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Teoría cuántica de la disipación basada en el plano complejo hiperbólico

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Abstract

This thesis develops a field theory for dissipative systems, employing a hypercomplex ring formalism that naturally reproduces dissipative systems' effective doubling field formulation in the context of thermal field theory. The system is defined by a non-canonical ansatz that ensures the conservation of unitarity. Asymptotically entangled states are developed to facilitate the analysis of ergodicity in systems experiencing entanglement. In addition, calculations of partition functions in a hyperbolic ring and a hypercomplex ring are performed. A significant part of this thesis is based on the reference [1]. The results shown in this thesis have already been published [2]. The development of the present work follows the steps outlined in [1], but introduces an extension of the formalism. The section devoted to the partition function in a pure hyperbolic ring in chapter 5 of this work represents an extension of the research presented in [1]. As for the partition function in the hypercomplex ring in chapter 6, this is unpublished content that is currently in the process of publication.

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Contents

Introduction

Quantum mechanics has a strange feature that is also surprising, entanglement. This phenomenon is present in various areas, such as quantum optics [3], quantum field theory (QFT) [4], AdS/CFT correspondence [5], it is an essential part of the development of quantum information theory, and for the emergence of new technologies. For instance, in [6] the authors develop new techniques to entangle disparate electromagnetic fields that range from microwave radiation to optical beams; if this entanglement were achieved, it would be closer to solving the problem that arises when entanglement is shared between two separate quantum computers. Another method that has been used to generate entanglement is the non-local Seebeck effect [7], which consists of generating a thermoelectric current through a temperature difference by using a quantum splitter. It is found that certain processes such as Copper pair splitting (CPS) and elastic co-tunneling (EC) can contribute to this current, achieving a tuning between CPS and EC. On the other hand, in addition to being able to generate the entanglement it is important to measure it; for example in [8], the authors propose a test for entanglement in the scattering of chemical reactions.

The entanglement has been generated artificially, but in most cases it is generated naturally, often having an unwanted effect. A natural way in which entanglement is generated is as an effect of dissipation, which causes populations of the quantum states to be modified due to the interaction with the environment, which in turn causes a phase shift [9]; these two actions correspond to the result of the intertwining between the degrees of freedom of the environment and the system of interest; in a system with these characteristics, the unitarity of the dynamics is lost [10].

On the other hand, any small alteration that interferes with the unitarity of the quantum evolution is undesirable, since quantum states lose their coherence. Decoherence is an effect that cannot be avoided, it arises when a quantum system is coupled to its environment; this phenomenon is often undesirable since it causes the information of the system of interest to be lost. But not everything is bad with decoherence, there is also some interest in this phenomenon, as discussed in [11].

For a long time, it has been of interest to study the quantum dynamics of a particle that is coupled with its environment, and several methods have been developed. A well-known and widely used method is the doubling of degrees of freedom; in [12] the authors make use of this duplication to review the general features of a dissipative quantum model of the brain and discuss how QFT phase correlations and entanglement are achieved for modeling functional brain activity. On the other hand, for systems coupled to an environment, the thermodynamic descriptions assume that a system of finite size, being in contact with a thermal reservoir with a temperature T, will reach a state of thermal equilibrium after a period of time. When there are dissipative processes that occur at very long times, it is normal to think that this thermalization occurs; however, this is not always the case. On the one hand, it is well known that in coherent quantum mechanics, there are time-periodic modulations that can lead to non-equilibrium asymptotic states, which are known as Floquet states. On the other hand, there are cycle-stationary states, and periodic states in time; the latter introduce the ergodic theory to study dissipative systems [13].

Statistical mechanics gives us a fundamental tool, the partition function. This function is essential to describe systems in thermodynamic equilibrium. With this function, it is possible to know the state functions of the system such as free energy, pressure, entropy, etc. However, it is important to highlight that studying a system from the perspective of the environment and the subsystem of interest may not always be feasible. In most cases, to analyze the behavior of each subsystem, it is necessary to plot one of their degrees of freedom, which implies losing a large amount of information about the total system. We have developed a partition function in this work to understand better what happens in each subsystem. This function is built on a hypercomplex ring $\mathbb{H}$, allowing us to observe the point of view of each subsystem without having to eliminate any of them.

Chapter 1

The hypercomplex numbers

The real numbers can be extended in a great quantity of non-commutative and commutative extensions that relate to more than one complex unit. The extension of the real numbers that will be used in this work is the hyperbolic form. Let be an arbitrary number $w = x + \imath y$; where "$\imath$" is a complex unit, with $\imath^2 = 1$, it is the hyperbolic case, when $\imath^2 = -1$, it is the elliptical case and the parabolic case is when $\imath^2 = 0$.

In the same way that complex (or elliptical) numbers are associated with Euclidean geometry, the other two two-dimensional complex number systems (parabolic and hyperbolic) can also be associated with geometries of physical relevance, parabolic numbers can be associated with the Galileo group, and hyperbolic numbers with Lorentz's special relativity group [89, 90]. These numbers serve not only to make Lorentzian geometry similar to Euclidean geometry, but there are various applications of hyperbolic numbers. In 1986, Fjelstad [88] presented an application of these numbers to interpret superluminal phenomena; Reany [87] also applied them but in second-order linear differential equations; Kunstatter has applied them to general relativity [80–86, 88]. In addition, several authors have provided extensions to n-dimensional space [77–79], including an analysis of hyperbolic Fourier transforms [76]. Band [75], obtained the essential mathematical results of Fjelstad from two simple postulates [74].

Despite the diversity of applications of hyperbolic numbers found in the literature, the majority of mathematicians and physicists are unaware of implicit mathematics and, therefore, neither are the possibilities that these numbers open to obtain more applications as opposed to ordinary complex numbers. This number system has a whole series of fundamental theorems, which will not be mentioned in this work.

This section will examine some geometric characteristics and fundamental properties of hyperbolic numbers. The fundamental bases that will be presented below are essential for developing this thesis and understanding the subsequent chapters. The definition of an algebraic ring is also provided since hyperbolic numbers satisfy all the properties that define a ring. Furthermore, this ring will arise naturally by constructing a Lagrangian with dissipation given in terms of a pure hyperbolic field.

1.1 The formalism of the hyperbolic ring $\mathbb{P}$

The complex hyperbolic unit will be denoted by j from this point. With this in mind, we begin to describe the algebraic structure of the hyperbolic complex plane. Ring $\mathbb{P}$ is defined as:

$$\mathbb{P} \equiv \left\{ w = x + jy \mid \quad j^2 = 1, \quad j \neq \pm 1, \quad \bar{j} = -j; \quad x, y \in \mathbb{R} \right\}. \tag{1.1}$$

Properties like closure concerning addition and multiplication are similar to the properties of usual complex numbers (ellipticals). Taking as a starting point the invariant form in the plane of the hyperbolic complex,

$$|w|^2 = w\bar{w} = x^2 - y^2, \tag{1.2}$$

the modulus (1.2) is invariant under the hyperbolic rotation $w \to we^{j\chi}$, with $e^{j\chi} = \cosh\chi + j\sinh\chi$, where χ is a noncompact parameter $\chi \in \mathbb{R}$. These continuous rotations correspond to the connected component of the Lie group $SO(1,1)$ that contains the identity element of the group I with $\chi = 0$. Let us take any two hyperbolic numbers $w_1 = x_1 + jy_1$ and $w_2 = x_2 + jy_2$ to generalize the module (1.2), and thus be able to construct a real quantity that is invariant under hyperbolic rotations,

$$w_1\bar{w}_2 + \bar{w}_1 w_2 = 2(x_1x_2 - y_1y_2). \tag{1.3}$$

On the other hand, it is observed that if $|w|^2$ does not vanish, the multiplicative inverse given by,

$$w^{-1} = \frac{\bar{w}}{|w|^2} = \frac{x - jy}{x^2 - y^2}, \tag{1.4}$$

is a well-defined inverse for w; that is, a hyperbolic number has an inverse if and only if, it is modulus is nonzero, $|w|^2 \neq 0$. Therefore w has no inverse, in general. Also, note that any hyperbolic number $w = x + jy$ with $x = \pm y$ is a zero divisor. When we have $x = \pm y$, these define two isotropic lines that separate the quadrants of the hyperbolic plane which, in turn, also define an idempotent base.

In a ring $\mathfrak{A}$, a nonzero element $a \in \mathfrak{A}$ is considered a right zero divisor if there exists another nonzero element $b \in \mathfrak{A}$ such that $ba = 0$. Left zero divisors are defined similarly [73]. In a commutative context, the definitions of right and left zero divisors are unified, allowing them to be referred to in a simplified way as zero divisors. That is, while $\mathbb{C}$ is a field, $\mathbb{P}$ is not an integral domain [72], it is an abelian ring [71]. When considering the cases where $x = \pm y$, two isotropic lines dividing the quadrants of the hyperbolic plane are generated (see figure 1.1). These lines also establish an idempotent basis. Appendix (6.2) gives a more complete description of the definition and characteristics of an algebraic ring.

1.2 Geometry associated with the hyperbolic ring $\mathbb{P}$

While $\mathbb{C}$ establishes the geometry of the Euclidean plane $\mathbb{R}^2$, the ring $\mathbb{P}$ lays the groundwork for the geometry of the two-dimensional Minkowski plane [69, 70]. The set of points $\{w : |w|^2 = a^2\}$ is a hyperbola for each nonzero value of a. This hyperbola has branches passing through the points $(a, 0)$ and $(-a, 0)$. When $a = 1$, we observe the existence of a unit hyperbola. The conjugate of this hyperbola is given by $\{w : |w|^2 = -a^2\}$, where its branches pass through the points $(0, a)$ and $(0, -a)$. Likewise, the hyperbola and its conjugate are delimited by two diagonal asymptotes that create a set of null elements, separating the quadrants of the hyperbolic plane. This set is $\{w : |w|^2 = 0\}$.

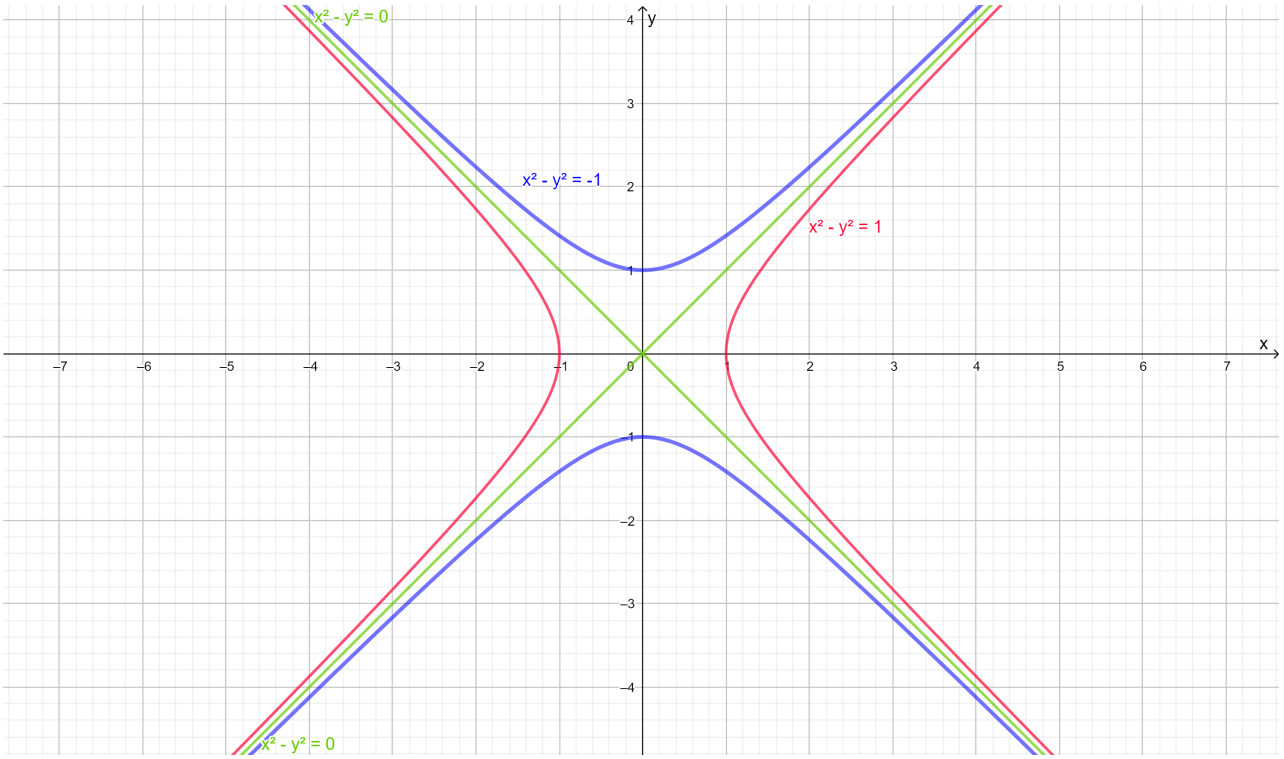

Figure 1.1: *Hyperbolas unitary when* $\|z\| = 1$, *conjugated when* $\|z\| = -1$ *and asymptotes when* $\|z\| = 0$, *this two lines are two lines are perpendicular at* $\mathbb{R}^2$ *and have slopes with values* ± 1.

Thus the zero divisors in $\mathbb{P}$ act as natural boundaries between the quadrants of the hyperbolic plane. The hyperbolic plane is often considered a two-dimensional projection of spacetime into 4-dimensions. For a hyperbolic number $w = x + jy$, we have that,

- If $|x| > |y| \Rightarrow w$ it is a temporary number.
- If $|x| < |y| \Rightarrow w$ it is a spatial number.
- If $|x| = |y| \Rightarrow w$ it is a light number.

For more in-depth information on hyperbolic numbers, one can explore topics such as hyperbolic polar transformations, hyperbolic rotations interpreted as Lorentz transformations, and the concept of the hyperbolic angle, all of which are detailed in [68]. In addition, hyperbolic algebra is explored, as well as hyperbolic geodesics [67], hyperbolic arc length, and hyperbolic volume. Application examples of hyperbolic numbers can also be found in [66].

1.3 Properties of the hyperbolic ring $\mathbb{P}$

Unlike the usual complexes $\mathbb{C}$, with their idempotent[1] elements, which are 0 and 1, the ring $\mathbb{P}$ also has non-trivial idempotent bases,

$$\begin{aligned} J^+ &= \frac{1}{2}(1+j), \quad (J^+)^n = J^+; \\ J^- &= \frac{1}{2}(1-j), \quad (J^-)^n = J^-, \quad n = 1, 2, 3, ...; \end{aligned} \tag{1.5}$$

which eliminate each other,

$$J^+ \cdot J^- = 0, \qquad \overline{J^+} = J^-, \quad \overline{J^-} = J^+. \tag{1.6}$$

This can be seen as a property of orthogonality between the bases, and thus the ring $\mathbb{P}$ is not an integral domain [72]. Furthermore, the hyperbolic unit j is absorbed by these idempotents,

$$jJ^+ = J^+, \qquad jJ^- = -J^-. \tag{1.7}$$

Using these bases, the hyperbolic rotations $e^{j\chi} = \cosh\chi + j\sinh\chi$, can be formulated as a combination of (J^+, J^-), which implies that,

$$e^{J^+} = J^+e^{\chi} + J^-, \qquad e^{J^-} = J^-e^{\chi} + J^+; \tag{1.8}$$

the quantities that present this structure will have their norm defined positively, given that,

$$\left|e^{J^+\chi}\right|^2 \equiv e^{J^+\chi}\, e^{J^-\chi} = e^{\chi}, \qquad \sqrt{\left|e^{J^+\chi}\right|^2} = e^{\frac{\chi}{2}}. \tag{1.9}$$

Therefore, the expressions in equation (1.8) are inverses of each other. It is also noted that the hyperbolic exponential can be expressed in terms of these idempotent bases or projectors,

$$e^{j\chi} = e^{\chi}J^+ + e^{-\chi}J^-, \qquad x \in \mathbb{R}. \tag{1.10}$$

[1] An element e in a ring $\mathfrak{A}$ is considered idempotent if it satisfies the condition $e^2 = e$. Two idempotents e and f are considered orthogonal if they satisfy the relation $ef = fe = 0$. The zero and the identity of any ring are idempotent elements. However, there can be numerous other idempotent elements [73]. Besides these elements, there also exist idempotent bases, such as the ones mentioned in (1.5).

