

$$
\begin{aligned}
\mathcal{H}'_0 \;=\; & \int [\frac{1}{2}\pi^{ij}\pi_{ij} - \frac{1}{4}\pi^i{}_i\pi^j{}_j + h_{ij}R^{ij} - \frac{1}{\mu}\epsilon^{ijk}(4K_i{}^l\partial_j K_{kl} + \partial^m h_{im}\partial_j\partial^l h_{kl} + \nabla^2 h_i{}^l\partial_j h_{kl})\\
& -\frac{4}{\mu^2}(2\partial^i K_{ij}\partial^j K^k{}_k + 2\partial^i K^j{}_k\partial_i K_j{}^k - 2\partial^j K^i{}_k\partial_i K_j{}^k - \partial^j K^i{}_k\partial^k K_{ij} - \partial_k K^i{}_i\partial^k K^j{}_j]d^3x.
\end{aligned}
$$

On the other hand, if we apply integrability conditions to the set (5.2.50)-(5.2.53) we obtain redundant identities

$$
\begin{aligned}
dH_7^{0i} &= 0, && (5.2.54)\\
dH_9 &= -\partial_i\partial_j\pi^{ij} = -\partial_i H_7^{0i} = 0, && (5.2.55)
\end{aligned}
$$

thus the fundamental differential (5.2.49) is final. Now, what physical info can be extracted from the fundamental differential? A direct calculation of the characteristic equations reveals the equations of motion and in consequence the physical degrees of freedom, the characteristic equations are:

$$
\begin{aligned}
dh_{00} &= d\theta^1_{00}, && (5.2.56)\\
dh_{0i} &= \frac{1}{2}d\theta^2_{0i}, && (5.2.57)\\
dh_{ij} &= [2K_{ij} + \partial_i h_{0j} + \partial_j h_{0i}]dt - \frac{1}{2}(\delta^k_i\partial_j + \delta^k_j\partial_i)d\theta^7_{0k}, && (5.2.58)\\
d\pi^{00} &= -R_{ij}{}^{ij}dt, && (5.2.59)\\
d\pi^{0i} &= \frac{1}{2}\partial_j\pi^{ij}dt, && (5.2.60)\\
d\pi^{ij} &= [\eta^{ij}\nabla^2 h_{00} - \partial^i\partial^j h_{00} - \eta^{ij}R_{kl}{}^{kl} - 2R^{ij} - \frac{1}{\mu}[(\epsilon^{ikl}\partial^j + \epsilon^{jkl}\partial^i)\partial_k\partial^m h_{lm}\\
&\quad -(\epsilon^{ikl}\eta^{jm} + \epsilon^{jkl}\eta^{im})\partial_k\nabla^2 h_{lm}]]dt + (\partial^i\partial^j - \eta^{ij}\nabla^2)d\theta^9, && (5.2.61)\\
dK_{ij} &= [-\frac{1}{2}\partial_i\partial_j h_{00} - R_{ij} + \frac{1}{4}\eta_{ij}R_{kl}{}^{kl}]dt + \frac{1}{2}\partial_i\partial_j d\theta_9, && (5.2.62)\\
dP^{ij} &= [0]dt, && (5.2.63)
\end{aligned}
$$

from the characteristic equations we can identify the fields h_{00} and h_{0i} as Lagrange multipliers, on the other hand, we can discard P^{ij} as a degree of freedom because its time evolution vanishes. Furthermore, by taking $d\omega^7_{0k} = 0$ and $d\omega^9 = 0$ we obtain the equations of motion of the theory

$$\begin{aligned} \dot{h}_{ij} &= 2K_{ij} + \partial_i h_{0j} + \partial_j h_{0i}, && (5.2.64)\\ \dot{\pi}^{ij} &= \eta^{ij}\nabla^2 h_{00} - \partial^i\partial^j h_{00} - \eta^{ij}R_{kl}{}^{kl} - 2R^{ij} - \frac{1}{\mu}[(\epsilon^{ikl}\partial^j + \epsilon^{jkl}\partial^i)\partial_k\partial^m h_{lm} \\ &\quad -(\epsilon^{ikl}\eta^{jm} + \epsilon^{jkl}\eta^{im})\partial_k\nabla^2 h_{lm}], && (5.2.65)\\ \dot{K}_{ij} &= -\frac{1}{2}\partial_i\partial_j h_{00} - R_{ij} + \frac{1}{4}\eta_{ij}R_{kl}{}^{kl}. && (5.2.66) \end{aligned}$$

The equation (5.2.66) corresponds to the definition of K_{ij}, thus, if we combine the time derivative of equation (5.2.64) with the equation (5.2.66) we obtain a second order time equation for h_{ij} as expected, thus we conclude that there are 6 physical degrees of freedom associated with the perturbation field. The total number of degrees of freedom is $DoF = \frac{1}{2}[(12) - (8)] = 2$ because we have 12 dynamical variables and 8 involutive Hamiltonians. In addition to these results, the characteristics can also reveal a set of transformations that leave the physical states intact, let's take $dt = 0$, in this manner the canonical transformations δX are obtained:

$$\begin{aligned} \delta h_{00} &= \delta\omega^1_{00}, && (5.2.67)\\ \delta h_{0i} &= \frac{1}{2}\delta\omega^2_{0i}, && (5.2.68)\\ \delta h_{ij} &= -\frac{1}{2}(\delta^k_i\partial_j + \delta^k_j\partial_i)\delta\omega^7_{0k}. && (5.2.69) \end{aligned}$$

We can go even further, we can ensure that the transformations (5.2.67)-(5.2.69) are gauge ones, this is acomplished by demanding that the action S is invariant under $\delta h_{\mu\nu}$, in other words, $\delta S = 0$ under (5.2.67)-(5.2.69). Now we will show that this requeriment imposes restrictions on the parameters $\omega's$. Let's take the variation of S

$$\begin{aligned} \delta S &= \left[\frac{\partial S}{\partial h_{\mu\nu}}\delta h_{\mu\nu} + \frac{\partial S}{\partial(\partial_\alpha h_{\mu\nu})}\delta(\partial_\alpha h_{\mu\nu}) + \frac{\partial S}{\partial(\partial_\alpha\partial_\beta h_{\mu\nu})}\delta(\partial_\alpha\partial_\beta h_{\mu\nu})\right] && (5.2.70)\\ &= \int [(-\Box h^{\mu\nu} + \Box h^\lambda{}_\lambda\eta^{\mu\nu} - \partial_\alpha\partial_\lambda h^{\alpha\lambda}\eta^{\mu\nu} - \partial^\mu\partial^\nu h^\lambda{}_\lambda + 2\partial^\mu\partial_\lambda h^{\nu\lambda} + \frac{1}{\mu}\epsilon^{0\mu\lambda\gamma}(\partial^\nu\partial_\alpha\partial_\lambda h^\alpha{}_\gamma \\ &\quad -\partial_\lambda\Box h^\nu{}_\gamma))\,\delta h_{\mu\nu}]\,d^4x, && (5.2.71) \end{aligned}$$

thus, by substituting (5.2.67)-(5.2.69) we get

$$\begin{aligned} \delta S &= \int [R_{ij}{}^{ij}\delta\omega^1_{00} + \frac{1}{2}[2\nabla^2 h_0{}^i + 2\partial^i \dot{h}^j{}_j - 2\partial^i\partial^j h_{0j} - 2\partial_j \dot{h}^{ij} + \frac{1}{\mu}\epsilon^{0ijk}(\partial_j \nabla^2 h_{0k} - \partial_j\partial^l \dot{h}_{kl})]\delta\omega^2_{0i} \\ &\quad -\frac{1}{2}[\ddot{h}^{ij} - \ddot{h}^k{}_k\eta^{ij} + 2\partial^k \dot{h}_{0k}\eta^{ij} - 2\partial^i \dot{h}_0{}^j + \partial^i\partial^j h_{00} - \nabla^2 h_{00}\eta^{ij} + 2R^{ij} - R_{kl}{}^{kl}\eta^{ij} \\ &\quad +\frac{1}{\mu}\epsilon^{0ikl}(\partial_k \ddot{h}^j{}_l - \partial^j\partial_k \dot{h}_{0l} + \partial^j\partial_k\partial^m h_{lm} - \partial_k\nabla^2 h^j{}_l)]\delta(\partial_i\omega^7_{0j} + \partial_j\omega^7_{0i})]d^4x. \end{aligned} \tag{5.2.72}$$

Without loss of generality we can define $\partial_0\xi \equiv \delta\omega^1_{00}$ and after a long algebraic work and some integration by parts we find that the variation of the action takes the form

$$\begin{aligned} \delta S &= \int [-\partial_j \dot{h}^{ij} + \partial^i h^j{}_j + \nabla^2 h_0{}^i - \partial^i\partial^j h_{0j} + \frac{1}{2\mu}\epsilon^{0ijk}(\partial_j\nabla^2 h_{0k} - \partial_j\partial^l \dot{h}_{kl})] \\ &\quad \times(-\partial_i\xi + \delta\omega^2_{0i} + \partial_0\delta\omega^7_{0i})d^4x. \end{aligned} \tag{5.2.73}$$