On the other hand, it will be essential to carry out the series expansion of the expression e^{χ}. This can be obtained by considering the usual expansion of a real function, this is,

$$\begin{aligned} e^{J^+\chi} &= J^+ \sum_{n=0}^{\infty} \frac{\chi^n}{n!} + J^- = J^+ + J^- + J^+ \sum_{n=1}^{\infty} \frac{\chi^n}{n!} = 1 + J^+ \sum_{n=1}^{\infty} \frac{\chi^n}{n!}, \\ e^{J^-\chi} &= J^- \sum_{n=0}^{\infty} \frac{\chi^n}{n!} + J^+ = J^- + J^+ + J^- \sum_{n=1}^{\infty} \frac{\chi^n}{n!} = 1 + J^- \sum_{n=1}^{\infty} \frac{\chi^n}{n!}, \end{aligned} \tag{1.11}$$

where the expressions (1.5) have been added. Since these series are designed for $n \geq 1$, they are perfectly defined when χ represents an operator. Later, the characterization of entangled states will require this type of expansion. In addition, the classical fields, quantum operators, and state vectors will be described in terms of these projectors (J^+, J^-) throughout this work.

1.4 The formalism of the hypercomplex ring $\mathbb{H}$

We have already described pure hyperbolic numbers and some of their characteristics. Now, we will explore hypercomplex numbers. These hypercomplex numbers are obtained from the hyperbolic ring by transforming the real components of a pure hyperbolic number into conventional complex numbers; the set of these numbers is denoted by $\mathbb{H}$. The ring $\mathbb{H}$ can be seen as an extension of the usual complex numbers [67], thus, the hypercomplex ring is defined as,

$$\mathbb{H} \equiv \{\xi = \sigma + j\varsigma | \sigma, \varsigma \in \mathbb{C}\} = \{x + iy + j(u + iv) | x, y, u, v \in \mathbb{R}\}. \tag{1.12}$$

The hypercomplex ring $\mathbb{H}$ is, like the hyperbolic ring $\mathbb{P}$, a commutative ring. Now, we will explore some of the distinctive features of hypercomplex numbers. Let us take a hypercomplex number ξ,

$$\xi = x + iy + ju + ijv, \qquad \overline{\xi} = x - iy - ju + ijv \qquad x, y, u, v \in \mathbb{R}; \tag{1.13}$$

which has the following characteristics in its complex units, for the hyperbolic unit one has $j^2 = 1$, and $\overline{j} = -j$, for the standard complex unit $i^2 = -1$, and $\overline{i} = -i$; additionally, we have a new complex unit, composed of the hybrid term ij, with the properties $(ij)^2 = -1$, and $\overline{ij} = ij$. The pure hyperbolic numbers $\mathbb{P}$ and the pure usual complex numbers $\mathbb{C}$ are subsets of the numbers $\mathbb{H}$.

As regards the conjugation that encompasses both complex units, the square of the hypercomplex number is expressed as follows [67],

$$\xi\bar{\xi} = x^2 + y^2 - u^2 - v^2 + 2ij(xv - yu). \tag{1.14}$$

The expression (1.14) is invariant under the rotations $e^{i\theta}$, corresponding to the Lie group $U(1)$, and under the hyperbolic rotations $e^{j\chi} = \cosh(\chi) + j\sinh(\chi)$, $\chi \in \mathbb{R}$, corresponding to the Lie group $SO(1,1)$. The product of phases $e^{i\theta}e^{j\chi}$ is an element that has form (1.14), this modulus is invariant under the transformation $\xi \to \xi\, e^{j\chi}e^{i\theta}$, and therefore corresponds to a $U(1) \times SO(1,1)$ invariant quantity. With this, the bi-complex phase can be expressed as,

$$e^{i\alpha}e^{j\beta} \equiv e^{i\alpha+j\beta} = \cos(\alpha)\cosh(\beta) + i\,\sin(\alpha)\cosh(\beta) + j\,\cos(\alpha)\sinh(\beta) + ij\,\sin(\alpha)\sinh(\beta). \tag{1.15}$$

The hypercomplex ring $\mathbb{H}$ encompasses the idempotent elements of the set $\mathbb{C}$, which are $(0,1)$, and also includes the idempotent bases of the hyperbolic ring $\mathbb{P}$, represented by (J^+, J^-). These idempotent bases act as projectors into the ring $\mathbb{H}$. Therefore, any element of the ring expressed as $\xi \equiv \xi_1 + i\xi_2 + j\xi_3 + ij \to_4$, can be factored into the product of a pure hyperbolic number and a conventional pure complex number,

$$\begin{aligned} J^+\xi &= J^+\left[\xi_1 + \xi_3 + i\left(\xi_2 + \xi_4\right)\right], \\ J^-\xi &= J^-\left[\xi_1 - \xi_3 + i\left(\xi_2 - \xi_4\right)\right]. \end{aligned} \tag{1.16}$$

If we combine these projections, we will obtain a complete number, a real number, that is,

$$J^+\xi + J^-\xi = \xi. \tag{1.17}$$

Hence, any hypercomplex number can be decomposed into the bases (J^+, J^-), using conventional complex numbers as its components. This represents the *spectral decomposition* of an element belonging to the ring $\mathbb{H}$.

An element of the ring $\mathbb{H}$ can also be written, using the idempotent bases as,

$$\begin{aligned}\xi = z_1 + jz_2 &= (x_1 + ix_2) + j(y_1 + iy_2) \\ &= [(x_1 + y_1) + i(x_2 + y_2)]\left(\frac{1+j}{2}\right) + [(x_1 - y_1) + i(x_2 - y_2)]\left(\frac{1-j}{2}\right) \\ &= z^+J^+ + z^-J^-; \quad z_1, z_2 \in \mathbb{C}.\end{aligned} \tag{1.18}$$

During the development of this work, this way of expressing an element of $\mathbb{H}$ in terms of the bases (J^+, J^-) will be frequently used. Below, we present some additional functions [64, 65] and properties [1, 2, 63] $\forall\, \xi \in \mathbb{H}$; that will be used in this work,

1. $||\xi||^2 = \frac{1}{2}\left[||z_1 + z_2||^2 + ||z_1 - z_2||^2\right] + \frac{1}{2}\left[||z_1 + z_2||^2 + ||z_1 - z_2||^2\right] j$,
2. $(z_1 + jz_2)^n = J^+(z_1 + z_2)^n + J^-(z_1 - z_2)^n$,
3. $e^{(z_1+z_2)}j = \frac{1}{2}\left(e^{z_1+z_2} + e^{z_1-z_2}\right) + \frac{1}{2}\left(e^{z_1+z_2} - e^{z_1-z_2}\right) j$,
4. $|z_1 + jz_2| = \frac{1}{2}\left(|z_1 + z_2| + |z_1 - z_2|\right) + \frac{1}{2}\left(|z_1 + z_2| - |z_1 - z_2|\right) j$,
5. $\ln\{z_1 + jz_2\} = \frac{1}{2}\ln\{z_1^2 + z_2^2\} + \frac{1}{2}\ln\left\{\frac{z_1+z_2}{z_1-z_2}\right\} j$; $\{z_1, z_2\} \in \mathbb{C}$; $(z_1^2 - z_2^2) > 0$, $(z_1 - z_2) \neq 0, \left(\frac{z_1+z_2}{z_1-z_2}\right) > 0$.

A more comprehensive exploration of hyperbolic numbers with complex coefficients is available in [62], while their detailed analysis is presented in [60, 61]. An analysis of n-dimensional hyperbolic-complex numbers is done in [59]. Bicomplex numbers have also been investigated in references [56–58]. Dual numbers with complex coefficients, also known as complex-dual numbers or dual-complex numbers, are presented in references [54, 55]. Hyperdual numbers, which are considered an extension of dual numbers, are also studied in references [48–53]. On the other hand, bihyperbolic numbers are examined in [44–47]. The literature contains numerous studies dealing with various combinations of coefficients concerning different types of numbers [42, 43].

Chapter 2

The dissipative system and its Lagrangian formalism

In this chapter, we will explore part of a work that I have done in collaboration with my advisors, which has been published [2]. The basis of this work is the formulation of thermal fields that doubles the number of degrees of freedom [41], in turn, this formulation is used in [1], where the total system is represented by a real field ϕ for the system of interest (A), and a second real field ψ representing the thermal bath or environment (B); here we will continue with that representation for the subsystems. In this thesis, we generalize the formulation in [1] by promoting the ϕ and ψ fields to charged fields, according to the generalization described in Eq. (1.12).

When placed within the context of the ϕ field, a dissipative process is revealed that can be described by an equation of motion [40],

$$(\partial_t^2 - \nabla^2)\phi + \gamma\partial_t\phi = 0. \tag{2.1}$$

The equation associated with ψ is analogous to that of ϕ; however, it is distinguished by a change of sign in the dissipative term,

$$(\partial_t^2 - \nabla^2)\psi - \gamma\partial_t\psi = 0; \tag{2.2}$$

where γ represents dissipation. The last equation states that the ψ field is a replica of the physical ϕ field, but its evolution will develop in the opposite direction over time. The dissipative behavior of the system in question is defined by the interaction between ψ and ϕ.

In this chapter a Lagrangian expressed in terms of four scalar fields will be presented, thus extending the Lagrangian proposed in [1]. The solution to the equation of motion of the system in question will be presented, and the implications of the Lagrangian and Hamiltonian formalism will be explored.

2.1 The dissipative Lagrangian

We will start with a Lagrangian that includes two real fields and then extend these fields to a pure hyperbolic field. Finally, we will arrive at the hypercomplex Lagrangian, which will consist of four real fields, this is, two charged fields.

A Lagrangian is available that describes the equation of motion (2.1):

$$\mathcal{L} = \int_{\Xi \in \mathbb{R}^d} dx^d \, \psi[(\partial_t^2 - \nabla^2)\phi + \gamma \partial_t \phi]; \tag{2.3}$$

applying integration by parts to this Lagrangian and omitting the terms on the boundary, we arrive at the following expression,

$$\mathcal{L} = \int_{\Xi \in \mathbb{R}^d} dx^d \, [\partial_\mu \phi \, \partial^\mu \psi + \frac{\gamma}{2}(\phi \, \partial_t \psi - \psi \, \partial_t \phi)]. \tag{2.4}$$

From this Lagrangian, the equations of motion for the field of interest (2.1) as well as for the environment (2.2) are obtained. Now, defining,

$$\Phi = \frac{\phi + \psi}{\sqrt{2}}, \quad \Psi = \frac{\phi - \psi}{\sqrt{2}}; \tag{2.5}$$

we rewrite the Lagrangian (2.4) as,

$$\mathcal{L}(\Phi, \Psi) = \frac{1}{2} \int dx^d \left[(\partial_\mu \Phi)^2 - (\partial_\mu \Psi)^2 + \gamma(\Psi \dot{\Phi} - \Phi \dot{\Psi}) \right]. \tag{2.6}$$

Where Φ and Ψ are real fields, each point in these fields symbolizes a time derivative. The Lagrangian presented corresponds to equation (10) in reference [1].

According to [1], the following definition of the hyperbolic field is established as,

$$\Omega = \Phi + j\Psi, \tag{2.7}$$

the Lagrangian presented in (2.6) can be reformulated as an invariant under the group $SO(1,1)$ as,

$$\mathcal{L}(\Omega,\overline{\Omega}) = \frac{1}{2}\int dx^d \left[\partial_\mu\Omega\cdot\partial^\mu\overline{\Omega} + \frac{\gamma}{2}(j\Omega\dot{\overline{\Omega}} + c.c) - m^2(\Omega\overline{\Omega})\right]. \tag{2.8}$$

Now the Lagrangian (2.6) is constructed with a pure hyperbolic field Ω and with a dissipative parameter γ. On the other hand, the Lagrangian (2.8) has objects invariant both under global hyperbolic rotations as well as under PT-type transformations, i.e.,

$$\begin{aligned} \partial_\mu\Omega\cdot\partial^\mu\overline{\Omega} &= (\partial_\mu\Phi)^2 - (\partial_\mu\Psi)^2, \quad && SO(1,d-1),\ \ SO(1,1), \\ \frac{1}{2}[j\Omega\dot{\overline{\Omega}} + c.c.] &= \Psi\dot{\Phi} - \Phi\dot{\Psi}, && SO(1,1). \end{aligned} \tag{2.9}$$

The introduction of dissipation gives rise to a significant problem: the breaking of Lorentz covariance. Dissipation is a typical process breaking boost invariance. Moreover, since dissipation establishes an "arrow of time", dissipative processes develop in a privileged natural setting, the thermal bath.

Now we generalize the hyperbolic Lagrangian (2.8) to obtain a hypercomplex Lagrangian. We do this by promoting the components of the field Ω as, $\Phi \to \phi_1 + i\phi_2$ and $\Psi \to \psi_1 + i\psi_2$, thus Ω takes the form of an element of the ring $\mathbb{H}$,

$$\Omega = \phi_1 + i\phi_2 + j\psi_1 + ij\psi_2; \qquad\qquad \phi_1,\phi_2,\psi_1,\psi_2 \in \mathbb{R}. \tag{2.10}$$

Obtaining our hypercomplex Lagrangian that includes a mass term, which will undergo dissipation that not was considered in [1]. The mass term has in general the form $m_1^2 + ijm_2^2$, but the corresponding dispersion relation will force $m_2 = 0$ (see Eq. 2.20),

$$\mathcal{L}(\Omega,\overline{\Omega}) = \frac{1}{2}\int dx^d \left[\partial_\mu\Omega\cdot\partial^\mu\overline{\Omega} + \frac{\gamma}{2}(j\Omega\dot{\overline{\Omega}} + c.c) - m^2(\Omega\overline{\Omega})\right]. \tag{2.11}$$

With the generalization of the Ω field to four components, a significant additional symmetry $U(1) \times SO(1,1)$ arises. To visualize it, we consider the objects that make up the Lagrangian presented in (2.11), but now expressed in terms of the four components,

$$\begin{aligned}
\partial_\mu \Omega \cdot \partial^\mu \overline{\Omega} &= (\partial_\mu \phi_1)^2 + (\partial_\mu \phi_2)^2 - (\partial_\mu \psi_1)^2 - (\partial_\mu \psi_2)^2 + 2ij\,(\partial_\mu \phi_1\, \partial_\mu \psi_2 - \partial_\mu \phi_2\, \partial_\mu \psi_1), \\
j\,\Omega\,\dot{\overline{\Omega}} + c.c &= (\psi_2\,\dot{\phi}_2 + \psi_1\,\dot{\phi}_1 - \phi_2\,\dot{\psi}_2 - \phi_1\,\dot{\psi}_1) + 2ij\,(\psi_1\,\dot{\psi}_2 + \phi_2\,\dot{\phi}_1 - \psi_2\,\dot{\psi}_1 - \phi_1\,\dot{\phi}_2) \\
\Omega\overline{\Omega} &= \left(\Phi\overline{\Phi} - \Psi\overline{\Psi}\right) + j\left(\Psi\overline{\Phi} - \Phi\overline{\Psi}\right) = (\phi_1^2 + \phi_2^2 - \psi_1^2 - \psi_2^2) + 2ij\,(\psi_2\phi_1 - \phi_2\psi_1).
\end{aligned} \tag{2.12}$$

Therefore, when we have the Lagrangian constructed in a hypercomplex way, it can be reinterpreted in terms of the two hypercomplexified field variables $(\Omega, \overline{\Omega})$ with the norm $\xi\overline{\xi}$, given in (1.14). In this formulation there will not be a free field theory, necessarily the interactions are present in the simplest model, this is because we have the simplest invariant quantity that can be constructed in $\mathbb{H}$, which is the norm. On the other hand, using the idempotent bases (1.5), the field (2.10) can be expressed as,

$$\Omega = J^+\Omega^+ + J^-\Omega^-. \tag{2.13}$$

where $\Omega^+ = (\Phi + \Psi)$ and $\Omega^- = (\Phi - \Psi)$. Throughout this work, only Ω will be used and sometimes Φ and Ψ, but the background content of this field must not be lost, which are it is four components.

Finally, the Lagrangian (2.11) can also be written in terms of these two complex fields (Φ, Ψ),

$$\begin{aligned}
\mathcal{L}(\Phi,\Psi;\overline{\Phi},\overline{\Psi}) = \frac{1}{2}\int dx^d \Big\{ &\partial_\mu\Phi\,\partial^\mu\overline{\Phi} - \partial_\mu\Psi\,\partial^\mu\overline{\Psi} + j\left(\partial_\mu\Psi\,\partial^\mu\overline{\Phi} - \partial_\mu\Phi\,\partial^\mu\overline{\Psi}\right) \\
&+\frac{\gamma}{2}[\Psi\overline{\Phi} - \Phi\overline{\Psi} + j(\Phi\vec{\Phi} - \Psi\overline{\Psi}) + c.c] - m^2[\Phi\overline{\Phi} - \Psi\overline{\Psi} + j(\Psi\overline{\Phi} - \Phi\overline{\Psi})]\Big\}.
\end{aligned} \tag{2.14}$$

2.2 The equation of motion

When the equations of motion are calculated in the basis (J^+, J^-), the correspondence of each equation of motion in such a basis is notorious, namely, one for the system of interest J^+ and, the other for the environment J^-; this is due to the annihilation property (1.6). First, we make the variation with respect to $\overline{\Omega}$,

$$\partial_\mu\,\partial^\mu\Omega + j\gamma\,\partial_t\Omega + m^2\Omega = 0. \tag{2.15}$$

When the variation is made with respect to Ω, this yields the complex conjugate of (2.14). Thus, we have the equation of motion for the entire system,

$$J^{+}\left[\partial_{\mu}\partial^{\mu}\Omega^{+}+\gamma\partial_{t}\Omega^{+}+m^{2}\Omega^{+}\right]+J^{-}\left[\partial_{\mu}\partial^{\mu}\Omega^{-}-\gamma\partial_{t}\Omega^{-}+m^{2}\Omega^{-}\right]=0. \tag{2.16}$$

We have explicitly obtained the corresponding part for the subsystem of interest that is accompanied by the idempotent J^{+}, and the part of the environment identified with J^{-}; the last one has a negative sign in the dissipative term γ, indicating the time inversion, besides that it is the mirror copy of the system of interest [3]. In [17] the authors constructed a solution that contains real exponentials, then they analytically extend that solution in the hyperbolic complex plane; it can also be complexified to the standard complex plane, and in both cases the relative sign between ω_k^2 and k^2 is not altered in the dispersion relation; however, there is a sign change in the dissipative term (γ^2), being negative ($-\gamma^2$) for the standard scheme and positive ($+\gamma^2$) for the scheme hyperbolic. Here we have the two imaginary units in this scheme, the product ij will appear in our solution, and as we will see, the sign for the dissipative term will be negative as in the standard scheme.

2.3 The solution to the equation of motion

The formal solution for the Eq.(2.15) has the form,

$$\eta(\boldsymbol{x},t)=\mathtt{a}e^{ip_1x^{\mu}}e^{jp_2x^{\nu}}+\mathtt{b}e^{-ip_1x^{\mu}}e^{-jp_2x^{\nu}}, \tag{2.17}$$

where $\mathtt{a},\mathtt{b}$ are arbitrary coefficients, and $\omega_{1,2}$ and $\boldsymbol{k}_{1,2}$ are real parameters. With this solution we extend $\omega\to\omega_1+i\omega_2$ and $k\to k_1+ik_2$; however, we can always find, through a Lorentz rotation, a system where we have $k_2=0$, that is, the spatial imaginary part for k vanishes; hence we are only concerned with the problem of temporal dissipation.

Using the property (1.10), and the mentioned criterion, we have that (2.14) can be rewritten as [2],

$$\eta(\boldsymbol{x},t)=\mathtt{a}\,e^{[(\alpha+i\omega_1)t-i\boldsymbol{k}\cdot\boldsymbol{x}]}J^{+}+\mathtt{b}\,e^{[(\alpha-i\omega_1 t)+i\boldsymbol{k}\cdot\boldsymbol{x}]}J^{-}. \tag{2.18}$$

[2]This solution can also be obtained by proposing a real exponential, $\Omega(\vec{x},t)=e^{\omega t-\vec{k}\cdot\vec{x}}$, and promoting $(\omega,\vec{k})\in\mathbb{H}$, and considering restrictions for obtaining convergent solutions.

With this, we obtain the solutions for the equation of motion (2.16),

$$\begin{aligned}\Omega^{+} &= \mathtt{a}_1\, e^{[(\Gamma_1+i\omega_1)t-i\boldsymbol{k}_1\cdot\boldsymbol{x}]} + \mathtt{b}_1\, e^{[(\Gamma_1-i\omega_1)t+i\boldsymbol{k}_1\cdot\boldsymbol{x}]},\\ \Omega^{-} &= \mathtt{a}_2\, e^{[(\Gamma_2+i\omega_2)t-i\boldsymbol{k}_2\cdot\boldsymbol{x}]} + \mathtt{b}_2\, e^{[(\Gamma_2-i\omega_2)t+i\boldsymbol{k}_2\cdot\boldsymbol{x}]};\end{aligned} \tag{2.19}$$

where $(\mathtt{a}_{1,2}, \mathtt{b}_{1,2})$ are arbitrary coefficients and the spectral parameters $(\omega, \boldsymbol{k})$ are real-valued. We will first calculate the solution for the part that corresponds to the system of interest, that is, the solution corresponding to J^{+} in (2.16). Specifically, we can determine of $J^{\pm}$ projections of the Eq.(2.16) and use the property (1.6); taking the solution Ω^{+} of (2.19) and performing the substitution into the equation of motion we obtain,

$$\begin{aligned}&e^{[(\Gamma_1+i\omega_1)t-i\boldsymbol{k}_1\cdot\boldsymbol{x}]}\left[\Gamma_1^2-\omega_1^2+k_1^2+\gamma\Gamma_1+m^2+i\omega_1(2\Gamma_1+\gamma)\right]\\ &+e^{[(\Gamma_1-i\omega_1)t+i\boldsymbol{k}_1\cdot\boldsymbol{x}]}\left[\Gamma_1^2-\omega_1^2+k_1^2+\gamma\Gamma_1+m^2-i\omega_1(2\Gamma_1+\gamma)\right]=0.\end{aligned} \tag{2.20}$$

A similar procedure is done for the solution Ω^{-}. From the imaginary part of these expressions, we obtain the dissipative coefficients,

$$\Gamma_1 = -\frac{\gamma}{2}, \qquad \Gamma_2 = \frac{\gamma}{2}; \tag{2.21}$$

and from the real part, we obtain the dispersion relations,

$$\omega_{1,2} = \pm\sqrt{k_{1,2}^2 + \underbrace{m^2-\frac{\gamma^2}{4}}_{modified\ \ mass}}\,. \tag{2.22}$$

It is necessary to notice some aspects of the frequencies of the system; the frequencies are real $(m^2-\frac{\gamma^2}{4}) \geq 0$ and therefore there is not an IR cut-off, for the case when $(m^2-\frac{\gamma^2}{4}) < 0$, we take the positive values of the radicand[3] in (2.22), therefore $k^2 \geq -(m^2-\frac{\gamma^2}{4})$, obtaining an IR cut-off. Therefore, the general solution is represented as plane waves damped by a decaying factor $e^{-\frac{\gamma}{2}t}$ for the system of interest and a growing factor $e^{\frac{\gamma}{2}t}$ for the environment; this is in accordance with the rules of TFD, since in this context, the environment evolves in the reverse direction of time.

[3]Otherwise, the radicand could take negative values and possibly lead to imaginary frequencies, which would imply superluminal speeds. In this work we only considered the real frequencies

Furthermore, we can write an arbitrary combination of solutions for the field Ω in terms of the (J^+, J^-) basis,

$$\Omega(\boldsymbol{x},t) = J^+ e^{-\frac{\gamma}{2}t}\left[\mathsf{a}_1 e^{i(\omega_1 - \boldsymbol{k}_1\cdot\boldsymbol{x})} + \overline{\mathsf{a}}_2 e^{-i(\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{x})}\right] + J^- e^{\frac{\gamma}{2}t}\left[\overline{\mathsf{b}}_1 e^{i(\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{x})} + \mathsf{b}_2 e^{-i(\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{x})}\right], \quad (2.23)$$

where $\mathsf{a}_{1,2}, \mathsf{b}_{1,2}$ are hypercomplex arbitrary coefficients; this combination allows us to construct the quantum fields in the next section.

2.4 Quantum fields and field commutators

In the spectral decomposition used in [1] the authors considered splitting the range of spectral parameters $\boldsymbol{k}$ into two parts, $(-\infty, 0) \leftrightarrow J^-$ for the environment, and $(0, +\infty) \leftrightarrow J^+$ for the system of interest; this is done to avoid the divergences that arise in the field operator commutators. In contrast to this, in the formulation at hand, we can take the full range $\boldsymbol{k} \in (-\infty, +\infty)$ since, due to the presence of the additional complex unit i, such divergences do not appear; therefore, the spectral decomposition for the field operator built with the solution (2.23) reads,

$$\begin{aligned}\widehat{\Omega}(\boldsymbol{x},t) = \Big\{ e^{-\frac{\gamma}{2}t} J^+ \int_{-\infty}^{\infty} \left[\hat{\mathsf{a}}_1(\boldsymbol{k}_1)\, e^{i\left(\omega_{k_1} t - \boldsymbol{k}_1\cdot\boldsymbol{x}\right)} + \hat{\mathsf{a}}_2^\dagger(\boldsymbol{k}_1)\, e^{-i\left(\omega_{k_1} t - \boldsymbol{k}_1\cdot\boldsymbol{x}\right)}\right] d\boldsymbol{k}_1 \\ + e^{\frac{\gamma}{2}t} J^- \int_{-\infty}^{\infty} \left[\hat{\mathsf{b}}_1^\dagger(\boldsymbol{k}_2)\, e^{i\left(\omega_{k_2} t - \boldsymbol{k}_2\cdot\boldsymbol{x}\right)} + \hat{\mathsf{b}}_2(\boldsymbol{k}_2)\, e^{-i\left(\omega_{k_2} t - \boldsymbol{k}_2\cdot\boldsymbol{x}\right)}\right] d\boldsymbol{k}_2 \Big\}.\end{aligned} \quad (2.24)$$

With this expression we can build the field commutator,

$$\begin{aligned}\left[\hat{\Omega}(\boldsymbol{x},t), \hat{\Omega}^\dagger(\boldsymbol{x}',t)\right] = \int_{-\infty}^{\infty} d\boldsymbol{k} \int_{-\infty}^{\infty} d\boldsymbol{k}' \Big\{ J^+ \Big(e^{i[(\omega_k - \omega_{k'})t - \boldsymbol{k}\cdot\boldsymbol{x} + \boldsymbol{k}'\cdot\boldsymbol{x}']} \left[\hat{\mathsf{a}}_1(\boldsymbol{k}), \hat{\mathsf{b}}_1(\boldsymbol{k}')\right] \\ + e^{i[(\omega_k + \omega_{k'})t - \boldsymbol{k}\cdot\boldsymbol{x} - \boldsymbol{k}'\cdot\boldsymbol{x}']} \left[\hat{\mathsf{a}}_1(\boldsymbol{k}), \hat{\mathsf{b}}_2^\dagger(\boldsymbol{k}')\right] \\ + e^{-i[(\omega_k + \omega_{k'})t - \boldsymbol{k}\cdot\boldsymbol{x} - \boldsymbol{k}'\cdot\boldsymbol{x}']} \left[\hat{\mathsf{a}}_2^\dagger(\boldsymbol{k}), \hat{\mathsf{b}}_1(\boldsymbol{k}')\right] \\ + e^{i[(\omega_{k'} - \omega_k)t + \boldsymbol{k}\cdot\boldsymbol{x} - \boldsymbol{k}'\cdot\boldsymbol{x}']} \left[\hat{\mathsf{a}}_2^\dagger(\boldsymbol{k}), \hat{\mathsf{b}}_2^\dagger(\boldsymbol{k}')\right] \Big) \\ + J^- \Big(e^{-i[(\omega_k - \omega_{k'})t + \boldsymbol{k}'\cdot\boldsymbol{x} - \boldsymbol{k}\cdot\boldsymbol{x}']} \left[\hat{\mathsf{b}}_1^\dagger(\boldsymbol{k}'), \hat{\mathsf{a}}_1^\dagger(\boldsymbol{k})\right] \\ + e^{-i[(\omega_{k'} + \omega_k)t - \boldsymbol{k}'\cdot\boldsymbol{x} - \boldsymbol{k}\cdot\boldsymbol{x}']} \left[\hat{\mathsf{b}}_2(\boldsymbol{k}'), \hat{\mathsf{a}}_1^\dagger(\boldsymbol{k})\right] \\ + e^{i[(\omega_{k'} + \omega_k)t - \boldsymbol{k}'\cdot\boldsymbol{x} - \boldsymbol{k}\cdot\boldsymbol{x}']} \left[\hat{\mathsf{b}}_1^\dagger(\boldsymbol{k}'), \hat{\mathsf{a}}_2(\boldsymbol{k})\right] \\ + e^{-i[(\omega_{k'} - \omega_k)t - \boldsymbol{k}'\cdot\boldsymbol{x} + \boldsymbol{k}\cdot\boldsymbol{x}']} \left[\hat{\mathsf{b}}_2(\boldsymbol{k}'), \hat{\mathsf{a}}_2(\boldsymbol{k})\right] \Big) \Big\}.\end{aligned} \quad (2.25)$$

One of the major difficulties that appear in the study of dissipative systems in quantum mechanics that involve the duplication of fields is that the commutation canonical relations are not preserved under the temporal evolution [38]. In the commutator (2.25), this time dependence has been eliminated due to the terms that have the damping factors vanish, $e^{-\frac{\gamma}{2}t}(J^+J^-)\left[\hat{\mathsf{a}},\hat{\mathsf{a}}^\dagger\right] = e^{+\frac{\gamma}{2}t}(J^+J^-)\left[\hat{\mathsf{b}},\hat{\mathsf{b}}^\dagger\right] = 0$, due to the property (1.6). On the other hand, there is also another elimination of the damping factors, since they have the form $e^{-\frac{\gamma}{2}}J^+ \cdot e^{+\frac{\gamma}{2}}J^+$.

The rest of the field commutators have a similar structure and the same characteristics as the expression (2.25), thus they do not have dissipative factors; however, there exists a special commutator with dissipative factors, namely,

$$\begin{aligned}\left[\hat{\Omega}^\dagger(\boldsymbol{x},t),\hat{\Pi}_\Omega(\boldsymbol{x}',t)\right] = \int_{-\infty}^{\infty}d\boldsymbol{k}\int_{-\infty}^{\infty}d\boldsymbol{k}'\Big\{ & e^{\gamma t}\,J^+\Big(e^{i[(\omega_{k'}-\omega_k)t+\boldsymbol{k}\cdot\boldsymbol{x}-\boldsymbol{k}'\cdot\boldsymbol{x}']}\left[\hat{\mathsf{b}}_1(\boldsymbol{k}),\hat{\mathsf{b}}_2^\dagger(\boldsymbol{k}')\right] \\ & -e^{-i[(\omega_{k'}-\omega_k)t+\boldsymbol{k}\cdot\boldsymbol{x}-\boldsymbol{k}'\cdot\boldsymbol{x}']}\left[\hat{\mathsf{b}}_2^\dagger(\boldsymbol{k}),\hat{\mathsf{b}}_1(\boldsymbol{k}')\right]\Big) \\ & +e^{-\gamma t}J^-\Big(e^{i[(\omega_{k'}-\omega_k)t-\boldsymbol{k}'\cdot\boldsymbol{x}'+\boldsymbol{k}\cdot\boldsymbol{x}]}\left[\hat{\mathsf{a}}_1^\dagger(\boldsymbol{k}),\hat{\mathsf{a}}_2(\boldsymbol{k}')\right] \\ & +e^{-i[(\omega_{k'}-\omega_k)t-\boldsymbol{k}'\cdot\boldsymbol{x}'+\boldsymbol{k}\cdot\boldsymbol{x}]}\left[\hat{\mathsf{a}}_2(\boldsymbol{k}),\hat{\mathsf{a}}_1^\dagger(\boldsymbol{k}')\right]\Big)\Big\}.\end{aligned} \tag{2.26}$$

The expression (2.26) contains the trivial canonical commutation relations $[\hat{a}_{\mathbf{p}}^\dagger,\hat{a}_{\mathbf{q}}^\dagger]=[\hat{a}_{\mathbf{p}},\hat{a}_{\mathbf{q}}]=0$, and the nontrivial relations $[\hat{a}_{\mathbf{p}},\hat{a}_{\mathbf{q}}^\dagger]$ and $[\hat{b}_{\mathbf{p}},\hat{b}_{\mathbf{q}}^\dagger]$; which do not vanish in general. However, in this work we will consider that the last commutators are vanishing $[\hat{a}_{\mathbf{p}},\hat{a}_{\mathbf{q}}^\dagger]=[\hat{b}_{\mathbf{p}},\hat{b}_{\mathbf{q}}^\dagger]=0$. The main reason is that, if we keep these commutators switched on, the damping coefficients $e^{\pm\gamma t}$ do not disappear, implying divergences, which bring us back to the main problem in of the dissipative dynamics. On the other hand, assuming for a moment that we do not have divergence problems when one uses these canonical relations, the use of these commutators does not allow us to identify the entanglement suffered by the subsystems due to dissipation, since they only describe the subsystems separately, through the pure commutators for type-a bosons for the subsystem of interest, and pure commutators for type-b bosons for the environment. Thus, we have a very special feature of the formulation at hand, since it is essential to have the non-vanishing commutator $[\hat{\mathsf{a}}(\boldsymbol{k}),\hat{\mathsf{b}}(\boldsymbol{k}')]\neq 0$ to do not trivialize the theory.