Now we demand that $\delta S = 0$, this implies the following equation

$$\delta\omega^2_{0i} = -\partial_0\delta\omega^7_{0i} + \partial_i\xi. \tag{5.2.74}$$

The expression (5.2.74) is a restriction on the gauge parameters in the same sense of (5.1.104). Now, with the equation (5.2.74) and with a proper definition of a 4-vector we can establish the variation of the perturbation in a covariant way. Let's define the vector ξ_μ such that

$$\xi_0 = \frac{1}{2}\xi \tag{5.2.75}$$

$$\xi_i = -\frac{1}{2}\delta\omega^7_{0i}, \tag{5.2.76}$$

the equations (5.2.75) and (5.2.76) are not restrictions, they only redefine the parameters. Hence, by combining equations (5.2.75), (5.2.76) and (5.2.74) we obtain

$$\frac{1}{2}\delta\omega^2_{0i} = \partial_0\xi_i + \partial_i\xi_0, \tag{5.2.77}$$

finally, from this last equation and from the definition of ξ_μ itself we get the desired result

$$\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu. \tag{5.2.78}$$

Chapter 6

Conclusions

In this work, a detailed Hamiltonian analysis of the higher-order modified gravity has been performed. By making a 3+1 decomposition we explicitly show that the model is a higher-order theory, the main procedures that we adopted (canonical formalism and HJ theory) were applied successfully; it was possible to identify and classify the constraints (Hamiltonians) of the theory, the classification of the constraints (Hamiltonians) allowed us to eliminate redundant degrees of freedom and to define a stable canonical Hamiltonian in the sense of Ostrogradsky. At the end, the structure of the first-class constraints and of the involutive Hamiltonians is very similar but we need to notice that they are not defined in the same phase-space

$$
\begin{aligned}
&\Gamma_2 = \pi^{00}, && H_1^{00} = \pi^{00},\\
&\Gamma_3^i = \pi^{0i} - \partial_j p^{ij} - \frac{1}{2\mu}\epsilon^{ijk}\partial_j\partial^l h_{kl}, && H_2^{0i} = \pi^{0i},\\
&\Gamma_4^i = \partial_j \pi^{ij} + \frac{1}{2\mu}\epsilon^{ijk}\partial_j\partial^l G_{kl}, && H_7^{0i} = \partial_j \pi^{ij},\\
&\Gamma_6 = \nabla^2 h^i{}_i - \partial_i\partial_j h^{ij} + \partial_i\partial_j p^{ij}, && H_9 = \nabla^2 h^i{}_i - \partial^i\partial^j h_{ij}.\\
&\Gamma_1 = p^{00}, &&\\
&\Gamma_5^i = p^{0i}. &&
\end{aligned}
$$

Moreover, now we can do a direct comparison of our results with the canonical structure of GR, this by taking the limit $\mu \longrightarrow \infty$ in the constraints and by making appropiate combinations of them. The resulting set is

$$\Theta_1 = \pi^{00}, \tag{6.0.1}$$

$$\Theta_2^i = \pi^{0i} + \partial^i h^j{}_j - \partial_j h^{ij}, \tag{6.0.2}$$

$$\Theta_3 = \nabla^2 h^i{}_i - \partial^i \partial^j h_{ij}, \tag{6.0.3}$$

$$\Theta_4^i = \partial_j \pi^{ij} - \partial^i \partial^j h_{0j} + \nabla^2 h_0{}^i. \tag{6.0.4}$$

These constraints are no more that the first-class constraints that arise in linearized gravity [30, 62]. On the other hand, once the second-class constraints (non-involutive Hamiltonians) served its purpose for the classification of constraints, then we eliminate them by the introduction of a generalized bracket: Dirac's bracket (HJ bracket). We can do a similar procedure by taking $\mu \longrightarrow \infty$ in the new brackets in order to obtain the well-known canonical brackets of linearized gravity

$$\begin{aligned} \{h_{ij}, \pi^{kl}\}_D &= \lim_{\mu\to\infty}\left[\frac{1}{2}(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k)\delta^3(x-y) - \frac{\mu^2}{4\mu^2(\nabla^2+\mu^2)(\nabla^2+\frac{\mu^2}{4})}\left[O(\mu^3)\right]\delta^3(x-y)\right] \\ &= \frac{1}{2}(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k)\delta^3(x-y). \end{aligned} \tag{6.0.5}$$

The remaining first-class constraints (involutive Hamiltonians) were the key feature to explore the gauge transformations, by taking two alternative paths we demonstrated that the extended model and GR shares the same gauge symmetries; one by following a systematic mechanism that takes advantage of the presence of third-generation constraints and other by demanding the invariance of the action. Both procedures reveal us that the gauge parameters are related by partial differential equations.

Once we have developed a Hamiltonian analysis in the perturbation field approximation it is worth to consider a full non-perturbative scenario of the Chern-Simons modified gravity, i.e. to study a background independent modified theory and then compare it with the canonical structure of GR. On the other hand, we built the essential ingredients to analyze the modified theory in the quantum context at least in the perturbative context, we can use our results together with the tools developed in the canonical quantization of field theories to make progress in this program. Either way, these are just prospects for future work.

Appendix A

Hamilton-Jacobi analysis of Chern-Simons gravity in three dimensions.

The content of this appendix is original and it can be found in doi.org/10.1016/j.cjph.2022.07.014.

The action that describes a Chern-Simons theory in three dimensions is given by

$$S_{CS}[g_{\mu\nu}] = \int_M \varepsilon^{\mu\nu\alpha} \left(\frac{1}{2} \Gamma^{\beta}_{\mu\gamma} \partial_\nu \Gamma^{\gamma}_{\alpha\beta} + \frac{1}{3} \Gamma^{\beta}_{\mu\gamma} \Gamma^{m}_{\nu\delta} \Gamma^{\delta}_{\alpha\beta} \right) d^3x, \tag{A.0.1}$$

where $g_{\mu\nu}$ is the 3-metric tensor, M is the space-time manifold, $\varepsilon^{\mu\nu\alpha}$ is the Levi-Civita tensor, $\Gamma^{\alpha}_{\mu\nu}$ are the Christoffel symbols given by

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\alpha} - \partial_\beta g_{\mu\nu}). \tag{A.0.2}$$

We are going to work in the perturbation field approximation, this is, by taking a perturbation $h_{\mu\nu}$ of the metric around the flat space-time geometry,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{A.0.3}$$

the signature $\eta_{\mu\nu} = (-1, 1, 1)$ is considered. The corresponding Lagrangian is

$$L_{CS} = \frac{1}{2} \epsilon^{\lambda\mu\nu} \left(\partial_\sigma h^{\rho}{}_{\lambda} \partial_\rho \partial_\mu h^{\sigma}{}_{\nu} - \partial_\sigma h^{\rho}{}_{\lambda} \partial^\sigma \partial_\mu h_{\rho\nu} \right), \tag{A.0.4}$$

by performing a $3+1$ decomposition we get

$$\begin{aligned} L_{CS} &= \epsilon^{ij}\Big(\ddot{h}_{ik}\partial^k h_{0j} + \ddot{h}_{ik}\partial_j h_0{}^k - \frac{1}{2}\ddot{h}_{ik}\dot{h}_j{}^k + \dot{h}_{ik}\partial_j\partial^k h_{00} + \dot{h}_{0k}\partial_i\partial^k h_{0j} \\ &+\frac{1}{2}\dot{h}_{0i}\nabla^2 h_{0j} - \frac{1}{2}\dot{h}_{ik}\nabla^2 h_j{}^k + \frac{1}{2}\dot{h}_i{}^k\partial_k\partial_l h_j{}^l + \nabla^2 h_{00}\partial_i h_{0j} \\ &-\partial_i h_{jk}\nabla^2 h_0{}^k - \partial_i\partial^l h_j{}^k\partial_k h_{0l}\Big). \end{aligned} \tag{A.0.5}$$

We can observe that the CS theory is a higher-order theory, we need to rewrite L by introducing the following variables

$$\xi_{\mu\nu} = h_{\mu\nu} \tag{A.0.6}$$

$$v_{\mu\nu} \equiv \dot{h}_{\mu\nu} \tag{A.0.7}$$

thus we have

$$\begin{aligned} L_{CS} &= \epsilon^{ij}\Big(\dot{v}_{ik}\partial^k \xi_{0j} + \dot{v}_{ik}\partial_j \xi_0{}^k - \frac{1}{2}\dot{v}_{ik}v_j{}^k + v_{ik}\partial_j\partial^k \xi_{00} + v_{0k}\partial_i\partial^k \xi_{0j} \\ &+\frac{1}{2}v_{0i}\nabla^2 \xi_{0j} - \frac{1}{2}v_{ik}\nabla^2 \xi_j{}^k + \frac{1}{2}v_{ik}\partial^k\partial_l \xi_j{}^l + \nabla^2 \xi_{00}\partial_i \xi_{0j} \qquad \text{(A.0.8)} \\ &-\partial_i \xi_{jk}\nabla^2 \xi_0{}^k - \partial_i\partial^l h_j{}^k\partial_k \xi_{0l}\Big) + \psi^{\mu\nu}\left(v_{\mu\nu} - \dot{\xi}_{\mu\nu}\right) \qquad \text{(A.0.9)} \end{aligned}$$

where $\psi_{\mu\nu}$ are the Lagrange multipliers. Now that we reduced the order of the Lagrangian we proceed to define the momenta

$$P = \frac{\partial L}{\partial \dot{Q}}, \tag{A.0.10}$$

where $Q = (\varepsilon_{00}, \xi_{0i}, \xi_{ij}, v_{00}, v_{0i}, v_{ij}, \psi^{00}, \psi^{0i}, \psi^{ij})$ are the canonical variables and $P = (\pi^{00}, \pi^{0i}, \pi^{ij}, \tilde{\pi}^{00}, \tilde{\pi}^{0i}, \tilde{\pi}^{ij}, p_{00}, p_{0i}, p_{ij})$ their corresponding momenta, we find the following

Hamiltonians:

$$
\begin{aligned}
H &\equiv H_0 + \Pi = 0, && \text{(A.0.11)}\\
\Omega^{00} &\equiv \pi^{00} + \psi^{00} = 0, && \text{(A.0.12)}\\
\Omega^{0i} &\equiv \pi^{0i} + \psi^{0i} = 0, && \text{(A.0.13)}\\
\Omega^{ij} &\equiv \pi^{ij} + \psi^{ij} = 0, && \text{(A.0.14)}\\
\tilde{\Omega}^{00} &\equiv \tilde{\pi}^{00} = 0, && \text{(A.0.15)}\\
\tilde{\Omega}^{0i} &\equiv \tilde{\pi}^{0i} = 0, && \text{(A.0.16)}\\
\tilde{\Omega}^{ij} &\equiv \tilde{\pi}^{ij} - \frac{1}{2}(\epsilon^{ik}\partial^j + \epsilon^{jk}\partial^i)\xi_{0k} - \frac{1}{2}(\epsilon^{ik}\eta^{jl} + \epsilon^{jk}\eta^{il})\partial_k\xi_{0l} && \text{(A.0.17)}\\
&\quad + \frac{1}{4}(\epsilon^{ik}\eta^{lj} + \epsilon^{jk})v_{kl} && \text{(A.0.18)}\\
\Sigma^{00} &\equiv p^{00} = 0, && \text{(A.0.19)}\\
\Sigma^{0i} &\equiv p^{0i} = 0, && \text{(A.0.20)}\\
\Sigma^{ij} &\equiv p^{ij} = 0, && \text{(A.0.21)}
\end{aligned}
$$

where $\Pi \equiv \partial_0 S$, S is the action and H_0 is the canonical Hamiltonian given by

$$
\begin{aligned}
H_0 &\equiv \dot{\xi}_{\mu\nu}\pi^{\mu\nu} + \dot{v}_{\mu\nu}\tilde{\pi}^{\mu\nu} + \dot{\psi}p^{\mu\nu} - L_{CS}\\
&= \epsilon^{ij}\Big(-v_{ik}\partial_j\partial^k\xi_{00} - v_{0k}\partial_i\partial^k\xi_{0j} - \frac{1}{2}v_{0i}\nabla^2\xi_{0j} - \frac{1}{2}v_{ik}\nabla^2{\xi_j}^k - \frac{1}{2}{v_i}^k\partial_k\partial_l{\xi_j}^l\\
&\quad -\partial_i\xi_{0j}\nabla^2\xi_{00} + \partial_i\xi_{jk}\nabla^2{\xi_0}^k + \partial_i\partial^l{\xi_j}^k\partial_k\xi_{0l}\Big) - v_{00}\psi^{00}\\
&\quad + v_{0i}(\pi^{0i} - 2\psi^{0i}) - v_{ij}\psi^{ij}. \qquad \text{(A.0.22)}
\end{aligned}
$$

With the Hamiltonians identified we can construct the differential that describes the evolution on the phase-space

$$
\begin{aligned}
df &= \int \Big[\{f,H\}dt + \{f,\Omega^{00}\}d\omega^1_{00} + \{f,\Omega^{0i}\}d\omega^1_{0i} + \{f,\Omega^{ij}\}d\omega^1_{ij} + \{f,\tilde{\Omega}^{00}\}d\omega^2_{00}\\
&\quad + \{f,\tilde{\Omega}^{0i}\}d\omega^2_{0i} + \{f,\tilde{\Omega}^{ij}\}d\omega^2_{ij} + \{f,\Sigma^{00}\}d\omega^3_{00} + \{f,\Sigma^{0i}\}d\omega^3_{0i}\\
&\quad + \{f,\Sigma^{ij}\}d\omega^3_{ij}\Big]\, d^2y, \qquad \text{(A.0.23)}
\end{aligned}
$$

where the $\omega' s$ are parameters associated to the Hamiltonians. The non-zero Poisson algebra between all Hamiltonians is given by

$$\{\Omega^{00}, \Sigma_{00}\} = \delta^2(x-y), \quad (A.0.24)$$

$$\{\Omega^{0i}, \Sigma_{0j}\} = \delta^i_j \delta^2(x-y), \quad (A.0.25)$$

$$\{\Omega^{ij}, \Sigma_{kl}\} = \frac{1}{2}(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k)\delta^2(x-y), \quad (A.0.26)$$

$$\{\Omega^{0i}, \tilde{\Omega}^{kl}\} = \frac{1}{4}(\epsilon^{ik}\eta^{lm} + \epsilon^{il}\eta^{km} + \epsilon^{km}\eta^{il} + \epsilon^{lm}\eta^{ik})\partial_m \delta^2(x-y), \quad (A.0.27)$$

$$\{\tilde{\Omega}^{ij}, \tilde{\Omega}^{kl}\} = \frac{1}{4}(\epsilon^{ik}\eta^{jl} + \epsilon^{il}\eta^{jk} + \epsilon^{jk}\eta^{il} + \epsilon^{jl}\eta^{ik})\delta^2(x-y). \quad (A.0.28)$$

We can observe that the Hamiltonians Ω^{00}, Ω^{0i}, Ω^{ij}, $\tilde{\Omega}^{ij}$, Σ_{00}, Σ_{0i} and Σ_{ij} are non-involutive. This is expected because these Hamiltonians are related to the unphysical variales $\psi^{\mu\nu}$, these Hamiltonians will be removed by introducing the generalized HJ brackets. The Hamiltonians $\tilde{\Omega}^{00}$ and $\tilde{\Omega}^{0i}$ are involutive. Now, we need to remove all non-involutive Hamiltonians, the corresponding matrix of PB between non-involutive Hamiltonians is

$$\Delta_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \Gamma^{i,kl} & 0 & \eta^{ij} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k) \\ 0 & -\Gamma^{kl,i} & 0 & \Lambda^{ij,kl} & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\eta^{ij} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k) & 0 & 0 & 0 & 0 \end{pmatrix} \delta^2(x-y), \quad (A.0.29)$$

where we defined

$$\Gamma^{i,kl} \equiv \frac{1}{4}(\epsilon^{ik}\eta^{lm} + \epsilon^{il}\eta^{km} + \epsilon^{km}\eta^{il} + \epsilon^{lm}\eta^{ik})\partial_m, \quad (A.0.30)$$

$$\Lambda^{ij,kl} \equiv \frac{1}{4}(\epsilon^{ik}\eta^{jl} + \epsilon^{il}\eta^{jk} + \epsilon^{jk}\eta^{il} + \epsilon^{jl}\eta^{ik}). \quad (A.0.31)$$

The matrix Δ_{ab} is not invertible, which means that the Hamiltonians are not independent. In fact, there are null vectors given by $\vec{v} = (0, 0, 0, \varpi\eta_{ij}, 0, 0, 0)$, ϖ is an arbitrary function. From the contraction of the null vectors with the Hamiltonians we identify a new Hamiltonian

$$\tilde{\Omega} \equiv \tilde{\pi}^i{}_i = 0. \quad (A.0.32)$$

$\tilde{\Omega}$ is involutive, this imply a new algebra between all Hamilltonians, the new matrix Δ'_{ab} is given by

$$\Delta'_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}\partial_1 & -\frac{1}{2}\partial_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k) \\ 0 & \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\partial_1 & \frac{1}{2}\partial_2 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k) & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A.0.33}$$

its inverse $(\Delta'_{ab})^{-1}$ is

$$(\Delta'_{ab})^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}\partial_2 & -\frac{1}{2}\partial_1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}\partial_1 & -\frac{1}{2}\partial_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k) \\ 0 & \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\partial_1 & \frac{1}{2}\partial_2 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k) & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \delta^2(x-y). \tag{A.0.34}$$