From (2.25) and (2.26) we need to propose the following commutation rules for the annihilation and creation operators,

$$\begin{array}{l}
\left[\hat{\mathsf{a}}_{1,2}\left(\boldsymbol{k}\right),\hat{\mathsf{b}}_{1,2}\left(\boldsymbol{k}'\right)\right]=\rho_i\,\delta\left(\boldsymbol{k}-\boldsymbol{k}'\right)\\
\left[\hat{\mathsf{b}}^{\dagger}_{1,2}\left(\boldsymbol{k}'\right),\hat{\mathsf{a}}^{\dagger}_{1,2}\left(\boldsymbol{k}\right)\right]=\overline{\rho}_i\,\delta\left(\boldsymbol{k}-\boldsymbol{k}'\right)\\
\left[\hat{\mathsf{a}}_{1,2}\left(\boldsymbol{k}\right),\hat{\mathsf{b}}^{\dagger}_{1,2}\left(\boldsymbol{k}'\right)\right]=\sigma_j\,\delta(\boldsymbol{k}+\boldsymbol{k}')\\
\left[\hat{\mathsf{b}}_{1,2}\left(\boldsymbol{k}'\right),\hat{\mathsf{a}}^{\dagger}_{1,2}\left(\boldsymbol{k}\right)\right]=\overline{\sigma}_j\,\delta(\boldsymbol{k}+\boldsymbol{k}'),\\
\left[\hat{\mathsf{b}}_{1,2}\left(\boldsymbol{k}\right),\hat{\mathsf{b}}^{\dagger}_{1,2}\left(\boldsymbol{k}'\right)\right]=0,\\
\left[\hat{\mathsf{a}}_{1,2}\left(\boldsymbol{k}\right),\hat{\mathsf{a}}^{\dagger}_{1,2}\left(\boldsymbol{k}'\right)\right]=0;
\end{array}\quad,\qquad i,j=1,2,3,4;\qquad\begin{array}{l}
[a_1,b_1]\rightarrow\rho_1,\\
[a_1,b_2]\rightarrow\rho_2,\\
[a_2,b_1]\rightarrow\rho_3,\\
[a_2,b_2]\rightarrow\rho_4,
\end{array}\tag{2.27}$$

where ρ_i and σ_j are arbitrary elements in the ring $\mathbb{H}$ and in general depend on $(\boldsymbol{k},\boldsymbol{k}')$; the column on the right-hand side indicates the identification (the correspondence) between the indices for the commutators and the coefficients ρ. Now considering that,

$$\left[\hat{\Omega}(\boldsymbol{x},t),\hat{\Omega}^{\dagger}(\boldsymbol{x}',t)\right]=\left[\hat{\Omega}(\boldsymbol{x},t),\hat{\Omega}^{\dagger}(\boldsymbol{x}',t)\right]^{\dagger}_{(\boldsymbol{x}\leftrightarrow\boldsymbol{x}')}\qquad\Rightarrow\qquad\sigma_j=0;\tag{2.28}$$

that is two commutation rules in (2.27) vanish. Therefore, using the commutation relations (2.28), the field commutator (2.25) reduces to,

$$\left[\widehat{\Omega}(\boldsymbol{x},t),\widehat{\Omega}^{\dagger}\left(\boldsymbol{x}',t\right)\right]=(2\pi)^n\left[J^{+}(\rho_1-\overline{\rho}_4)+J^{-}(\overline{\rho}_1-\rho_4)\right]\delta_n(\boldsymbol{x}'-\boldsymbol{x}).\tag{2.29}$$

In [1] this field commutator diverges and convergence criteria are considered, by splitting the full spectral parameter $(-\infty,+\infty)$ into two parts, as mentioned at the beginning of this section. Here the integration converges due to we have the additional complex unit i, which allows us to obtain the Dirac delta; this same field commutator in [1] has a more complicated mathematical expression; however, in both formulations, this commutator has the same physically acceptable asymptotic limits,

$$\lim_{\boldsymbol{x}'-\boldsymbol{x}\rightarrow 0}\left[\widehat{\Omega}(\boldsymbol{x},t),\widehat{\Omega}^{\dagger}(\boldsymbol{x}',t)\right]=\infty;\qquad\qquad\lim_{\boldsymbol{x}'-\boldsymbol{x}\rightarrow\infty}\left[\widehat{\Omega}(\boldsymbol{x},t),\widehat{\Omega}^{\dagger}(\boldsymbol{x}',t)\right]=0;\tag{2.30}$$

note that the field commutator (2.29) does not depend on either the modified mass or the dissipative parameter γ.

Now we obtain the conjugate canonical moment from Eq.(2.11),

$$\Pi_\Omega \equiv \frac{\partial \mathcal{L}}{\partial \dot{\Omega}} = \dot{\overline{\Omega}} - j\frac{\gamma}{2}\overline{\Omega}; \tag{2.31}$$

then we construct the conjugate momentum operator,

$$\begin{aligned}\widehat{\Pi}_\Omega(\boldsymbol{x},t) = -\Big\{ & e^{\frac{\gamma}{2}t} J^+ \int_{-\infty}^{\infty} d\boldsymbol{k}_2 \; i\omega_{k_2} \left[\hat{\mathtt{b}}_1\left(\boldsymbol{k}_2\right) e^{-i\left(\omega_{k_2}t - \boldsymbol{k}_2\cdot\boldsymbol{x}\right)} - \hat{\mathtt{b}}_2^\dagger\left(\boldsymbol{k}_2\right) e^{i\left(\omega_{k_2}t - \boldsymbol{k}_2\cdot\boldsymbol{x}\right)}\right] \\ & + e^{-\frac{\gamma}{2}t} J^- \int_{-\infty}^{\infty} d\boldsymbol{k}_1 \; i\omega_{k_1} \left[\hat{\mathtt{a}}_1^\dagger\left(\boldsymbol{k}_1\right) e^{-i\left(\omega_{k_1}t - \boldsymbol{k}_1\cdot\boldsymbol{x}\right)} - \hat{\mathtt{a}}_2\left(\boldsymbol{k}_1\right) e^{i\left(\omega_{k_1}t - \boldsymbol{k}_1\cdot\boldsymbol{x}\right)}\right] \Big\},\end{aligned} \tag{2.32}$$

therefore, we have the following equal-time field commutator,

$$\left[\widehat{\Pi}_\Omega(\boldsymbol{x},t), \widehat{\Pi}_\Omega^\dagger\left(\boldsymbol{x}',t\right)\right] = (2\pi)^n \left[J^+(\rho_1 - \overline{\rho}_4) + J^-(\overline{\rho}_1 - \rho_4)\right] \left[\delta_n''\left(\boldsymbol{x}' - \boldsymbol{x}\right) - \left(m^2 - \frac{\gamma^2}{4}\right)\delta_n(\boldsymbol{x}' - \boldsymbol{x})\right], \tag{2.33}$$

where δ'' is the second derivative of the delta function. Again the dependence on dissipative factors $e^{\pm\frac{\gamma}{2}t}$ has been eliminated due to the condition (1.6). In contrast to [1], here we have the presence of the modified mass; with the following limits,

$$\lim_{(m^2 - \frac{\gamma^2}{4}) \to 0} \left[\widehat{\Pi}_\Omega(\boldsymbol{x},t), \widehat{\Pi}_\Omega^\dagger\left(\boldsymbol{x}',t\right)\right] = (2\pi)^n \left[J^+(\rho_1 - \overline{\rho}_4) + J^-(\overline{\rho}_1 - \rho_4)\right] \delta_n''\left(\boldsymbol{x}' - \boldsymbol{x}\right). \tag{2.34}$$

$$\lim_{m \to 0} \left[\widehat{\Pi}_\Omega(\boldsymbol{x},t), \widehat{\Pi}_\Omega^\dagger\left(\boldsymbol{x}',t\right)\right] = (2\pi)^n \left[J^+(\rho_1 - \overline{\rho}_4) + J^-(\overline{\rho}_1 - \rho_4)\right] \left(\delta_n''\left(\boldsymbol{x}' - \boldsymbol{x}\right) + \frac{\gamma^2}{4}\delta_n(\boldsymbol{x}' - \boldsymbol{x})\right). \tag{2.35}$$

In both formulations, the one developed in [1] and in the present work, the field commutator (2.33) satisfies the same limits that we mentioned in equation (2.30). Similarly, we construct another equal-time commutator,

$$\left[\widehat{\Omega}(\boldsymbol{x},t), \widehat{\Pi}_\Omega\left(\boldsymbol{x}',t\right)\right] = -i \int_{-\infty}^{\infty} d\boldsymbol{k}\; \omega_{\boldsymbol{k}} \left[\left(J^+\rho_1 + J^-\bar{\rho}_1\right) e^{i\boldsymbol{k}\cdot(\boldsymbol{x}' - \boldsymbol{x})} + \left(J^+\bar{\rho}_4 - J^-\rho_4\right) e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}' - \boldsymbol{x})}\right]. \tag{2.36}$$

This field commutator is the only one (of the three ones in this formulation) that does not vanish in both, the standard scheme and in the present scheme. In the standard scheme, this field commutator is simply a Dirac delta, without providing more information.

The integral in the commutator (2.36) can be solved both numerically and analytically, considering certain restrictions for each case. We solve this integral in one spatial dimension (or $1+1$ background space-time), and due to the presence of the frequencies, we use the dispersion relation (2.22) with $(m^2 - \frac{\gamma^2}{4}) > 0$, for obtaining,

$$\left[\widehat{\Omega}(x,t), \widehat{\Pi}_{\Omega}(x',t)\right] = -2i\left[J^{+}(\rho_1 - \overline{\rho}_4) + J^{-}(\overline{\rho}_1 - \rho_4)\right] \frac{\sqrt{m^2 - \frac{\gamma^2}{4}}}{(x'-x)} K_1\left(\left|m^2 - \frac{\gamma^2}{4}\right|(x'-x)\right) \quad (2.37)$$

where $K_1(x)$ is the modified Bessel function. The integral (2.36) generally does not converge. It can also be solved numerically when the constraint $(m^2 - \frac{\gamma^2}{4}) < 0$ is considered. On the other hand, we can see in Fig.2.1 the behavior of the commutator (2.37), for the the following cases: it converges to zero as $m_{mod} \to +\infty$ and the field commutator diverges as $m_{mod} \to 0$.

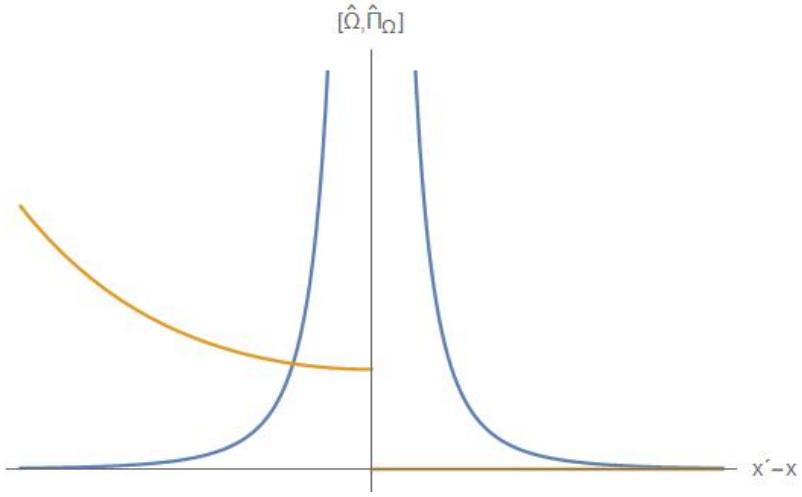

Figure 2.1: Graphics corresponding to the real part $\left(\frac{\sqrt{m^2-\frac{\gamma^2}{4}}}{(x'-x)}K_1\right)$ of (2.37); where K is a function of $(x'-x)$ for a fixed m_{mod}. The blue line corresponds to the real part and the orange line corresponds to the imaginary part. Both lines are not continuous at $(x'-x)=0$. Note that this field commutator behaves similarly to a Dirac delta function as would be expected in standard quantum field theory. The most notable feature between the expression (2.37) and a Dirac delta is that the value of the Dirac delta is zero everywhere except at the origin, where it is infinite; in Eq.(2.37) we have that at the origin it has an infinite value and it has a finite value for $(x'-x) \neq 0$; furthermore, the commutator tends to zero as $(x'-x) \to \infty$.

Similar to the limit computed for the field commutator (2.34), we also compute the limit when $(m^2 - \frac{\gamma^2}{4}) \to 0$ for the field commutator (2.37), which shows us that this limit diverges.

On the other hand, we can leave the γ-parameter intact and see the following limits,

$$\lim_{m\to 0}\left[\widehat{\Omega}(x,t),\widehat{\Pi}_{\Omega}\left(x',t\right)\right]=\left[J^{+}(\rho_1-\overline{\rho}_4)+J^{-}(\overline{\rho}_1-\rho_4)\right]|\gamma|K_1\left[\frac{\gamma^4}{16}(x'-x),\right] \tag{2.38}$$

where we have the explicit dependence of the dissipative parameter γ. We also have the following limits,

$$\lim_{m\to\infty}\left[\widehat{\Omega}(x,t),\widehat{\Pi}_{\Omega}\left(x',t\right)\right]=0, \qquad \lim_{(m^2-\frac{\gamma^2}{4})\to\infty}\left[\widehat{\Omega}(x,t),\widehat{\Pi}_{\Omega}\left(x',t\right)\right]=0. \tag{2.39}$$

Again, in both formulations, namely in [1] and in this work, the field commutator (2.37) satisfies the limits (2.30).

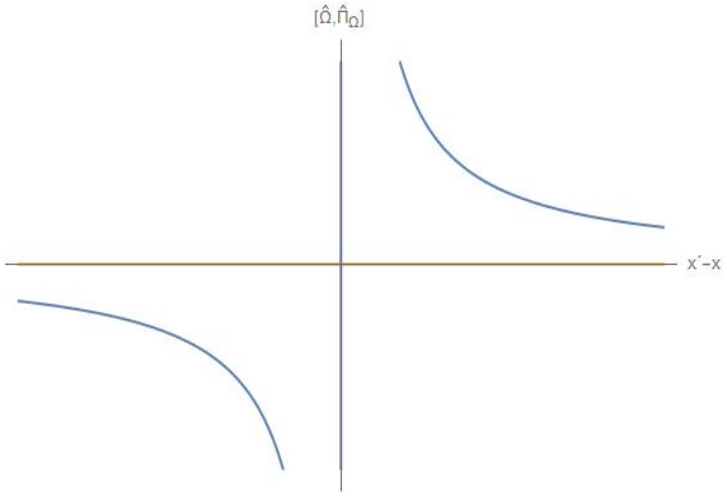

Figure 2.2: Graphics corresponding to the commutator $\left[\widehat{\Omega},\widehat{\Pi}_{\Omega}\right]$ in (2.37) for a fixed $(x'-x)>0$. The orange line corresponds to the imaginary part and the blue line represents the real part; both lines are continuous throughout the real interval. For this field commutator, in the work [1] the imaginary part vanishes in this region $(x'-x)>0$, and as commented, the imaginary part that does not vanish will lead to a commutator belonging to the extended ring.

We can see that the integrals defining all field commutators depend in general on the spatial dimension; moreover, the field commutators (2.33) and (2.36) have an explicit dependency on γ through the modified mass and the frequencies ω_k; however, the field commutator (2.29) does not depend on any of these quantities.

Chapter 3

The formalism of the dissipative Hamiltonian

In this section we will obtain the Hamiltonian operator which, as will be seen, is independent of the exponential dissipative factor and, which in turn will help to build the evolution operator and the charge operator, and in turn a conserved current, according to Noether's theorem. In addition, in this chapter, we will present the natural emergence of a function, which plays a fundamental role in addressing a crucial property in the study of ergodic theory, the geometry of the total system. This will allow us to connect quantum field theory with ergodic theory, addressing a topic that is not yet fully explored and has little information in the literature.

3.1 The Hamiltonian

In [1], the function $G(k, k'; system)$ is introduced, and depends on the spatial configurations for the subsystem and the environment; this function is real, which leads to maintaining a double integration. In the formulation at hand we obtain a similar function, also identified with the total geometry of the system; our function now contains the imaginary unit i, which allows us to simplify the Hamiltonian operator, as discussed in [1]; we will describe this simplification for specific geometries.