Once we have $(\Delta'_{ab})^{-1}$ we can compute (3.4.4) for the phase-space coordinates

$$\{\xi_{00}, \pi^{00}\}^* = \delta^2(x-y), \tag{A.0.35}$$

$$\{\xi_{0i}, \pi^{0k}\}^* = \frac{1}{2}\delta^k_i\delta^2(x-y), \tag{A.0.36}$$

$$\{\xi_{ij}, \pi^{kl}\}^* = \frac{1}{2}(\delta^k_i\delta^l_j + \delta^l_i\delta^k_j)\delta^2(x-y), \tag{A.0.37}$$

$$\{\pi^{0i}, \pi^{0k}\}^* = \frac{1}{2}\epsilon^{ik}\nabla^2\delta^2(x-y), \tag{A.0.38}$$

$$\{\pi^{0i}, v_{kl}\}^* = -\frac{1}{2}[(\delta^1_k\delta^2_l + \delta^2_l\delta^1_k)(\epsilon^{1i}\partial^1 + \eta^{1i}\partial^2) - \delta^1_l\delta^1_m(\epsilon^{1i}\partial^2 + \epsilon^{2i}\partial^1 \tag{A.0.39}$$

$$+\eta^{2i}\partial^2 - \eta^{1i}\partial^1)]\delta^2(x-y). \tag{A.0.40}$$

With the introduction of the generalized brackets, the non-involutive Hamiltonians can be removed, then, the fundamental differential will be

$$df = \int \left(\{f, H\}^* dt + \{f, \tilde{\Omega}^{00}\}^* d\omega_{00}^2 + \{f, \tilde{\Omega}^{0i}\}^* d\omega_{0i}^2 + \{f, \tilde{\Omega}^* d\omega^2 \right) d^2y. \tag{A.0.41}$$

Once the generalized brackets are introduced, we could perform the substitution of the fields ψ by the momenta π. The canonical Hamiltonian takes the form

$$\begin{aligned} H_0 &= \epsilon^{ij} \left(-v_{ik}\partial_j\partial^k\xi_{00} - v_{0k}\partial_i\partial^k\xi_{0j} - \frac{1}{2}v_{0i}\nabla^2\xi_{0j} - \frac{1}{2}v_{ik}\nabla^2\xi_j{}^k - \frac{1}{2}v_i{}^k\partial_k\partial_l\xi_j{}^l \right. \\ &\left. -\partial_i\xi_{0j}\nabla^2\xi_{00} + \partial_i\xi_{jk}\nabla^2\xi_0{}^k + \partial_i\partial^l\xi_j{}^k\partial_k\xi_{0l} \right) - v_{00}\pi^{00} + 2v_{0i}\pi^{0i} - v_{ij}\pi^{ij}. \end{aligned} \tag{A.0.42}$$

From integrability conditions, which ensure the integrability of the system, the following Hamiltonians emerge

$$\begin{aligned} d\tilde{\Omega}^{00} &= \int \left(\{\tilde{\Omega}^{00}, H\}^* dt + \{\tilde{\Omega}^{00}, \tilde{\Omega}^{00}\}^* d\omega_{00}^2 + \{\tilde{\Omega}^{00}, \tilde{\Omega}^{0i}\}^* d\omega_{0i}^2 + \{\tilde{\Omega}^{00}, \tilde{\Omega}^* d\omega^2 \right) d^2y = 0 \\ &\Rightarrow \tilde{\Omega}_2^{00} \equiv \pi^{00} = 0, \qquad \text{(A.0.43)} \\ d\tilde{\Omega}^{0i} &= \int \left(\{\tilde{\Omega}^{0i}, H\}^* dt + \{\tilde{\Omega}^{0i}, \tilde{\Omega}^{00}\}^* d\omega_{00}^2 + \{\tilde{\Omega}^{0i}, \tilde{\Omega}^{0k}\}^* d\omega_{0k}^2 + \{\tilde{\Omega}^{0i}, \tilde{\Omega}^* d\omega^2 \right) d^2y = 0 \\ &\Rightarrow \tilde{\Omega}_2^{0i} \equiv \pi^{0i}\frac{1}{2}\epsilon^{jk}\partial^i\partial_j\xi_{0k} - \frac{1}{4}\epsilon^{ij}\nabla^2\xi_{0j} = 0, \qquad \text{(A.0.44)} \\ d\tilde{\Omega} &= \int \left(\{\tilde{\Omega}, H\}^* dt + \{\tilde{\Omega}, \tilde{\Omega}^{00}\}^* d\omega_{00}^2 + \{\tilde{\Omega}, \tilde{\Omega}^{0k}\}^* d\omega_{0k}^2 + \{\tilde{\Omega}, \tilde{\Omega}^* d\omega^2 \right) d^2y = 0 \\ &\Rightarrow \tilde{\Omega}_2 \equiv \pi^i{}_i - \frac{1}{2}\epsilon^{ij}\partial_i\partial^k\xi_{jk}. \qquad \text{(A.0.45)} \end{aligned}$$

The Hamiltonians $\tilde{\Omega}_2^{00}$, $\tilde{\Omega}_2^{0i}$ and $\tilde{\Omega}_2$ are involutive because the PB with $\tilde{\Omega}^{00}$, $\tilde{\Omega}^{0i}$, $\tilde{\Omega}$ and with themselves vanishes. Since the new Hamiltonians are involutive, we will add them to the fundamental differential, then its integrability will be calculated. Thus we obtain:

$$\begin{aligned}
d\tilde{\Omega}^{00}_2 &= \int(\{\tilde{\Omega}^{00}_2, H\}^* dt + \{\tilde{\Omega}^{00}_2, \tilde{\Omega}^{00}\}^* d\omega^2_{00} + \{\tilde{\Omega}^{00}_2, \tilde{\Omega}^{0i}\}^* d\omega^2_{0i} + \{\tilde{\Omega}^{00}_2, \tilde{\Omega}\}^* d\omega_2 \\
&\quad \{\tilde{\Omega}^{00}_2, \tilde{\Omega}^{00}_2\}^* d\tilde{\omega}_{00} + \{\tilde{\Omega}^{00}_2, \tilde{\Omega}^{0i}_2\}^* d\tilde{\omega}_{0i} + \{\tilde{\Omega}^{00}_2, \tilde{\Omega}_2\}^* d\tilde{\omega}) d^2y \\
&\Rightarrow \tilde{\Omega}^{00}_3 \equiv \epsilon^{ij}\partial_j\partial^k v_{ik} + \epsilon^{ij}\partial_i\nabla^2\xi_{0j} = 0, \qquad (A.0.46) \\
d\tilde{\Omega}^{0i}_2 &= \int(\{\tilde{\Omega}^{0i}_2, H\}^* dt + \{\tilde{\Omega}^{0i}_2, \tilde{\Omega}^{00}\}^* d\omega^2_{00} + \{\tilde{\Omega}^{0i}_2, \tilde{\Omega}^{0i}\}^* d\omega^2_{0i} + \{\tilde{\Omega}^{0i}_2, \tilde{\Omega}\}^* d\omega_2 \\
&\quad \{\tilde{\Omega}^{0i}_2, \tilde{\Omega}^{00}_2\}^* d\tilde{\omega}_{00} + \{\tilde{\Omega}^{0i}_2, \tilde{\Omega}^{0i}_2\}^* d\tilde{\omega}_{0i} + \{\tilde{\Omega}^{0i}_2, \tilde{\Omega}_2\}^* d\tilde{\omega}) d^2y \\
&\Rightarrow \tilde{\Omega}^{0i}_3 \equiv \partial_j\pi^{ij} - \frac{1}{4}\epsilon^{jk}\partial_j\nabla^2\xi^i{}_k - \frac{1}{4}\epsilon^{jk}\partial^i\partial_j\partial_l\xi^l{}_k, \qquad (A.0.47) \\
d\tilde{\Omega}_2 &= \int(\{\tilde{\Omega}_2, H\}^* dt + \{\tilde{\Omega}_2, \tilde{\Omega}^{00}\}^* d\omega^2_{00} + \{\tilde{\Omega}_2, \tilde{\Omega}^{0i}\}^* d\omega^2_{0i} + \{\tilde{\Omega}_2, \tilde{\Omega}\}^* d\omega_2 \\
&\quad \{\tilde{\Omega}_2, \tilde{\Omega}^{00}_2\}^* d\tilde{\omega}_{00} + \{\tilde{\Omega}_2, \tilde{\Omega}^{0i}_2\}^* d\tilde{\omega}_{0i} + \{\tilde{\Omega}_2, \tilde{\Omega}_2\}^* d\tilde{\omega}) d^2y \\
&\Rightarrow \tilde{\Omega}^{00}_3 = 0, \qquad (A.0.48)
\end{aligned}$$