We start with the classical Hamiltonian operator,

$$H=\int dx^{d}\left[2\Pi_{\Omega}\Pi_{\overline{\Omega}}+\frac{1}{2}\partial_{i}\Omega\partial_{i}\overline{\Omega}+j\frac{\gamma}{2}\left(\overline{\Omega}\Pi_{\overline{\Omega}}-\Omega\Pi_{\Omega}\right)+\frac{1}{2}\left(m^{2}-\frac{\gamma^{2}}{4}\right)\Omega\overline{\Omega}\right]; \tag{3.1}$$

each term is a $U(1)\times SO(1,1)$-invariant; and again, the dissipative factors $e^{\pm\gamma t}$ do not appear due to property (1.6). Thus the Hamiltonian operator is,

$$\hat{\mathcal{H}}\left(\gamma;t\right)=\int_{-\infty}^{\infty}d\boldsymbol{k}\int_{-\infty}^{\infty}d\boldsymbol{k}'\left\{H_{\gamma}\,G(\boldsymbol{k},\boldsymbol{k}';\gamma;t)\left[J^{+}\left\{\hat{\mathtt{a}}_{1}(\boldsymbol{k}),\hat{\mathtt{b}}_{1}(\boldsymbol{k}')\right\}+J^{-}\left\{\hat{\mathtt{b}}_{2}(\boldsymbol{k}'),\hat{\mathtt{a}}_{2}(\boldsymbol{k})\right\}\right]+h.c\right\}, \tag{3.2}$$

where the following complex functions have been defined,

$$G(\boldsymbol{k},\boldsymbol{k}';\gamma;t)\equiv e^{i(\omega_{k}-\omega_{k'})t}\int\limits_{Complete\ System}d\boldsymbol{x}^{n}\;e^{-i\boldsymbol{x}\cdot(\boldsymbol{k}-\boldsymbol{k}')}=e^{i(\omega_{k}-\omega_{k'})t}\;\mathtt{I}\left(\boldsymbol{k}-\boldsymbol{k}'\right), \tag{3.3}$$

$$H_{\gamma}\equiv2\,\omega_{k'}\omega_{k}+\frac{1}{2}\boldsymbol{k}'\cdot\boldsymbol{k}+\frac{i\gamma}{2}(\omega_{k'}+\omega_{k})+\frac{1}{2}\left(m^{2}-\frac{\gamma^{2}}{4}\right). \tag{3.4}$$

In (3.2) appears the conjugate of (3.3) and (3.4). The function $\mathtt{I}(\boldsymbol{k}-\boldsymbol{k}')$ and its conjugate, are defined as integrals over the total system, and they will depend on the geometry.

3.2 Evolution operator and the Hamiltonian in $(1+1)$ space-time

In the expression (3.2) there is not yet a defined geometry for the total system. Different geometries can be considered; some specific configurations are shown in Fig.3.1 (see [37] for other geometries). We focus first on the a)-geometry in the Fig.3.1. Therefore, the integration interval for the momenta (3.1) will be $(-\infty,\infty)$. Hence, the $\mathtt{I}(\boldsymbol{k}-\boldsymbol{k}')$ functions will be reduced to Dirac deltas; thus the Hamiltonian operator takes the form,

$$\hat{\mathcal{H}}\left(m;\gamma;t\right)=\int_{-\infty}^{\infty}d\boldsymbol{k}\left\{H_{\gamma}\left[J^{+}\left\{\hat{\mathtt{a}}_{1}\left(\boldsymbol{k}\right),\hat{\mathtt{b}}_{1}\left(\boldsymbol{k}\right)\right\}+J^{-}\left\{\hat{\mathtt{b}}_{2}\left(\boldsymbol{k}\right),\hat{\mathtt{a}}_{2}\left(\boldsymbol{k}\right)\right\}\right]+h.c\right\}. \tag{3.5}$$

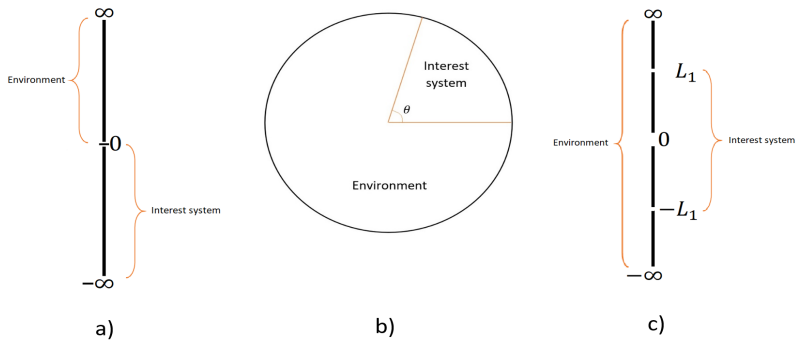

Figure 3.1: Some examples of one-dimension geometries for the total system.

Due to the presence of the complex unit i the Hamiltonian operator could be reduced to a single k-integration, as we mentioned at the beginning of this section; however, we see that the Dirac delta only arises when we have an infinite total system. Later we will consider finite geometries, where the finite geometry of the total system will play an important role.
The unitary evolution operator can be constructed as the exponential of the Hamiltonian operator by using the complex unit i or, as in the case of [1], with the complex unit j. We can also think of constructing the evolution operator with a hyper-complex exponential, that is, containing the two complex units $e^{ij\hat{\mathcal{H}}}$; however, we will use the exponential with the standard unit i since the hyperbolic complex unit in the exponential is always implicit due to the bases (J^+, J^-). Thus our time evolution operator is,

$$\begin{aligned} e^{i\hat{\mathcal{H}}t} &\equiv e^{it\int d\boldsymbol{k}\left[H_{\gamma^+}\left(J^+\{\hat{\mathtt{a}}_1(\boldsymbol{k}),\hat{\mathtt{b}}_1(\boldsymbol{k})\}+J^-\{\hat{\mathtt{b}}_2(\boldsymbol{k}),\hat{\mathtt{a}}_2(\boldsymbol{k})\}\right)+H_{\gamma^-}\left(J^+\{\hat{\mathtt{b}}_2^\dagger(\boldsymbol{k}),\hat{\mathtt{a}}_2^\dagger(\boldsymbol{k})\}+J^-\{\hat{\mathtt{a}}_1^\dagger(\boldsymbol{k}),\hat{\mathtt{b}}_1^\dagger(\boldsymbol{k})\}\right)\right]} \\ &= J^+e^{it\int d\boldsymbol{k}\left[H_{\gamma^+}\{\hat{\mathtt{a}}_1(\boldsymbol{k}),\hat{\mathtt{b}}_1(\boldsymbol{k})\}+H_{\gamma^-}\{\hat{\mathtt{b}}_2^\dagger(\boldsymbol{k}),\hat{\mathtt{a}}_2^\dagger(\boldsymbol{k})\}\right]}+J^-e^{it\int d\boldsymbol{k}\left[H_{\gamma^+}\{\hat{\mathtt{b}}_2(\boldsymbol{k}),\hat{\mathtt{a}}_2(\boldsymbol{k})\}+H_{\gamma^-}\{\hat{\mathtt{a}}_1^\dagger(\boldsymbol{k}),\hat{\mathtt{b}}_1^\dagger(\boldsymbol{k})\}\right]}, \end{aligned} \tag{3.6}$$

where the decomposition property (1.10), and the annihilation property (1.6), have been used. We can see that with this operator, the transition to the statistical case can be made.

3.3 The charge operator

The following conservation law follows from the invariance of the Lagrangian under the action of $U(1)\times SO(1,1)$,

$$\partial_t(\bar{\Omega}\dot{\Omega}-\Omega\dot{\bar{\Omega}}+j\gamma\Omega\bar{\Omega})+\partial_i\left(\bar{\Omega}\partial^i\Omega-\Omega\partial^i\bar{\Omega}\right)=0; \tag{3.7}$$

with the charge density $j_0 = (\overline{\Omega}\dot{\Omega} - \Omega\dot{\overline{\Omega}} + j\gamma\Omega\overline{\Omega})$ and the current $j^i = \left(\bar{\Omega}\partial^i\Omega - \Omega\partial^i\overline{\Omega}\right)$. Note the term $j\gamma\Omega\overline{\Omega}$ in the charge density, which is a new term that appears due to dissipation. Therefore, the charge Q associated with this hyper-complex current is given by the following space integral,

$$Q = \int dx^n \, j_0 = \int dx^n \left(\overline{\Omega}\Pi_{\overline{\Omega}} - \Omega\Pi_{\Omega}\right); \tag{3.8}$$

in terms of the four real components of the fields, the charge is:

$$Q = \int \left[i(\dot{\phi}_2\phi_1 - \dot{\phi}_1\phi_2 + \dot{\psi}_1\psi_2 - \dot{\psi}_2\psi_1) + j(\dot{\psi}_1\phi_1 + \dot{\psi}_2\phi_2 - \dot{\phi}_1\psi_1 - \dot{\phi}_2\psi_2) \right] dx^n, \tag{3.9}$$

note that in the limit when $\psi_1 = \psi_2 = 0$, the $U(1)$-charge is recovered, which is given by $i(\dot{\phi}_2\phi_1 - \dot{\phi}_1\phi_2)$. Using the expressions (2.24) and (2.31) of the fields in terms of the $\hat{\mathtt{a}}, \hat{\mathtt{a}}^\dagger, \hat{\mathtt{b}}, \hat{\mathtt{b}}^\dagger$ operators, the charge operator can be written as,

$$\begin{aligned}
\hat{Q}(\gamma;t) &= -j \int dx^n \left[\left\{ \widehat{\Omega}, \widehat{\Pi}_{\Omega} \right\} j + c.c \right] \\
&= -2i \int_{-\infty}^{\infty} d\boldsymbol{k} \, \omega_k \left[J^+ \left\{ \hat{\mathtt{a}}_1(\boldsymbol{k}), \hat{\mathtt{b}}_1(\boldsymbol{k}) \right\} + J^- \left\{ \hat{\mathtt{a}}_1^\dagger(\boldsymbol{k}), \hat{\mathtt{b}}_1^\dagger(\boldsymbol{k}) \right\} \right. \\
&\quad \left. - \left(J^+ \left\{ \hat{\mathtt{b}}_2^\dagger(\boldsymbol{k}), \hat{\mathtt{a}}_2^\dagger(\boldsymbol{k}) \right\} + J^- \left\{ \hat{\mathtt{b}}_2(\boldsymbol{k}), \hat{\mathtt{a}}_2(\boldsymbol{k}) \right\} \right) \right].
\end{aligned} \tag{3.10}$$

We can observe that the charge operator is anti-Hermitian $\hat{Q}^\dagger = -\hat{Q}$. We remark on the differences concerning the results in [1]. First notice that in our expression (3.9) we have a $\boldsymbol{k}$-integral, since, once again, the imaginary unit i allows us to perform an integration; on the other hand, in our charge operator there are two additional anti-commutators $\left\{ \hat{\mathtt{b}}_2^\dagger, \hat{\mathtt{a}}_2^\dagger \right\}$, and $\left\{ \hat{\mathtt{b}}_2, \hat{\mathtt{a}}_2 \right\}$, which correspond to the copy-system.

Chapter 4

Asymptotic entangled states

In classical mechanics, two or more systems that do not interact with each other and are completely isolated do not exert any force on each other. This means that experiments performed on one of the systems will not influence experiments performed on the other system. In contrast, quantum mechanics recognizes that it is possible to establish correlations even without physical interactions. This correlation phenomenon is known as *entanglement*. Entanglement refers to the non-factorizability of the state vector representing a system with at least two distinct degrees of freedom [36].

Experiments exploring quantum superpositions and entanglement face a major challenge: decoherence. A quantum system would retain coherence if it were in perfect isolation, making its manipulation and study unfeasible. In contrast, if the isolation is not total, coherence is exchanged with the environment during a measurement, which would cause a gradual loss over time. This phenomenon is known as *quantum decoherence*. Decoherence can be understood as the loss of information of a system in relation to its environment, which is often modeled as a thermal bath [35]. This occurs because each system is intrinsically related to the energy state that surrounds it. When viewed in isolation, the dynamics of the system are not unitary. However, when analyzing the system together with its environment, it can be observed that its evolution is uniform [34]. Therefore, the dynamics of the system, by itself, turns out to be irreversible. As with any coupling, entanglement is generated between the system and the environment.

In this chapter, we begin by defining the vacuum as a coherent state for operators including two annihilation operators [63]. The representation of coherent states is the most suitable tool to carry out comparisons between quantum and classical theories. These states present the minimum uncertainty according to the Heisenberg uncertainty principle and, by evolving along a classical trajectory, maintain their shape as time passes. The purpose of this is to present the observables related to the Hamiltonian and the conserved charge. The effect of the time evolution operator on the vacuum state is examined in depth, leading to the creation of entangled states. This analysis reveals that entanglement holds, even in the absence of dissipation. In addition, we will explore how the geometry of the total dissipative system plays a fundamental role, acting as a bridge between quantum field theory and ergodic theory.

4.1 Vacuum as a coherent state in the $\mathbb{H}$ ring

According to [1] and [63] a definition with linear expressions in operators of annihilation, leads to a trivial quantum field theory. The vacuum will be defined as a coherent state for the following operators that involve two annihilation operators,

$$J^{+}\left\{\hat{\mathtt{a}}_1(\boldsymbol{k}),\hat{\mathtt{b}}_1(\boldsymbol{k}')\right\}|0\rangle = J^{+}\lambda_1|0\rangle\,, \qquad J^{-}\left\{\hat{\mathtt{b}}_2(\boldsymbol{k}'),\hat{\mathtt{a}}_2(\boldsymbol{k})\right\}|0\rangle = J^{-}\lambda_2|0\rangle \tag{4.1}$$

for all $\boldsymbol{k},\boldsymbol{k}'$ and $\lambda_{1,2}$ are elements of the ring $\mathbb{H}$. With subscript 1 we are representing the definition of the vacuum for the subsystem of interest and with subscript 2 for the environment. Now, considering the quadratic combination of the creation and annihilation operators in the observables (3.5) and (3.10) on the vacuum state, we have,

$$\begin{aligned} &J^{-}\left\{\hat{\mathtt{a}}_1(\boldsymbol{k}),\hat{\mathtt{b}}_1(\boldsymbol{k}')\right\}^{\dagger}|0\rangle = J^{-}\hat{\mathtt{a}}_1^{\dagger}(\boldsymbol{k})\hat{\mathtt{b}}_1^{\dagger}(\boldsymbol{k}')|0\rangle + J^{-}\hat{\mathtt{b}}_1^{\dagger}(\boldsymbol{k}')\hat{\mathtt{a}}_1^{\dagger}(\boldsymbol{k})|0\rangle = J^{-}\left(|^1\mathtt{a}_k,{}^1\,\mathtt{b}_{k'}\rangle + |^1\mathtt{b}_{k'},{}^1\,\mathtt{a}_k\rangle\right),\\ &J^{+}\left\{\hat{\mathtt{b}}_2(\boldsymbol{k}'),\hat{\mathtt{a}}_2(\boldsymbol{k})\right\}^{\dagger}|0\rangle = J^{+}\left(|^2\mathtt{b}_{k'},{}^2\,\mathtt{a}_k\rangle + |^2\mathtt{a}_{k'},{}^2\,\mathtt{b}_k\rangle\right). \end{aligned} \tag{4.2}$$

To obtain the vacuum expectation values corresponding to the Hamiltonian and the charge operator we use the expression (4.1),

$$\langle 0|\hat{\mathcal{H}}|0\rangle = \langle 0|0\rangle\int_{-\infty}^{\infty} d\boldsymbol{k}H_{\gamma}\left[\left(J^{+}\lambda_1 + c.c.\right) + \left(J^{-}\lambda_2 + c.c.\right)\right], \tag{4.3}$$

$$\langle 0|\hat{Q}|0\rangle = -\langle 0|0\rangle \int_{\infty}^{\infty} d\boldsymbol{k} \left[\left(J^{+}\lambda_1 + c.c.\right) - \left(J^{-}\lambda_2 + c.c.\right)\right]. \tag{4.4}$$

The integrals in (4.3) and (4.4) are divergent, but these divergences can be eliminated. To visualize this, we write the part on the right hand side of (4.1) as,

$$J^{+}\lambda_1 + c.c. = \lambda_{\mathtt{I}} + \lambda_{\mathtt{II}}, \qquad\qquad J^{-}\lambda_2 + c.c. = \lambda_{\mathtt{III}} - \lambda_{\mathtt{IV}}. \tag{4.5}$$

Now we can write to (4.3) and (4.4) in terms of the coefficients $(\lambda_{\mathtt{I},\mathtt{II},\mathtt{III},\mathtt{IV}})$, and separating the real part and imaginary, we have,

$$\langle 0|\hat{\mathcal{H}}|0\rangle = \langle 0|0\rangle \int_{-\infty}^{\infty} d\boldsymbol{k} \left[H_{\boldsymbol{k}'}\left(\lambda_{\mathtt{I}} + \lambda_{\mathtt{II}} + \lambda_{\mathtt{III}} - \lambda_{\mathtt{IV}}\right) + 2ij\omega_{\boldsymbol{k}}\gamma\left(\lambda_{\mathtt{I}} + \lambda_{\mathtt{II}} - \lambda_{\mathtt{III}} + \lambda_{\mathtt{IV}}\right)\right], \tag{4.6}$$

$$\langle 0|\hat{Q}|0\rangle = -2i\,\langle 0|0\rangle \int_{\infty}^{\infty} d\boldsymbol{k}\left(\omega_{\boldsymbol{k}}\right)\left[\lambda_{\mathtt{I}} + \lambda_{\mathtt{II}} - \lambda_{\mathtt{III}} + \lambda_{\mathtt{IV}}\right]; \tag{4.7}$$

where $H_{\boldsymbol{k}'}$ is,

$$H_{\boldsymbol{k}'} = 2\omega_k^2 + \frac{1}{2}k^2 + \frac{1}{2}\left(m^2 - \frac{\gamma^2}{4}\right). \tag{4.8}$$

Therefore, by imposing the restrictions,

$$\lambda_{\mathtt{I}} + \lambda_{\mathtt{II}} = 0, \qquad\qquad \lambda_{\mathtt{III}} - \lambda_{\mathtt{IV}} = 0; \tag{4.9}$$

it means, imposing the vanishing of the projections (4.5), hence, the vanishing of the v.e.v'.s (4.6) and (4.7) are achieved. The first condition corresponds to the case of $\lambda_1 \sim J^{-}$ and the second corresponds to $\lambda_2 \sim J^{+}$ in the definitions (4.1), where $\sim$ means proportional to.

4.2 Entangled states

Considering the action of the evolution operator in the form (3.6) on the vacuum state defined above, one obtains,

$$\begin{aligned}|0(t)\rangle &\equiv e^{i\hat{\mathcal{H}}t}\,|0\rangle \\ &= e^{i\int_{-\infty}^{\infty}d\boldsymbol{k}\left[H_{\gamma^+}\left(J^+\left\{\hat{\mathtt{a}}_1(\boldsymbol{k}),\hat{\mathtt{b}}_1(\boldsymbol{k})\right\}+J^-\left\{\hat{\mathtt{b}}_2(\boldsymbol{k}),\hat{\mathtt{a}}_2(\boldsymbol{k})\right\}\right)+H_{\gamma^-}\left(J^+\left\{\hat{\mathtt{b}}_2^\dagger(\boldsymbol{k}),\hat{\mathtt{a}}_2^\dagger(\boldsymbol{k})\right\}+J^-\left\{\hat{\mathtt{a}}_1^\dagger(\boldsymbol{k}),\hat{\mathtt{b}}_1^\dagger(\boldsymbol{k})\right\}\right)\right]t}\,|0\rangle\,;\end{aligned} \tag{4.10}$$

if the original vacuum is normalized, the evolved state (4.10) will also be a normalized state for any t:

$$\langle 0(t)|0(t)\rangle \equiv \langle 0|\,e^{-i\hat{\mathcal{H}}t}\cdot e^{i\hat{\mathcal{H}}t}\,|0\rangle = \langle 0|0\rangle\,. \tag{4.11}$$

As well known in quantum mechanics the representations of the canonical commutation relations are all unitarily equivalent to each other (Stone-Von Neuman theorem); thus the evolved vacuum leaves the original space of states, in a finite volume case; since $t\to\infty$, this yields an asymptotic state orthogonal to the initial state $|0\rangle$ [33]. On the other hand, in a QFT the number of degrees of freedom is infinite, thus, there are infinite unitarily non-equivalent representations of the canonical commutation relations, this allows describing different systems that can be in different phases [32]. Using this, a dissipative quantum model of the brain has been studied where there is a non-unit time evolution [31].

Considering the evolved vacuum (4.10), we see that this state evolves always aligned with the original vacuum state,

$$\begin{aligned}\langle 0|0(t)\rangle &= e^{\int_{-\infty}^{\infty}d\boldsymbol{k}\left[H_{\gamma^+}J^+\lambda_1-c.c.\right]t-\int_{-\infty}^{\infty}d\boldsymbol{k}\left[H_{\gamma^+}J^-\lambda_2-c.c.\right]t}\,\langle 0|0\rangle \\ &= \langle 0|0\rangle\exp\left\{\int_{-\infty}^{\infty}d\boldsymbol{k}\left[iH_k\left(\lambda_{\mathtt{I}}+\lambda_{\mathtt{II}}+\lambda_{\mathtt{III}}-\lambda_{\mathtt{IV}}\right)+j2\omega_k\gamma\left(\lambda_{\mathtt{I}}+\lambda_{\mathtt{II}}+\lambda_{\mathtt{III}}-\lambda_{\mathtt{IV}}\right)\right]t\right\}\neq 0.\end{aligned} \tag{4.12}$$

The usual imaginary part and the hyperbolic imaginary part of the second line in Eq.(4.12) have been separated. The expression in (4.12) has the form of a bi-complex phase $e^{i\alpha}e^{j\beta}$, then the property (1.15) can be used,

$$\langle 0|0(t)\rangle = \langle 0|0\rangle\left(\cos[\alpha(t)]\cosh[\beta(t)]+i\sin[\alpha(t)]\cosh[\beta(t)]+j\cos[\alpha(t)]\sinh[\beta(t)]+ij\sin[\alpha(t)]\sinh[\beta(t)]\right), \tag{4.13}$$

whit,

$$\begin{aligned}\beta(t) &= 2\gamma t \int_{-\infty}^{\infty} d\boldsymbol{k}\, \omega_k \left(\lambda_{\mathtt{I}} + \lambda_{\mathtt{II}} + \lambda_{\mathtt{III}} - \lambda_{\mathtt{IV}}\right) = 0, \\ \alpha(t) &= t \int_{-\infty}^{\infty} d\boldsymbol{k}\, H_k \left(\lambda_{\mathtt{I}} + \lambda_{\mathtt{II}} + \lambda_{\mathtt{III}} - \lambda_{\mathtt{IV}}\right) = 0,\end{aligned} \tag{4.14}$$

the vanishing is due to the condition (4.9), which ensures that the temporal evolution does not leave the original Hilbert space. Furthermore, as we know, the proportionality conditions $(\lambda_1 \sim J^-), (\lambda_2 \sim J^+)$, will lead to the entanglement dynamics is constructed in both directions, namely, for the subsystem of interest (J^+) and, for the environment (J^-) perspective. With these constraints in mind, and using the expansion $e^{J^{\pm}\chi} = 1 + J^{\pm}\sum_{n=1}^{\infty}\frac{\chi^n}{n!}$, we have that the evolved state (4.10) can be rewritten as,

$$\begin{aligned}|0(t)\rangle &= \exp\left\{t\int_{\infty}^{\infty} d\boldsymbol{k}\, H_\gamma \left(J^+\left\{\hat{\mathtt{b}}_2^\dagger, \hat{\mathtt{a}}_2^\dagger\right\} - J^-\left\{\hat{\mathtt{a}}_1^\dagger, \hat{\mathtt{b}}_1^\dagger\right\}\right)\right\}|0\rangle \\ &= |0\rangle + J^+ t\int_{-\infty}^{\infty} d\boldsymbol{k}\, H_\gamma\left(|^2\mathtt{b}_k, ^2\,\mathtt{a}_k\rangle + |^2\mathtt{a}_k, ^2\,\mathtt{b}_k\rangle\right) - J^- t\int_{-\infty}^{\infty} d\boldsymbol{k}\, H_\gamma\left(|^1\mathtt{a}_k, ^1\,\mathtt{b}_k\rangle + |^1\mathtt{b}_k, ^1\,\mathtt{a}_k\rangle\right) + ...;\end{aligned} \tag{4.15}$$

the notation (4.1) for excited states is used. This state is then entangled in moments since it cannot be factorized into the product of single modes. In the next section, we will see that entanglement is present even without dissipation.

4.3 Entangled asymptotic states

In this section, we return to the general Hamiltonian operator (3.5), in which the geometry of the total system has been not specified; now, we consider different geometrical configurations and we shall construct the entangled states associated. We begin by considering the following evolved state,

$$|0(t)\rangle \equiv e^{i\hat{\mathcal{H}}t}|0\rangle = \exp\left\{i\int d\boldsymbol{k}\int_{-\infty}^{+\infty} d\boldsymbol{k}'\left[H_\gamma G\left(J^+\left\{\hat{\mathtt{b}}_2^\dagger(\boldsymbol{k}'), \hat{\mathtt{a}}_2^\dagger(\boldsymbol{k})\right\} + J^-\left\{\hat{\mathtt{a}}_1^\dagger(\boldsymbol{k}), \hat{\mathtt{b}}_1^\dagger(\boldsymbol{k}')\right\}\right)\right] t\right\}|0\rangle\,, \tag{4.16}$$

where we used $\hat{\mathcal{H}}$ given in (3.5), and the function G is defined in (3.3). Also, we have used the conditions given in Eq.(4.9), that is, the eigenvalues in Eq.(4.1) have disappeared.

The above expression can be rewritten as,

$$e^{i\hat{\mathcal{H}}t}\,|0\rangle = \exp\left\{ i \int d\boldsymbol{k} \int_{-\infty}^{+\infty} d\boldsymbol{k}'\; H_{\gamma}(\boldsymbol{k},\boldsymbol{k}')\, t\, e^{i(\omega_{k'}-\omega_k)t}\, \mathtt{I}^{+}(\boldsymbol{k}-\boldsymbol{k}')\, \hat{\mathtt{W}}(\boldsymbol{k},\boldsymbol{k}') \right\} |0\rangle\,, \tag{4.17}$$

where,

$$\hat{\mathtt{W}}(\boldsymbol{k},\boldsymbol{k}') \equiv J^{+}\left\{ \hat{\mathtt{b}}_2^{\dagger}(\boldsymbol{k}'), \hat{\mathtt{a}}_2^{\dagger}(\boldsymbol{k}) \right\} + J^{-}\left\{ \hat{\mathtt{a}}_1^{\dagger}(\boldsymbol{k}), \hat{\mathtt{b}}_1^{\dagger}(\boldsymbol{k}') \right\}, \tag{4.18}$$

and $\mathtt{I}(\boldsymbol{k}-\boldsymbol{k}')$ is given in Eq.(3.3) and depending also on the geometry of the total system. An important aspect to consider in the expression (4.17) is that, when integrating over the total system, the expression $\mathtt{I}(\boldsymbol{k}-\boldsymbol{k}')$ will have different forms when considering different geometries. An example, if the total system has the geometry of the area of a disk, then the integration will give us a modified Bessel function, which must be considered when performing the $\boldsymbol{k}$-integrations in (4.17).

Since we have incorporated the imaginary unit i in the present work, this has a great implication, since it opens the way for us to begin to study the ergodicity of systems that undergo entanglement, a characteristic that contrasts with [1], where it is not possible to explore this area. As it is well known, asymptotic states are closely related to the ergodic theory; this theory has various applications in different areas of mathematics [30][29] and physics [28][27], where ergodicity criteria and the physical properties are established for specific configurations of a system. These configurations can have well-defined asymptotic states (states that reach thermal equilibrium) or cyclo-stationary asymptotic states (states that do not thermalize, but relax to periodic states) [13].

This section is the beginning of a long way to go to study the ergodicity in entangled quantum systems with a new approach, the hyper-complex formalism.

4.3.1 An asymptotic entangled state (finite total system)

For this subsection we consider the total one-dimensional system as represented in Fig.4.1 with finite ranges, that is, the range for the environment is $(L_2, 0)$ and the range for the system of interest is $(0, L_1)$, thus the total system is contained in the interval (L_1, L_2).

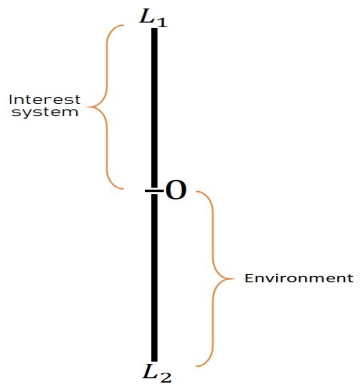

Figure 4.1: Total system contained in a finite interval.

For the state (4.16), distribution for a discrete-time sequence can be identified, namely $te^{i(\omega_{k'}-\omega_k)t}$, as the generating function of the delta function $\delta_t[i(\omega_{k'}-\omega_k)]$; this delta sequence satisfies:

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n(is)f(s)ds=-if(0). \tag{4.19}$$

Hence, we have the following asymptotic state with dissipation,

$$\begin{aligned}\lim_{t\to\infty}e^{i\hat{\mathcal{H}}t}\,|0\rangle &= \exp\left\{i\lim_{t\to\infty}\int dk\int_{-\infty}^{+\infty}dk'\;H_\gamma(k,k')\,t\,e^{i(\omega_{k'}-\omega_k)t}\,\mathtt{I}^{+}(k-k')\,\hat{\mathtt{W}}(k,k')\right\}|0\rangle\\ &= \exp\left\{5\,(L_2-L_1)\int_{-\infty}^{\infty}dk\left[\frac{\omega_k^3}{k}-i\frac{2\omega_k^2\gamma}{k}\right]\hat{\mathtt{W}}(k,k)\right\}|0\rangle\,,\end{aligned} \tag{4.20}$$

where the function $H_\gamma(k,k')$ given in Eq.(3.4) has been considered, and the function $\mathtt{I}(k-k')$ reduces, in this case, to (L_2-L_1), which is the total length of the one-dimensional system; in addition, the real part and the imaginary part have been separated. Furthermore, using the expansion (1.11), and defining,

$$\eta_k\equiv\frac{\omega_k^3}{k}-i\frac{2\omega_k^2\gamma}{k}, \tag{4.21}$$

the state (4.20) can be described as,

$$\begin{aligned}\lim_{t\to\infty}e^{i\hat{\mathcal{H}}t}\,|0\rangle = |0\rangle &+ 5J^{+}(L_2-L_1)\int_{-\infty}^{\infty}dk\;\eta_k\;\left(|^2\mathtt{b}_k,^2\,\mathtt{a}_k\rangle+|^2\mathtt{a}_k,^2\,\mathtt{b}_k\rangle\right)+\cdots\\ &+\;5J^{-}(L_2-L_1)\int_{-\infty}^{\infty}dk\;\eta_k\;\left(|^1\mathtt{a}_k,^1\,\mathtt{b}_k\rangle+|^1\mathtt{b}_k,^1\,\mathtt{a}_k\rangle\right)+\cdots\end{aligned} \tag{4.22}$$

The notation given in Eq.(4.2) for excited states has been used. In expression (4.22) we can identify what the subsystem of interest and the environment observe, by considering the projections of (4.22) on the basis (J^+, J^-). For the point of view of the subsystem of interest, we project J^+ on Eq.(4.22),

$$J^+ \lim_{t\to\infty} e^{i\hat{\mathcal{H}}t}\,|0\rangle = J^+\left(|0\rangle + 5(L_2-L_1)\int_{-\infty}^{\infty} dk\;\eta_k\;\left(|^2\mathtt{b}_k,^2\,\mathtt{a}_k\rangle + |^2\mathtt{a}_k,^2\,\mathtt{b}_k\rangle\right) + \cdots\right); \tag{4.23}$$

therefore, the subsystem of interest observes the creation of bosons of type-$\mathtt{a}$ and type-$\mathtt{b}$, corresponding to the environment. Now, if we project Eq.(4.22) on the basis J^-, we will have the point of view of the environment, where it observes the creation of bosons corresponding to the subsystem of interest, represented with the superscript 1. Furthermore, we also note that this state is then entangled at the moment since it cannot be factored into the product of individual modes.

Now, if the dissipative parameter γ disappears, the following asymptotic state is obtained,

$$\lim_{t\to\infty} e^{i\hat{\mathcal{H}}t}\,|0\rangle = \exp\left\{5(L_2-L_1)\int_{-\infty}^{\infty} dk\,\frac{(k^2+m^2)^{3/2}}{k}\hat{\mathtt{W}}(k,k)\right\}|0\rangle\,; \tag{4.24}$$

where $\hat{\mathtt{W}}(k,k')$ has been defined in Eq.(4.18). Similarly to Eq.(4.22), an expansion can be done for the state (4.24), obtaining,

$$\begin{aligned}\lim_{t\to\infty} e^{i\hat{\mathcal{H}}t}\,|0\rangle = |0\rangle &+ 5J^+(L_2-L_1)\int_{-\infty}^{\infty} dk\,\frac{(k^2+m^2)^{3/2}}{k}\left(|^2\mathtt{b}_k,^2\,\mathtt{a}_k\rangle + |^2\mathtt{a}_k,^2\,\mathtt{b}_k\rangle\right) + \cdots \\ &+ 5J^-(L_2-L_1)\int_{-\infty}^{\infty} dk\,\frac{(k^2+m^2)^{3/2}}{k}\left(|^1\mathtt{a}_k,^1\,\mathtt{b}_k\rangle + |^1\mathtt{b}_k,^1\,\mathtt{a}_k\rangle\right) + \cdots\,;\end{aligned} \tag{4.25}$$

similarly to (4.23), we can project the state (4.24) on the basis (J^+, J^-) to obtain the point of view of the subsystem of interest and the environment, respectively. Furthermore, we can observe that the entanglement is present even in the absence of dissipation, since a large number of degrees of freedom present in free quantum field theories induce entanglement with other degrees of freedom along the boundary [26].