the new third generation of Hamiltonians $\tilde{\Omega}^{00}_3$ and $\tilde{\Omega}^{0i}_3$ are involutive ones and from their integrability we find no further Hamiltonians. The final fundamental differential is given by

$$\begin{aligned}
df &= \int[\{f, H\}^* dt + \{f, \tilde{\Omega}^{00}\} d\omega^2_{00} + \{f, \tilde{\Omega}^{0i}\}^* d\omega^2_{0i} + \{f, \tilde{\Omega}\}^* d\omega^2 + \{f, \tilde{\Omega}^{00}_2\}^* d\tilde{\omega}_{00} \\
&\quad + \{f, \tilde{\Omega}^{0i}_2\}^* d\tilde{\omega}_{0i} + \{f, \tilde{\Omega}_2\}^* d\tilde{\omega} + \{f, \tilde{\Omega}^{00}_3\}^* d\tilde{\omega}^{00}_3 + \{f, \tilde{\Omega}^{0i}_3\}^* d\tilde{\omega}^3_{0i}] d^2y \qquad (A.0.49)
\end{aligned}$$

where the involutive Hamiltonians are:

$$\begin{aligned}
\tilde{\Omega}^{00} &\equiv \tilde{\pi}^{00} = 0, && (A.0.50) \\
\tilde{\Omega}^{0i} &\equiv \tilde{\pi}^{0i} = 0, && (A.0.51) \\
\tilde{\Omega} &\equiv \tilde{\pi}^i{}_i = 0, && (A.0.52) \\
\tilde{\Omega}^{00}_2 &\equiv \pi^{00} = 0, && (A.0.53) \\
\tilde{\Omega}^{0i}_2 &\equiv \pi^{0i} - \frac{1}{2}\epsilon^{jk}\partial^i\partial_j\xi_{0k} - \frac{1}{4}\epsilon^{ij}\nabla^2\xi_{0j} = 0, && (A.0.54) \\
\tilde{\Omega}_2 &\equiv \pi^i{}_i - \frac{1}{2}\epsilon^{ij}\partial_i\partial^k\xi_{jk} = 0, && (A.0.55) \\
\tilde{\Omega}^{00}_3 &\equiv \epsilon^{ij}\partial_j\partial^k v_{ik} + \epsilon^{ij}\partial_i\nabla^2\xi_{0j} = 0, && (A.0.56) \\
\tilde{\Omega}^{0i}_3 &\equiv \partial_j\pi^{ij} - \frac{1}{4}\epsilon^{jk}\partial_j\nabla^2\xi^i{}_k - \frac{1}{4}\epsilon^{jk}\partial^i\partial_j\partial_l\xi^l{}_k. && (A.0.57)
\end{aligned}$$

From the fundamental differential we can calculate the characteristics which will reveal the symmetries of the theory. The characteristic equations are:

$$
\begin{aligned}
d\xi_{00} &= -v_{00}dt - d\tilde{\omega}_{00}, &\text{(A.0.58)}\\
d\xi_{0i} &= v_{0i}dt + \frac{1}{2}d\tilde{\omega}_{0i}, &\text{(A.0.59)}\\
d\xi_{ij} &= v_{ij}dt + \eta_{ij}d\tilde{\omega} - \frac{1}{2}\partial_i d\tilde{\omega}^3_{0j} - \frac{1}{2}\partial_j d\tilde{\omega}^3_{0i}. &\text{(A.0.60)}
\end{aligned}
$$

If we set $dt = 0$ we obtain a set of canonical transformations that can be related to the gauge ones;

$$
\begin{aligned}
\delta\xi_{00} &= -\delta\tilde{\omega}_{00}, &\text{(A.0.61)}\\
\delta\xi_{0i} &= \frac{1}{2}\delta\tilde{\omega}_{0i}, &\text{(A.0.62)}\\
\delta\xi_{ij} &= \eta_{ij}\delta\tilde{\omega} - \frac{1}{2}\partial_i\delta\tilde{\omega}^3_{0i} - \frac{1}{2}\partial_j\delta\tilde{\omega}^3_{0i}. &\text{(A.0.63)}
\end{aligned}
$$

In order to identify the corresponding gauge symmetries its necessary to find the conditions in which (A.0.61)-(A.0.63) acts into the Lagrangian, such conditions arise if the Lagrangian is invariant under these transformations. The variation of the Lagrangian is

$$\delta L_{CS} = \int dtd^2x\epsilon^{\alpha\mu\nu}[\partial^\rho\partial_\rho\partial_\mu\xi^\beta{}_\nu - \partial^\sigma\partial^\beta\partial_\mu\xi_{\sigma\nu}]\delta\xi_{\alpha\beta} = 0, \tag{A.0.64}$$

this will lead to the following relations between the $\omega's$ parameters:

$$
\begin{aligned}
\delta\tilde{\omega}_{00} &= -2\partial_0\zeta_0, &\text{(A.0.65)}\\
\delta\tilde{\omega}_{0i} &= 2(\partial_0\zeta_i + \partial_i\zeta_0), &\text{(A.0.66)}\\
\eta_{ij}\delta\tilde{\omega} &= 2(\partial_i\zeta_j + \partial_j\zeta_i), &\text{(A.0.67)}\\
\delta\tilde{\omega}^3_{0i} &= 2\zeta_i, &\text{(A.0.68)}
\end{aligned}
$$

by substituting these relations in the canonical transformations we get

$$\delta\xi_{\mu\nu} = \partial_\mu\zeta_\nu + \partial_\nu\zeta_\mu. \tag{A.0.69}$$

Appendix B

Conformal invariance of $\sqrt{-g}\mathcal{C}^{ab}$

Let be the tensor $\sqrt{-g}\mathcal{C}^{\mu\nu}$

$$\sqrt{-g}\mathcal{C}^{\mu\nu} \equiv v_\sigma(\epsilon^{\sigma\mu\alpha\beta}D_\alpha R^\nu{}_\beta + \epsilon^{\sigma\nu\alpha\beta}D_\alpha R^\mu{}_\beta) + v_{\sigma\tau}({}^*R^{\tau\mu\sigma\nu} + {}^*R^{\tau\nu\sigma\mu}), \tag{B.0.1}$$

in order to show that this tensor is conformal invariant in the infinitesimal sense is sufficient to show that

$$\delta C^{\mu\nu} = 0, \tag{B.0.2}$$

under the change

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = (1+\epsilon\pi)g_{\mu\nu}, \qquad \epsilon \ll 1, \tag{B.0.3}$$

where π is a function on the space-time manifold. The function π is directly related to the *conformal factor* λ that defines the conformal transformation: $\tilde{g}_{\mu\nu} = \lambda g_{\mu\nu} = \mathrm{e}^{\epsilon\pi} g_{\mu\nu}$.
Now, we need to specify the variation of the metric tensor and the variation of all objects that are made of $g_{\mu\nu}$, we have that

$$\delta g_{\mu\nu} = \tilde{g}_{\mu\nu} - g_{\mu\nu} = \epsilon\pi g_{\mu\nu}. \tag{B.0.4}$$

The Christoffel symbol $\Gamma_{\mu\nu}$ variate as follows

$$\begin{aligned}
\delta\Gamma^\tau_{\mu\nu} &= \delta\left[\frac{1}{2}g^{\tau\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})\right] \\
&= \frac{1}{2}(\delta g^{\tau\rho})(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) + \frac{1}{2}g^{\tau\rho}\delta(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \\
&= -\frac{\epsilon}{2}\pi g^{\tau\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) + \frac{\epsilon}{2}g^{\tau\rho}[\partial_\mu(\pi g_{\rho\nu}) + \partial_\nu(\pi g_{\rho\mu}) - \partial_\rho(\pi g_{\mu\nu})] \\
&= -\epsilon\pi\Gamma^\tau_{\mu\nu} + \epsilon\pi\Gamma^\tau_{\mu\nu} + \frac{\epsilon}{2}g^{\tau\rho}[(\partial_\mu\pi)g_{\rho\nu} + (\partial_\nu\pi)g_{\rho\mu} - (\partial_\rho\pi)g_{\mu\nu}] \\
&= \frac{\epsilon}{2}g^{\tau\rho}[(\partial_\mu\pi)g_{\rho\nu} + (\partial_\nu\pi)g_{\rho\mu} - (\partial_\rho\pi)g_{\mu\nu}]
\end{aligned}$$

$$\Rightarrow \delta\Gamma^{\tau}_{\mu\nu} = \frac{\epsilon}{2} g^{\tau\rho}[(\partial_\mu \pi) g_{\rho\nu} + (\partial_\nu \pi) g_{\rho\mu} - (\partial_\rho \pi) g_{\mu\nu}]. \quad \text{(B.0.5)}$$