4.3.2 An asymptotic entangled state (infinite total system)

Now we consider the geometry in which the one-dimensional subsystems are semi-infinite; this spatial configuration is represented in Fig.4.2.

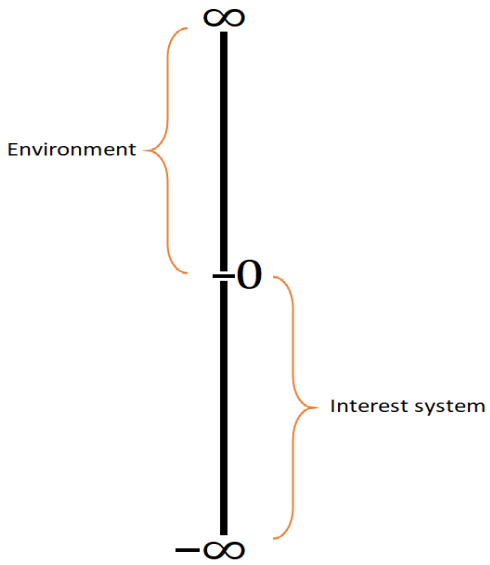

Figure 4.2: Total system contained in an infinite interval.

For this case one can show that the function $\mathtt{I}^{+}(k-k')$ reduces to a Dirac delta; thus the state in (4.16) takes the form,

$$\begin{aligned}\lim_{t\to\infty} e^{i\hat{\mathcal{H}}t}\,|0\rangle &= \exp\left\{5\pi \lim_{t\to\infty} t\int_{-\infty}^{\infty} dk\left[\frac{\gamma\omega_k}{5}+i\omega_k^2\right]\hat{\mathtt{W}}(k,k)\right\}|0\rangle \\ &= \exp\left\{\lim_{t\to\infty}\gamma t\int_{-\infty}^{\infty} dk\,\omega_k\,\hat{\mathtt{W}}(k,k)\right\}\left[\cos\left(5\pi\lim_{t\to\infty} t\int_{-\infty}^{\infty} dk\,\omega_k^2\,\hat{\mathtt{W}}(k,k)\right)+i\sin\left(...\right)\right]|0\rangle\,.\end{aligned} \tag{4.26}$$

The sine function has the same argument as the cosine. There is no asymptotic state defined since in general the expression (4.26) diverges. However, there are works where these asymptotic states are analyzed in more detail [40]. Here, the authors consider the product γt to be finite, giving a holographic interpretation between two AdS boundaries; from this, they make a comparison between their results that were applied to the entropy of the system with the results of finite temperature (interpreting γ as temperature). This intriguing comparison hints at ergodicity in the thermodynamic limit. The theory generates a remarkable result, the holographic counterpart of these two asymptotically AdS boundaries precisely relates to the BTZ black hole, with precisely

defined fields. When $t = 0$ in expression (4.26), the result is the usual vacuum state. Furthermore, if the dissipative parameter γ vanishes in the same expression, a cyclo-stationary state is achieved, which means an oscillating state,

$$\lim_{t\to\infty} e^{i\hat{\mathcal{H}}t}\,|0\rangle = \exp\left\{5i\pi \lim_{t\to\infty} t \int_{-\infty}^{\infty} dk\,\left(k^2+m^2\right)\hat{\mathtt{W}}(k,k)\right\}|0\rangle\,, \tag{4.27}$$

and analogously to the expression (4.23), we can project the state (4.27) on the basis (J^+, J^-) to have the point of view of the subsystem of interest or the environment; for the subsystem of interest, one obtains,

$$J^+ \lim_{t\to\infty} e^{i\hat{\mathcal{H}}t}\,|0\rangle = J^+ \exp\left[5i\pi \lim_{t\to\infty} t \int_{-\infty}^{\infty} dk\,\left(k^2+m^2\right)\left(|^2\mathtt{b}_{k'},{}^2\,\mathtt{a}_k\rangle + |^2\mathtt{a}_{k'},{}^2\,\mathtt{b}_k\rangle\right)\right]|0\rangle\,. \tag{4.28}$$

We can notice that, when we have the point of view of the subsystem of interest, we see bosons corresponding to the environment (subscript 2) and when we observe from the perspective of the environment, we see bosons that correspond to the subsystem of interest (subscript 1). Furthermore, the oscillation occurs when the systems are semi-infinite and the fields are free. In [40] the corresponding states with $\gamma = 0$ and $t = 0$ are equivalent; however, we see that in this work they are totally different; a more detailed explanation can be seen in [1]. As the authors comment in [13], the absence of thermalization is unusual when dissipation is absent; explaining it with an example, with the Langevin equation for a free Brownian particle of mass m, where by using the condition of zero integral friction, that is, an asymptotic condition on the dissipation kernel in the Laplace domain, they can show the absence of thermalization when $\gamma = 0$. In addition, this is accompanied by another phenomenon, the superdiffusion.

4.4 The usual weighted measure

Dealing with quantum field theories it is usual to see that the expansion used for the momentum contains the weight $\frac{1}{\sqrt{\omega_k}}$ as a measure of integration, a convenient choice for the normalization of coefficients of type $a_{\mathbf{p}}$. This choice imposes the well-known equal-time commutation relation $[\phi(t,\mathbf{x}),\Pi(t,\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y})$. In the development of this work, we did not impose this weight on our field; however, if we follow the usual path found in the literature, we will not obtain significant changes in our field commutators (2.29), (2.33) and (2.36).

In this section, we will discuss the results obtained by adding this weight. Considering the weight $\frac{1}{\sqrt{\omega_k}}$ in the measure for the field (2.24) and its moment (2.30), we obtain,

$$\left[\hat{\Omega}(\boldsymbol{x},t),\hat{\Omega}^{\dagger}(\boldsymbol{x}',t)\right] \;= 2\left[J^{+}(\rho_1-\overline{\rho}_4)+J^{-}(\overline{\rho}_1-\rho_4)\right]K_0\left(\left|m^2-\tfrac{\gamma^2}{4}\right|(\boldsymbol{x}'-\boldsymbol{x})\right), \tag{4.29}$$

$$\left[\hat{\Pi}_{\Omega}(\boldsymbol{x},t),\hat{\Pi}^{\dagger}(\boldsymbol{x}',t)\right] \;= -2\left[J^{+}(\rho_1-\overline{\rho}_4)+J^{-}(\overline{\rho}_1-\rho_4)\right]\frac{\sqrt{m^2-\frac{\gamma^2}{4}}}{(x'-x)}K_1\left(\left|m^2-\tfrac{\gamma^2}{4}\right|(x'-x)\right), \tag{4.30}$$

$$\left[\hat{\Omega}(\boldsymbol{x},t),\hat{\Pi}_{\Omega}(\boldsymbol{x}',t)\right] \;= i\,\delta_n(\boldsymbol{x}'-\boldsymbol{x})\left[J^{+}(\rho_1-\overline{\rho}_4)+J^{-}(\overline{\rho}_1-\rho_4)\right]. \tag{4.31}$$

Where $K_0(x)$ and $K_1(x)$ are modified Bessel functions of the second kind. We can notice that, by adding the weight $(1/\sqrt{\omega_k})$, the value of the field commutator $[\hat{\Omega},\hat{\Omega}^{\dagger}]$ (Eq.(2.29)) reduces to the field commutator $[\hat{\Omega},\hat{\Pi}_{\Omega}]$ above, that is $[\hat{\Omega},\hat{\Omega}^{\dagger}]\rightarrow[\hat{\Omega},\hat{\Pi}_{\Omega}]$. Similarly we have the mapping $[\hat{\Pi}_{\Omega},\hat{\Pi}^{\dagger}_{\Omega}]$ $\rightarrow[\hat{\Omega},\hat{\Pi}_{\Omega}]$ (Eq.(2.36)), etc.

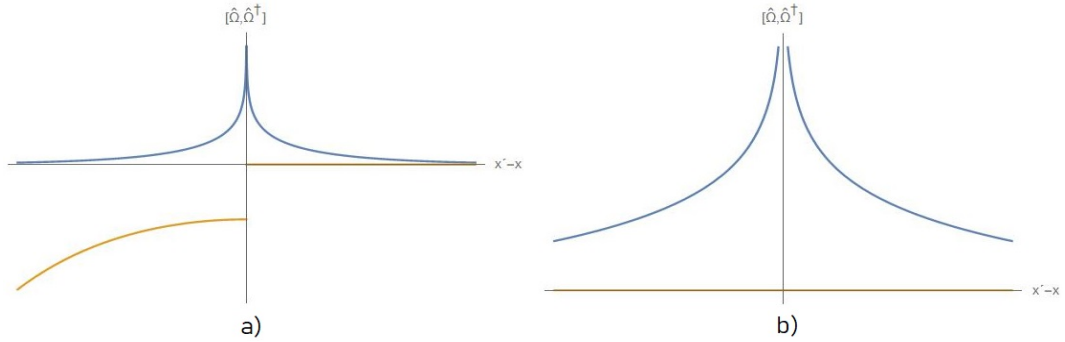

Figure 4.3: Graph corresponding to the field commutator $[\hat{\Omega},\hat{\Omega}^{\dagger}]$ in (4.28). The blue line corresponds to the real part and the orange line corresponds to the imaginary part. In graph a), K is a function of $(x'-x)$ for a fixed m_{mod} and in graph b), $(x'-x)>0$ is fixed.

The integration process of (4.30) was the same as that performed to obtain (2.37), considering $(1+1)$ background space-time.

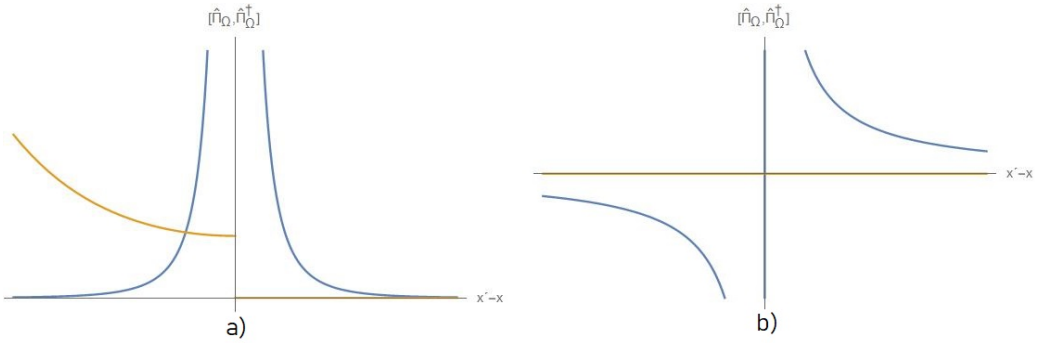

Figure 4.4: Graph corresponding to the field commutator $[\hat{\Pi}_\Omega, \hat{\Pi}_\Omega^\dagger]$ in (4.30). These graphs are the same as those obtained for the field commutator given in (2.37). The blue line corresponds to the real part and the orange line corresponds to the imaginary part. In graph a), K is a function of $(x' - x)$ for a fixed m_{mod} and in graph b), $(x' - x) > 0$ is fixed.

Thus, by using different measures, the field commutators are interchanged with each other; in the case of the field commutator $[\hat{\Omega}, \hat{\Omega}^\dagger]$, we get a more pronounced "peak" for the second choice for the measure. However, this weight in the integration measure is important, since the exponent of this weight can lead to different theories, even for closed systems [30]. By considering the new measure, it is enough to add the corresponding weight in quantities such as the Hamiltonian operator Eq.(3.2), the charge operator Eq.(3.10), etc; (moreover, the adding of the new weight does not help to eliminate the divergences in the observables Eq.(4.3) and Eq.(4.4)) and in the entangled states.

Chapter 5

Thermal Field Theory

This chapter presents the main and fundamental ideas that constitute quantum field theory (QFT) and thermal field theory (TFT), which will be used in parts of this thesis. We begin by establishing the fundamental concepts of QFT, then we show the relationship-connection between QFT and statistical mechanics, which will be done through the path integral (also known as the Feynman integral). More detailed information about QFT can be found in [Peskin]. For this section, we will use natural units.

5.1 Partition function in the path integral formalism

The QFT is a theoretical framework that merges the principles of quantum mechanics with special relativity. In this context, particles and fundamental interactions are handled as physical fields, which can be quantized. On the other hand, speaking in the mathematical sense, that is, transforming the fields that are in the Lagrangian density into field operators; in turn, this leads to the possibility of studying and describing physical systems with n degrees of freedom with creation and annihilation operators, so, using these operators and the following (anti) commutation relations,

$$\begin{aligned} [\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}] = (2\pi)^n \delta^n(\mathbf{p}-\mathbf{q}), \quad [a_{\mathbf{p}}, a_{\mathbf{q}}] = [a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = 0, \quad & for\, bosons, \\ \{a_{\mathbf{p}}, a_{\mathbf{q}}\} = (2\pi)^n \delta^n(\mathbf{p}-\mathbf{q}), \quad \{a_{\mathbf{p}}, a_{\mathbf{q}}\} = \{a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{q}}\} = 0, \quad & for\, fermions, \end{aligned} \tag{5.1}$$

it is possible to describe a variety of particles by constructing field operators.

The quantum field theory incorporates the principles of special relativity by considering equal-time commutation relations in field operators. A free scalar field can be used to visualize this incorporation of these two theories. To achieve this, such a field is written in terms of its Fourier modes, and it is also required to be a solution to the Klein-Gordon equation $(\partial_t^2+\mathbf{p}^2+m^2)\tilde{\phi}(\mathbf{p},t)=0$. In order to quantize a field, the field amplitudes are replaced with the annihilation and creation operators (5.1) (from this point on we will only consider bosons), and we obtain

$$\phi(x)=\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ipx}+a_{\mathbf{p}}^{\dagger}e^{ipx}\right)\Big|_{p^0=E_p}, \qquad E_{\mathbf{p}}=\sqrt{\mathbf{p}^2+m^2} \tag{5.2}$$

where its field commutator is given by,

$$[\phi(x),\phi(y)]=\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(e^{-ip(x-y)}-e^{ip(x-y)}\right)\begin{cases}=0, & (x-y)^2<0,\\ \neq 0, & (x-y)^2>0.\end{cases} \tag{5.3}$$

The commutation relation (5.3) tells us that when two fields are casually connected, this is, separated by a time interval, these operators cannot be commutative. This means that it is not possible to determine both operators simultaneously with arbitrary precision. However, when separated by a space-like interval, the operators will commute $(=0)$[4], which implies that they can be measured simultaneously with relative precision, because they are not casually connected.

On the other hand, it is well known that the propagation of particles through space is given by a Green function for the differential equation that represents the field of interest, such as: the Klein-Gordon equation (scalar fields), the Dirac equation (spinor fields), the Maxwell equation (vector fields), etc. This is known as the time-ordered product, where the fields are projected onto the base state in a free theory. This propagation can be represented mathematically, receiving the name of *Feynman propagator*,

$$\langle 0|\,T\phi(x)\phi(y)\,|0\rangle = D_F(x-y)=\int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip(x-y)}}{p^2+m^2-i\epsilon}. \tag{5.4}$$

The expression (5.4) is the mathematical form for a Feynman propagator for a real scalar field. This way works very well for free theories; however, when working with interacting theories, the

[4]A Lorentz transformation can be performed on a space-like interval, $(x-y)^2\to-(x-y)^2$, with which the integrand in (5.3) would disappear, but for time-like intervals, there is no similar transformation.

propagator calculation is done in an analogous way to the free theory, a problem arises, it is not known a priori how the field operators act in the vacuum state of the interacting theory. One way to perform the propagator calculation of an interacting theory is to separate it into two parts; express the part containing the interaction in the Hamiltonian in the interaction image and express the rest in terms of the vacuum state of the free theory, thus,

$$\langle\Omega|\, T\{\phi(x)\phi(y)\}\,|\Omega\rangle = \lim_{t\to(1-i\epsilon)} \frac{\langle 0|\, T\left\{\phi_{\mathtt{I}}(x)\phi_{\mathtt{I}}(y)\exp\left[-i\int_{-t}^{t} dt' H_{\mathtt{I}}(t')\right]\right\}|0\rangle}{\langle 0|\, T\left\{\exp\left[-i\int_{-t}^{t} dt' H_{\mathtt{I}}(t')\right]\right\}|0\rangle}. \tag{5.5}$$

In expression (5.5) one can obtain an infinite sum of products of Feynman propagators. This is obtained by expanding the exponentials in Taylor series in the coupling constant and by using Wick's theorem [G.C. Wick. The evaluation of the collision Matrix. Phys. Rev., 80:268, 1950]. Then, the LSZ reduction formula is used so that the scattering matrix can be expressed in terms of a series of Feynman diagrams. This method is used to calculate physical quantities such as particle decay rates explicitly. On the other hand, there is an alternative formalism to QFT, this formalism is the path integral, which generalizes the action principle of classical field mechanics. The path integral says that the evolution of a system that changes from one state to another is represented by the sum of all the different (infinite) trajectories that arise in the phase space. All the details of the above can be consulted in the extensive literature on QFT's. Following [15], we have the mathematical form,

$$\langle\Omega|\, T\{\phi(x_1)\bullet\cdots\bullet\phi(x_n)\}\,|\Omega\rangle = \lim_{t\to(1-i\epsilon)} \frac{\int \mathcal{D}\phi\ \phi(x_1)\bullet\cdots\bullet\phi(x_n)\exp\left[i\int_{-t}^{t} dt\int d^3\mathbf{x}\,\mathcal{L}\right]}{\int \mathcal{D}\phi\exp\left[i\int_{-t}^{t} dt\int d^3\mathbf{x}\,\mathcal{L}\right]}. \tag{5.6}$$

At this point, the differences between quantum field theory and the path integral become evident. The n-point function is clearly Lorentz invariant, because the exponent of the integrand has the Lagrangian density and not the Hamiltonian density. On the other hand, the classical fields enter the path integral, replacing the field operators. We also have the difference between the path integral and the free theory. In the path integral, the projection is made on the vacuum state of the theory with interaction $|\Omega\rangle$, in addition to the fact that the Lagrangian also contains the interaction part and is not separated as previously mentioned. It was mentioned that one difference between the two approaches is that a function of n points arises.