In the same manner, the Riemann tensor changes an ammount $\delta R^{\tau}{}_{\mu\sigma\nu}$,

$$\delta R^{\tau}{}_{\mu\sigma\nu} = \partial_\sigma \delta\Gamma^{\tau}_{\nu\mu} - \partial_\nu \delta\Gamma^{\tau}_{\sigma\mu} + (\delta\Gamma^{\tau}_{\sigma\eta})\Gamma^{\eta}_{\nu\mu} + \Gamma^{\tau}_{\sigma\eta}(\delta\Gamma^{\eta}_{\nu\mu}) - (\delta\Gamma^{\tau}_{\nu\eta})\Gamma^{\eta}_{\sigma\mu} - \Gamma^{\tau}_{\nu\eta}(\delta\Gamma^{\eta}_{\sigma\mu}) \quad \text{(B.0.6)}$$

but

$$\begin{aligned}
\partial_\sigma \delta\Gamma^{\tau}_{\nu\mu} &= \frac{\epsilon}{2}[(\partial_\sigma g^{\tau\rho})[(\partial_\nu \pi) g_{\rho\mu} + (\partial_\mu \pi) g_{\rho\nu} - (\partial_\rho \pi) g_{\nu\mu}] + g^{\tau\rho}[(\partial_\sigma \partial_\nu \pi) g_{\rho\mu} + (\partial_\sigma \partial_\mu \pi) g_{\rho\nu} - (\partial_\sigma \partial_\rho \pi) g_{\nu\mu}] \\
&\quad + g^{\tau\rho}[(\partial_\nu \pi)(\partial_\sigma g_{\rho\mu}) + (\partial_\mu \pi)(\partial_\sigma g_{\rho\nu}) - (\partial_\rho \pi)(\partial_\sigma g_{\nu\mu})] \\
&= \frac{\epsilon}{2} g^{\tau\rho}[(\partial_\sigma \partial_\nu \pi) g_{\rho\mu} + (\partial_\sigma \partial_\mu \pi) g_{\rho\nu} - (\partial_\sigma \partial_\rho \pi) g_{\nu\mu}]
\end{aligned}$$

$$\Rightarrow \partial_\sigma \delta\Gamma^{\tau}_{\nu\mu} = \frac{\epsilon}{2} g^{\tau\rho}[(\partial_\sigma \partial_\nu \pi) g_{\rho\mu} + (\partial_\sigma \partial_\mu \pi) g_{\rho\nu} - (\partial_\sigma \partial_\rho \pi) g_{\nu\mu} - (\partial_\rho \pi)\partial_\sigma (g_{\mu\nu} g^{\tau\rho})] \quad \text{(B.0.7)}$$

and

$$\partial_\nu \delta\Gamma^{\tau}_{\sigma\mu} = \frac{\epsilon}{2} g^{\tau\rho}[(\partial_\nu \partial_\sigma \pi) g_{\rho\mu} + (\partial_\nu \partial_\mu \pi) g_{\rho\sigma} - (\partial_\nu \partial_\rho \pi) g_{\sigma\mu} - (\partial_\rho \pi)\partial_\nu (g_{\mu\sigma} g^{\tau\rho})]. \quad \text{(B.0.8)}$$

On the other hand

$$\begin{aligned}
(\delta\Gamma^{\tau}_{\sigma\eta})\Gamma^{\eta}_{\nu\mu} &= \frac{\epsilon}{2} g^{\tau\rho}[(\partial_\sigma \pi) g_{\rho\eta} + (\partial_\eta \pi) g_{\rho\sigma} - (\partial_\rho \pi) g_{\sigma\eta}]\Gamma^{\eta}_{\nu\mu} \\
&= \frac{\epsilon}{2}[(\partial_\sigma \pi)\Gamma^{\tau}_{\nu\mu} + (\partial_\eta \pi)\Gamma^{\eta}_{\nu\mu}\delta^{\tau}_{\sigma} - g^{\tau\rho} g_{\sigma\eta}(\partial_\rho \pi)\Gamma^{\eta}_{\nu\mu}], \quad \text{(B.0.9)}
\end{aligned}$$

$$\begin{aligned}
\Gamma^{\tau}_{\sigma\eta}(\delta\Gamma^{\eta}_{\nu\mu}) &= \frac{\epsilon}{2}\Gamma^{\tau}_{\sigma\eta} g^{\eta\rho}[(\partial_\nu \pi) g_{\rho\mu} + (\partial_\mu \pi) g_{\rho\nu} - (\partial_\rho \pi) g_{\nu\mu}] \\
&= \frac{\epsilon}{2}[(\partial_\nu \pi)\Gamma^{\tau}_{\sigma\mu} + (\partial_\mu \pi)\Gamma^{\tau}_{\sigma\nu} - g^{\eta\rho} g_{\nu\mu}(\partial_\rho \pi)\Gamma^{\tau}_{\sigma\eta}], \quad \text{(B.0.10)}
\end{aligned}$$

$$(\delta\Gamma^{\tau}_{\nu\eta})\Gamma^{\eta}_{\sigma\mu} = \frac{\epsilon}{2}[(\partial_\nu \pi)\Gamma^{\tau}_{\sigma\mu} + (\partial_\eta \pi)\Gamma^{\eta}_{\sigma\mu}\delta^{\tau}_{\nu} - g^{\tau\rho} g_{\nu\eta}(\partial_\rho \pi)\Gamma^{\eta}_{\sigma\mu}], \quad \text{(B.0.11)}$$

$$\Gamma^{\tau}_{\nu\eta}(\delta\Gamma^{\eta}_{\sigma\mu}) = \frac{\epsilon}{2}[(\partial_\sigma \pi)\Gamma^{\tau}_{\nu\mu} + (\partial_\mu \pi)\Gamma^{\tau}_{\nu\sigma} - g^{\eta\rho} g_{\sigma\mu}(\partial_\rho \pi)\Gamma^{\tau}_{\nu\eta}], \quad \text{(B.0.12)}$$

by substituting (B.0.7), (B.0.8) and (B.0.9)-(B.0.12) in the variation of the Riemann tensor we get

$$\begin{aligned}
\delta R^{\tau}{}_{\mu\sigma\nu} &= \frac{\epsilon}{2}\{(\partial_\sigma\partial_\mu\pi)\delta^\tau_\nu - g^{\tau\rho}g_{\nu\mu}(\partial_\sigma\partial_\rho\pi) - (\partial_\nu\partial_\mu\pi)\delta^\tau_\sigma + g^{\tau\rho}g_{\sigma\mu}(\partial_\nu\partial_\rho\pi) - (\partial_\rho\pi)[\partial_\sigma(g_{\mu\nu}g^{\tau\rho}) \\
&\quad -\partial_\nu(g_{\mu\sigma}g^{\tau\rho})] + (\partial_\sigma\pi)\Gamma^\tau_{\nu\mu} + (\partial_\eta\pi)\Gamma^\eta_{\nu\mu}\delta^\tau_\sigma - g^{\tau\rho}g_{\sigma\eta}(\partial_\rho\pi)\Gamma^\eta_{\nu\mu} + (\partial_\nu\pi)\Gamma^\tau_{\sigma\mu} + (\partial_\mu\pi)\Gamma^\tau_{\sigma\nu} \\
&\quad -g^{\eta\rho}g_{\nu\mu}(\partial_\rho\pi)\Gamma^\tau_{\sigma\eta} - (\partial_\nu\pi)\Gamma^\tau_{\sigma\mu} - (\partial_\eta\pi)\Gamma^\eta_{\sigma\mu}\delta^\tau_\nu + g^{\tau\rho}g_{\nu\eta}(\partial_\rho\pi)\Gamma^\eta_{\sigma\mu} - (\partial_\sigma\pi)\Gamma^\tau_{\nu\mu} - (\partial_\mu\pi)\Gamma^\tau_{\nu\sigma} \\
&\quad +g^{\eta\rho}g_{\sigma\mu}(\partial_\rho\pi)\Gamma^\tau_{\nu\eta}\} \\
&= \frac{\epsilon}{2}\{(\partial_\sigma\partial_\mu\pi - \Gamma^\eta_{\sigma\mu}\partial_\eta\pi)\delta^\tau_\nu - (\partial_\nu\partial_\mu\pi - \Gamma^\eta_{\nu\mu}\partial_\eta\pi)\delta^\tau_\sigma - g_{\mu\nu}[g^{\tau\rho}(\partial_\sigma\partial_\rho\pi) + \Gamma^\tau_{\sigma\eta}g^{\eta\rho}(\partial_\rho\pi)] \\
&\quad +g_{\mu\sigma}[g^{\tau\rho}(\partial_\nu\partial_\rho\pi) + \Gamma^\tau_{\nu\eta}g^{\eta\rho}(\partial_\rho\pi)] - (\partial_\rho\pi)[\partial_\sigma(g_{\mu\nu}g^{\tau\rho}) - \partial_\nu(g_{\mu\sigma}g^{\tau\rho})] - g^{\tau\rho}g_{\sigma\eta}(\partial_\rho\pi)\Gamma^\eta_{\nu\mu} \\
&\quad +g^{\tau\rho}g_{\nu\eta}(\partial_\rho\pi)\Gamma^\eta_{\sigma\mu}\} \\
&= \frac{\epsilon}{2}\{(\partial_\sigma\partial_\mu\pi - \Gamma^\eta_{\sigma\mu}\partial_\eta\pi)\delta^\tau_\nu - (\partial_\nu\partial_\mu\pi - \Gamma^\eta_{\nu\mu}\partial_\eta\pi)\delta^\tau_\sigma - g_{\mu\nu}[\partial_\sigma(g^{\tau\rho}\partial_\rho\pi) - (\partial_\sigma g^{\tau\rho})(\partial_\rho\pi) \\
&\quad +\Gamma^\tau_{\sigma\eta}g^{\eta\rho}\partial_\rho\pi] + g_{\mu\sigma}[\partial_\nu(g^{\tau\rho}\partial_\rho\pi) - (\partial_\nu g^{\tau\rho})(\partial_\rho\pi) + \Gamma^\tau_{\nu\eta}g^{\eta\rho}\partial_\rho\pi] - (\partial_\rho\pi)[\partial_\sigma(g_{\mu\nu}g^{\tau\rho}) \\
&\quad -\partial_\nu(g_{\mu\sigma}g^{\tau\rho})]\} - g^{\tau\rho}g_{\sigma\eta}(\partial_\rho\pi)\Gamma^\eta_{\nu\mu} + g^{\tau\rho}g_{\nu\eta}(\partial_\rho\pi)\Gamma^\eta_{\sigma\mu} \\
&= \frac{\epsilon}{2}\{(\partial_\sigma\partial_\mu\pi - \Gamma^\eta_{\sigma\mu}\partial_\eta\pi)\delta^\tau_\nu - (\partial_\nu\partial_\mu\pi - \Gamma^\eta_{\nu\mu}\partial_\eta\pi)\delta^\tau_\sigma - g_{\mu\nu}[\partial_\sigma(g^{\tau\rho}\partial_\rho\pi) + \Gamma^\tau_{\sigma\eta}(g^{\eta\rho}\partial_\rho\pi)] \\
&\quad +g_{\mu\nu}(\partial_\sigma g^{\tau\rho})(\partial_\rho\pi) + g_{\mu\sigma}[\partial_\nu(g^{\tau\rho}\partial_\rho\pi) + \Gamma^\tau_{\nu\eta}(g^{\eta\rho}\partial_\rho\pi)] - g_{\mu\sigma}(\partial_\nu g^{\tau\rho})(\partial_\rho\pi) \\
&\quad -(\partial_\rho\pi)[\partial_\sigma(g_{\mu\nu}g^{\tau\rho}) - \partial_\nu(g_{\mu\sigma}g^{\tau\rho})]\} - g^{\tau\rho}g_{\sigma\eta}(\partial_\rho\pi)\Gamma^\eta_{\nu\mu} + g^{\tau\rho}g_{\nu\eta}(\partial_\rho\pi)\Gamma^\eta_{\sigma\mu} \\
&= \frac{\epsilon}{2}[\nabla_\sigma\partial_\mu\pi\delta^\tau_\nu - \nabla_\nu\partial_\mu\pi\delta^\tau_\sigma - g_{\mu\nu}\nabla_\sigma(g^{\tau\rho}\partial_\rho\pi) + g_{\mu\sigma}\nabla_\nu(g^{\tau\rho}\partial_\rho\pi) + g_{\mu\nu}(\partial_\sigma g^{\tau\rho})(\partial_\rho\pi) \\
&\quad -g_{\mu\sigma}(\partial_\nu g^{\tau\rho})(\partial_\rho\pi) - (\partial_\rho\pi)[\partial_\sigma(g_{\mu\nu}g^{\tau\rho}) - \partial_\nu(g_{\mu\sigma}g^{\tau\rho})] - g^{\tau\rho}g_{\sigma\eta}(\partial_\rho\pi)\Gamma^\eta_{\nu\mu} + g^{\tau\rho}g_{\nu\eta}(\partial_\rho\pi)\Gamma^\eta_{\sigma\mu}] \\
&= \frac{\epsilon}{2}[\nabla_\sigma(\partial_\mu\pi)\delta^\tau_\nu - \nabla_\nu(\partial_\mu\pi)\delta^\tau_\sigma - g_{\mu\nu}g^{\tau\rho}\nabla_\sigma(\partial_\rho\pi) + g_{\mu\sigma}g^{\tau\rho}\nabla_\nu(\partial_\rho\pi) - (\partial_\rho\pi)[\partial_\sigma(g_{\mu\nu}g^{\tau\rho}) \\
&\quad -\partial_\nu(g_{\mu\sigma}g^{\tau\rho})] + g_{\mu\nu}(\partial_\sigma g^{\tau\rho})(\partial_\rho\pi) - g_{\mu\sigma}(\partial_\nu g^{\tau\rho})(\partial_\rho\pi) - g^{\tau\rho}(\partial_\rho\pi)\Gamma_{\sigma\nu\mu} + g^{\tau\rho}(\partial_\rho\pi)\Gamma_{\nu\sigma\mu}] \\
&= \frac{\epsilon}{2}\{\nabla_\sigma(\partial_\mu\pi)\delta^\tau_\nu - \nabla_\nu(\partial_\mu\pi)\delta^\tau_\sigma - g_{\mu\nu}g^{\tau\rho}\nabla_\sigma(\partial_\rho\pi) + g_{\mu\sigma}g^{\tau\rho}\nabla_\nu(\partial_\rho\pi) - (\partial_\rho\pi)[\partial_\sigma(g_{\mu\nu}g^{\tau\rho}) \\
&\quad -\partial_\nu(g_{\mu\sigma}g^{\tau\rho})] + (\partial_\rho\pi)[g_{\mu\nu}(\partial_\sigma g^{\tau\rho}) - g_{\mu\sigma}(\partial_\nu g^{\tau\rho}) - g^{\tau\rho}\Gamma_{\sigma\nu\mu} + g^{\tau\rho}\Gamma_{\nu\sigma\mu}]\} \\
&= \frac{\epsilon}{2}\{\nabla_\sigma(\partial_\mu\pi)\delta^\tau_\nu - \nabla_\nu(\partial_\mu\pi)\delta^\tau_\sigma - g_{\mu\nu}g^{\tau\rho}\nabla_\sigma(\partial_\rho\pi) + g_{\mu\sigma}g^{\tau\rho}\nabla_\nu(\partial_\rho\pi) - (\partial_\rho\pi)[\partial_\sigma(g_{\mu\nu}g^{\tau\rho}) \\
&\quad -\partial_\nu(g_{\mu\sigma}g^{\tau\rho})] + (\partial_\rho\pi)[\partial_\sigma(g_{\mu\nu}g^{\tau\rho}) - \partial_\nu(g_{\mu\sigma}g^{\tau\rho})]\} \\
&= \frac{\epsilon}{2}[\nabla_\sigma(\partial_\mu\pi)\delta^\tau_\nu - \nabla_\nu(\partial_\mu\pi)\delta^\tau_\sigma - g_{\mu\nu}g^{\tau\rho}\nabla_\sigma(\partial_\rho\pi) + g_{\mu\sigma}g^{\tau\rho}\nabla_\nu(\partial_\rho\pi)]
\end{aligned}$$

thus we have

$$\delta R^{\tau}{}_{\mu\sigma\nu} = \frac{\epsilon}{2}[\nabla_\sigma(\partial_\mu\pi)\delta^\tau_\nu - \nabla_\nu(\partial_\mu\pi)\delta^\tau_\sigma - g_{\mu\nu}g^{\tau\rho}\nabla_\sigma(\partial_\rho\pi) + g_{\mu\sigma}g^{\tau\rho}\nabla_\nu(\partial_\rho\pi)]. \tag{B.0.13}$$

With equation (B.0.13) its straightforward to obtain the variaton of the Ricci $R_{\mu\nu} = R^{\tau}{}_{\mu\tau\nu}$ tensor and of the scalar curvature $R = R^{\mu}{}_{\mu}$;

$$\begin{aligned} \delta R_{\mu\nu} &= \frac{\epsilon}{2}[\nabla_\tau(\partial_\mu\pi)\delta^\tau_\nu - \nabla_\nu(\partial_\mu\pi)\delta^\tau_\tau - g_{\mu\nu}g^{\tau\rho}\nabla_\tau(\partial_\rho\pi) + g_{\mu\tau}g^{\tau\rho}\nabla_\nu(\partial_\rho\pi)] \\ &= \frac{\epsilon}{2}[-\nabla_\nu(\partial_\mu\pi) - g_{\mu\nu}\nabla_\alpha(\partial^\alpha\pi)], \end{aligned} \tag{B.0.14}$$

$$\begin{aligned} \delta R &= \delta(g^{\mu\nu}R_{\mu\nu}) \\ &= -\epsilon\pi g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\frac{\epsilon}{2}[-\nabla_\nu(\partial_\mu\pi) - g_{\mu\nu}\nabla_\alpha(\partial^\alpha\pi)] \\ &= -\epsilon\pi R - 2\epsilon\nabla_\alpha(\partial^\alpha\pi). \end{aligned} \tag{B.0.15}$$

Finally, the variation of the tensor $\sqrt{-g}\mathcal{C}_{\mu\nu}$ goes as follows

$$\begin{aligned} \delta\sqrt{-g}C^{\mu\nu} &= -\frac{1}{2\sqrt{-g}}[v_\sigma(\epsilon^{\sigma\mu\alpha\beta}D_\alpha\delta R^{\nu}{}_{\beta} + \epsilon^{\sigma\nu\alpha\beta}D_\alpha\delta R^{\mu}{}_{\beta}) + v_{\sigma\tau}(\frac{1}{2}\epsilon^{\sigma\nu\alpha\beta}\delta R^{\tau\mu}{}_{\alpha\beta} + \frac{1}{2}\epsilon^{\sigma\mu\alpha\beta}\delta R^{\tau\nu}{}_{\alpha\beta})] \\ &= -\frac{\epsilon}{4\sqrt{-g}}[v_\sigma[\epsilon^{\sigma\mu\alpha\beta}D_\alpha(-D_\beta(\partial^\nu\pi) - g^{\nu}{}_{\beta}D_\rho(\partial^\rho\pi)) + \epsilon^{\sigma\nu\alpha\beta}D_\alpha(-D_\beta(\partial^\mu\pi) - g^{\mu}{}_{\beta}D_\rho(\partial^\rho\pi))] \\ &\quad +\frac{1}{2}v_{\sigma\tau}[\epsilon^{\sigma\nu\alpha\beta}(D_\alpha(\partial^\mu\pi)\delta^{\tau}{}_{\beta} - D_\beta(\partial^\mu\pi)\delta^{\tau}{}_{\alpha} - g^{\mu}{}_{\beta}g^{\tau\rho}D_\alpha(\partial_\rho\pi) + g^{\mu}{}_{\alpha}g^{\tau\rho}D_\beta(\partial_\rho\pi)) \\ &\quad +\epsilon^{\sigma\mu\alpha\beta}(D_\alpha(\partial^\nu\pi)\delta^{\tau}{}_{\beta} - D_\beta(\partial^\nu\pi)\delta^{\tau}{}_{\alpha} - g^{\nu}{}_{\beta}g^{\tau\rho}D_\alpha(\partial_\rho\pi) + g^{\nu}{}_{\alpha}g^{\tau\rho}D_\beta(\partial_\rho\pi))]] \\ &= -\frac{\epsilon}{4\sqrt{-g}}[v_\sigma[-\epsilon^{\sigma\mu\alpha\beta}g^{\nu}{}_{\beta}D_\alpha D_\rho(\partial^\rho\pi) - \epsilon^{\sigma\nu\alpha\beta}g^{\mu}{}_{\beta}D_\alpha D_\rho(\partial^\rho\pi)] + \frac{1}{2}v_{\sigma\tau}[\epsilon^{\sigma\nu\alpha\beta}(-g^{\mu}{}_{\beta}g^{\tau\rho}D_\alpha(\partial_\rho\pi) \\ &\quad +g^{\mu}{}_{\alpha}g^{\tau\rho}D_\beta(\partial_\rho\pi)) + \epsilon^{\sigma\mu\alpha\beta}(-g^{\nu}{}_{\beta}g^{\tau\rho}D_\alpha(\partial_\rho\pi) + g^{\nu}{}_{\alpha}g^{\tau\rho}D_\beta(\partial_\rho\pi))]] \\ &= \frac{\epsilon}{4\sqrt{-g}}[v_\sigma[\epsilon^{\sigma\mu\alpha\nu}D_\alpha D_\rho(\partial^\rho\pi) + \epsilon^{\sigma\nu\alpha\mu}D_\alpha D_\rho(\partial^\rho\pi)] + \frac{1}{2}v_{\sigma\tau}[\epsilon^{\sigma\nu\alpha\mu}D_\alpha(\partial^\tau\pi) - \epsilon^{\sigma\nu\mu\beta}D_\beta(\partial^\tau\pi)) \\ &\quad +\epsilon^{\sigma\mu\alpha\nu}D_\alpha(\partial^\tau\pi) - \epsilon^{\sigma\mu\nu\beta}D_\beta(\partial^\tau\pi)]] \\ &= \frac{\epsilon}{4\sqrt{-g}}[v_\sigma[\epsilon^{\sigma\mu\alpha\nu}D_\alpha D_\rho(\partial^\rho\pi) - \epsilon^{\sigma\mu\alpha\nu}D_\alpha D_\rho(\partial^\rho\pi)] + \frac{1}{2}v_{\sigma\tau}[\epsilon^{\sigma\nu\alpha\mu}D_\alpha(\partial^\tau\pi) + \epsilon^{\sigma\nu\alpha\mu}D_\alpha(\partial^\tau\pi)) \\ &\quad -\epsilon^{\sigma\nu\alpha\mu}D_\alpha(\partial^\tau\pi) - \epsilon^{\sigma\nu\alpha\mu}D_\alpha(\partial^\tau\pi)]] \\ &= 0. \end{aligned}$$

$$\Rightarrow \delta\sqrt{-g}C^{\mu\nu} = 0. \tag{B.0.16}$$

Therefore, the tensor $\sqrt{-g}C^{\mu\nu}$ is invariant under a local conformal transformation $g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = (1+\epsilon\pi)g_{\mu\nu}$.

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Asunto: **Oficio de modalidad de titulación.**

MTRO. RICARDO VALDERRAMA VALDEZ
Director de Administración Escolar
Benemérita Universidad Autónoma de Puebla
Presente

El que suscribe, Director del Instituto de Física "Ing. Luis Rivera Terrazas", le informo que **JESÚS ALDAIR PANTOJA GONZÁLEZ**, matrícula: 219570768, presentará y defenderá su examen de grado de **DOCTORADO EN CIENCIAS (FÍSICA)** en la **MODALIDAD DE PRESENTACIÓN DE TESIS**, cuyo título es: **"ANÁLISIS DE LAS SIMETRÍAS DE TEORÍAS MODIFICADAS DE GRAVEDAD: MODIFICACIÓN DE CHERN-SIMONS CUATRIDIMENSIONAL"**, que se llevará a cabo el día **martes 11 de julio de 2023 a las 11:00 horas**, en el auditorio de este Instituto. El Jurado Examinador estará integrado por:

Dr. Hugo Aurelio Morales Técotl.	**Presidente– en línea -**
Dr. Gerardo Francisco Torres del Castillo.	**Secretario**
Dr. Alfredo Herrera Aguilar.	**Vocal**
Dr. Valentín García Vázquez.	**Vocal**
Dr. Alberto Escalante Hernández	**Vocal**

Sin otro asunto que el particular, aprovecho la ocasión para enviarle un cordial saludo.

A T E N T A M E N T E
"Pensar Bien, Para Vivir Mejor"
Puebla, Pue., a 03 de julio de 2023

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DR. FELIPE PÉREZ RODRÍGUEZ
DIRECTOR

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MTRO. ALFREDO AVENDAÑO ARENAZA
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PRESENTE

El que suscribe, Director del Instituto de Física "Ing. Luis Rivera Terrazas" de la Benemérita Universidad Autónoma de Puebla, **AVALA** con base en los archivos que se encuentran en este Instituto que la tesis del **M.C. JESÚS ALDAIR PANTOJA GONZÁLEZ**, matrícula: 219570768, cuyo título es: **"ANÁLISIS DE LAS SIMETRÍAS DE TEORÍAS MODIFICADAS DE GRAVEDAD: MODIFICACIÓN DE CHERN-SIMONS CUATRIDIMENSIONAL"**, ha sido *revisada* y *aprobada* por el Comité respectivo para presentar su examen de grado de **DOCTORADO EN CIENCIAS (FÍSICA)** el próximo día **martes 11 de julio de 2023 a las 11:00 horas** en el auditorio del IFUAP. Dicho Comité está integrado por los siguientes profesores:

Dr. Hugo Aurelio Morales Técotl	**Presidente– en línea -**
Dr. Gerardo Francisco Torres del Castillo.	**Secretario**
Dr. Alfredo Herrera Aguilar.	**Vocal**
Dr. Valentín García Vázquez.	**Vocal**
Dr. Alberto Escalante Hernández	**Vocal**

Se informa también que dicha tesis se encuentra lista para su impresión.

Sin otro asunto que el particular, reciba mi más alto reconocimiento.

A T E N T A M E N T E
"Pensar Bien, Para Vivir Mejor"
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