If a generating function [16] is defined, these functions can be obtained systematically:

$$Z[J] = \int \mathcal{D}\phi \exp\left[i \int d^4x (\mathcal{L} - J\phi)\right]. \tag{5.7}$$

The field J which is in the integrand of the exponential argument is called the *source term*. Using functional derivatives with respect to J, it is possible to generate n-point functions such as,

$$\langle\Omega| T\{\phi(x_1)\cdots\phi(x_n)\} |\Omega\rangle = Z[J]^{-1}\left(-i\frac{\delta}{\delta J(x_1)}\right)\cdots\left(-i\frac{\delta}{\delta J(x_n)}\right) Z[J]\Big|_{J=0}. \tag{5.8}$$

At this point, the expression (5.7) can be identified with the partition function given by statistical mechanics, it is practically the same, except for one difference. In (5.7) the exponential is a pure imaginary and positive quantity, in contrast to statistical mechanics, where the exponential is real and negative; however, both expressions have the same structure and both represent a sum over all possible configurations of a statistical weight. It is possible to establish a relationship between these two formalisms, for this, the *Matsubara formalism* is used, whose procedure is similar to that used in a Wick rotation, where the time coordinate is defined as a pure imaginary, that is, $\mathbf{x}_E \equiv \mathbf{x}$ and $x_4 \equiv ix_0$ (or $t \to i\tau$). By using this, the expression (5.7) takes the form,

$$\int \mathcal{D}\phi \exp\left[i \cdot (-i) \int d^4x_E \left(\mathcal{L}_E - J\phi\right)\right], \tag{5.9}$$

where,

$$\begin{aligned} -\mathcal{L}_E &= \frac{1}{2}\left(\partial_0\phi\partial^0\phi - \nabla\phi\cdot\nabla\phi\right) - V(\phi) \\ &= -\left[\frac{1}{2}\left(\partial_4\phi\partial_4\phi + \nabla\phi\cdot\nabla\phi\right) + V(\phi)\right] \\ &= -\left[\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + V(\phi)\right], \end{aligned} \tag{5.10}$$

which now allows us to write (5.9) as,

$$Z[J=0] = C\int \mathcal{D}\phi \exp\left[-\int d^4x \mathcal{L}_E\right], \tag{5.11}$$

obtaining a real and negative exponential; in addition, the Euclidean Lagrangian ($\mathcal{L}_E$) corresponds to the sum of the kinetic energy with the potential energy $V(\phi)$, which represents the total energy of the entire system.

The partition function for a real scalar field is shown below (the complete development of this procedure is shown in the appendix [Partition Function-Scalar Field]),

$$\begin{aligned} Z &= \mathtt{Tr}\left[e^{-\beta\hat{H}}\right] \\ &= \int\limits_{\phi(\beta,\mathbf{x})=\phi(0,\mathbf{x})} \prod_{\mathbf{x}} \left[C\,\mathcal{D}\phi(\tau,\mathbf{x})\right] \exp\left\{-\int_0^\beta d\tau \int d^3\mathbf{x}\,\mathcal{L}_E\right\}. \end{aligned} \tag{5.12}$$

The conclusion to end this introduction is that both the generating function of the quantum field theory and the partition function of statistical mechanics have practically the same structure, except for one difference, which distinguishes one from the other; the integration limits in the time integral in the argument of the exponential function; since this time integral goes from zero to the inverse of the temperature, it also has an additional requirement, the periodicity of the fields in the time coordinate.

5.2 Partition function on the hyperbolic ring $\mathbb{P}$

In section 5.3 of the paper [1], the authors have a grand partition function in the formalism of a hyperbolic ring, where an "artificial" mass term appears due to dissipation. The authors wonder if this system leads to Bose-Einstein condensation. In this section, we will give more details about this partition function, where we will involve a modified mass, and we will answer this question. During the development we will anticipate the Bose-Einsten condensation; where the condensate represents an expected value in vacuum, with which the $U(1)$ symmetry is broken.

We start with the partition function of [1], where we have the additional term of the modified mass,

$$Z = \int D\Phi D\Psi \exp\left\{-\frac{1}{2}\int_0^\beta d\tau \int d^3x \left[\left(\frac{\partial\Phi}{\partial\tau} - i\nu\Psi\right)^2 - \left(\frac{\partial\Psi}{\partial\tau} - i\nu\Phi\right)^2 + (\nabla\Phi)^2 - (\nabla\Psi)^2 + \left(m^2 - \frac{\gamma^2}{4}\right)(\Phi^2 - \Psi^2)\right]\right\} \tag{5.13}$$

where $\nu = \frac{\gamma}{2} - \mu$ represents the adjustment to the chemical potential due to dissipation. The authors of [1] followed [22, 23] to calculate the partition function (5.13), we will also follow him for the development of this section.

The Fourier transform for the fields Φ and Ψ are, respectively,

$$\begin{aligned}\Phi &= \varsigma\cosh\theta + \sqrt{\frac{\beta}{V}}\sum_n\sum_{\mathbf{p}} e^{i(\mathbf{p}\cdot\mathbf{x}+\omega_n\tau)}\,\Phi_n(\mathbf{p}),\\ \Psi &= \varsigma\sinh\theta + \sqrt{\frac{\beta}{V}}\sum_n\sum_{\mathbf{p}} e^{i(\mathbf{p}\cdot\mathbf{x}+\omega_n\tau)}\,\Psi_n(\mathbf{p});\end{aligned} \tag{5.14}$$

anticipating the Bose-Einstein condensation, we separate the zero-momentum mode; that is, $\Phi_0(\mathbf{p}{=}\mathbf{0}) = \Psi_0(\mathbf{p}{=}\mathbf{0}) = 0$. In (5.14) ς and θ do not depend on $(\mathbf{x},\tau)$ and determine the full infrared behavior of the fields. We carry out an integration by parts in (5.13), then we introduce (5.14) in (5.13), with this the expression (5.13) is written as,

$$Z = \left(\prod_n\prod_{\mathbf{p}}\int d\Phi_n(\mathbf{p})\,d\Psi_n(\mathbf{p})\right)e^S, \tag{5.15}$$

where,

$$S = \frac{1}{2}\beta V\varsigma^2\left(\mu(\gamma-\mu)-m^2\right)\left(\cosh^2\theta-\sinh^2\theta\right) - \frac{1}{2}\sum_n\sum_{\mathbf{p}}\left[\Phi_{-n}(-\mathbf{p}),\Psi_{-n}(-\mathbf{p})\right]D\begin{pmatrix}\Phi_n(\mathbf{p})\\ \Psi_n(\mathbf{p}),\end{pmatrix} \tag{5.16}$$

and,

$$D = \beta^2\begin{pmatrix}\omega_n^2+\omega^2+\nu^2-\frac{\gamma^2}{4} & -2\nu\omega_n\\ 2\nu\omega_n & -\left(\omega_n^2+\omega^2+\nu^2-\frac{\gamma^2}{4}\right)\end{pmatrix}. \tag{5.17}$$

Carrying out the integrations,

$$\ln Z = \frac{1}{2}\beta V\varsigma^2\left[\mu(\gamma-\mu)-m^2\right]\left(\cosh^2\theta-\sinh^2\theta\right) + \ln\left(\det D\right)^{-1/2}. \tag{5.18}$$

Now we handle only the second term of (5.18):

$$\begin{aligned}\ln\left(\det D\right) &= \ln\left\{\prod_n\prod_{\mathbf{p}} -\beta^4\left(\omega_n^2+\omega^2+\frac{1}{4}(\gamma-2\mu)^2-\frac{\gamma^2}{4}\right)^2+\omega_n^2(\gamma-2\mu)^2\right\}\\ &= \ln\left\{\prod_n\prod_{\mathbf{p}}\beta^2\left[(\omega_n+\nu)^2+E_{\mathbf{p}}^2\right]\right\} + \ln\left\{\prod_n\prod_{\mathbf{p}}-\beta^2\left[(\omega_n-\nu)^2+E_{\mathbf{p}}^2\right]\right\},\end{aligned} \tag{5.19}$$

where we defined,

$$E_{\mathbf{p}} \equiv \sqrt{\mathbf{p}^2 + m_{mod}^2} \quad . \tag{5.20}$$

Then (5.18) takes the form,

$$\begin{aligned} \ln Z = \frac{1}{2}\beta V \varsigma^2 \left[\mu(\gamma - \mu) - m^2\right] - \frac{1}{2}\sum_n \sum_{\mathbf{p}} \ln\left\{\beta^2\left[(\omega_n + \nu)^2 + E_{\mathbf{p}}^2\right]\right\} \\ - \frac{1}{2}\sum_n \sum_{\mathbf{p}} \ln\left\{-\beta^2\left[(\omega_n - \nu)^2 + E_{\mathbf{p}}^2\right]\right\}. \end{aligned} \tag{5.21}$$

Now we need to evaluate the Matsubara sums, we will do this in the next section.

5.3 Summation over bosonic Matsubara frequencies

This section is based on a literature appendix [14], and in this thesis we have delved deeper into the calculations obtained from that reference. The Matsubara frequency sum is a main tool in thermal field theory, which consists of the sum of discrete imaginary frequencies [21–23],

$$S_\eta = \frac{1}{\beta}\sum_{i\omega_n} g(i\omega_n), \tag{5.22}$$

where $\beta = \hbar/k_B T$, ($\hbar = 1$), is the inverse temperature and the frequencies $\omega_n (n \in \mathbb{Z})$ can be:

$$\begin{aligned} \omega_n = \tfrac{2n\pi}{\beta} : \quad & bosonic \;\; frequencies, \\ \omega_n = \tfrac{(2n+1)\pi}{\beta} : \quad & fermionic \;\; frequencies \end{aligned} \tag{5.23}$$

The summation (5.22) will converge if $g(z = i\omega)$ tend to 0 in $z \to \infty$. Convergence problems sometimes occur when calculations are performed on this representation; to solve this, we must add a convergence-generating factor, which is obtained from the way the integral was built, but there are cases in which this will not help, since the divergences may have a physical origin (*more detail see*[21]).

To evaluate the Matsubara frequency sum we need to use the residue theorem, where the sum can be replaced by a contour integral, which surrounds the imaginary axis (*more detail see* [19–21]):

$$S_\eta = \frac{1}{\beta}\sum_{i\omega} g(i\omega) = \frac{1}{2\pi i\beta}\oint_C g(z)\, h_\eta(z)\, dz; \tag{5.24}$$

where h_η is a weight function that has simple poles located at $z = i\omega$. Sometimes you can not deal with the integral along a contour in the close neighborhood of the poles of h; however, we can deform the integration contour to be able to do the integral. The simple case where $g(z)$ has isolated singularities at z_0, we get,

$$S_\eta = -\frac{1}{\beta}\sum_{z_0 \in g(z)\, poles} Res\; g(z_0)h_\eta(z_0) \tag{5.25}$$

where the contour integral surrounds the singularities of $g(iz)$, giving a negative sign because the poles closed in a clockwise direction.

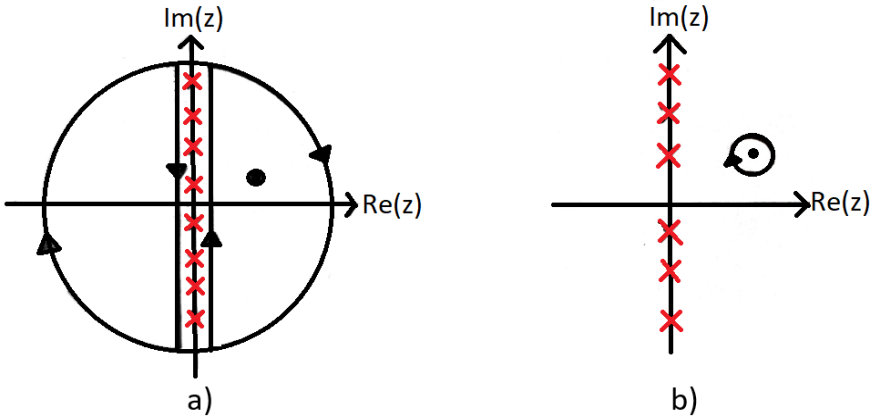

Figure 5.1: a) The integration contour used for the sum (5.24). When the integration is carried out, the residue of the poles (crosses) produced by the weighting function on the imaginary axis is collected, equivalent to the sum. b) The deformed integration contour.

Weight functions can be distinguished by the sign, $\eta = +1$ for bosons, and $\eta = -1$ for fermions. In this case, we are going to be working with bosons, then to produce single poles at boson frequencies, we can choose one of the following Matsubara weight functions,

$$\begin{aligned} h_B^{(1)}(z) &= \frac{1}{1-e^{-\beta z}} = \beta\, n_B(-z) = \beta[1+n_B(z)]\ , \\ h_B^{(2)}(z) &= \frac{-\beta}{1-e^{\beta z}} = \beta n_B(z), \end{aligned} \tag{5.26}$$

where $n_B(z) = (e^{\beta z}-1)^{-1}$ is the Bose-Einstein distribution function. The difference between $h_B^{(1)}$ and $h_B^{(2)}$, is that the first controls the divergence in the left half plane ($Re(z) < 0$), and the second controls the divergence in the right half plane ($Re(z) > 0$).

The goal of this section is to be able to evaluate the following Matsubara frequency sum,

$$\sum_n \ln\left\{\beta^2(\omega_n \pm \nu)^2 + E_{\mathbf{p}}^2\right\}; \tag{5.27}$$

for this, we will first evaluate a particular case, for ω and then we can obtain a generalization where $\omega \to \omega \pm \nu$. We start with the following Matsubara frequency sum,

$$\sum_n \ln\left\{\beta^2(\omega_n^2 + E_p^2)\right\}, \tag{5.28}$$

where, in our case, $E_p^2 = p^2 + m_{mod}^2$; for this calculation we only need E_p to be a real number. We can rewrite a (5.28) as,

$$\sum_n \ln\left\{\beta^2(\omega_n^2 + E_p^2)\right\} = \int_1^{(\beta E_p)^2} dx^2 \sum_n \frac{1}{(2\pi n)^2 + x^2} + \sum_n \ln\left\{1 + (2\pi n)^2\right\}, \tag{5.29}$$

and denoting $E_p \equiv Tx$, we can rewrite the sum that is in the integrand as,

$$\frac{1}{T}\sum_n \frac{1}{(2\pi n)^2 + x^2} = T\sum_n \frac{1}{\omega_n^2 + E_p^2}; \tag{5.30}$$

we can now apply (5.24) to this expression, which results in,

$$T\sum_n \frac{1}{\omega_n^2+E_p^2} = -\frac{1}{4\pi i}\oint \frac{1}{\omega^2-E_p^2}\cdot \coth\left(\frac{\omega}{2T}\right) d\omega. \tag{5.31}$$

Where was it used,

$$\begin{aligned} g(z)\to g(i\omega)\to \frac{1}{\omega_n^2+E_p^2} &= -\frac{1}{\omega^2-E_p^2}, \\ h_\eta(z)=h_{+1}(z)=\frac{-\beta}{1-e^{\beta z}} &= \frac{\beta}{2}\coth\left(\frac{\beta z}{2}\right). \end{aligned} \tag{5.32}$$

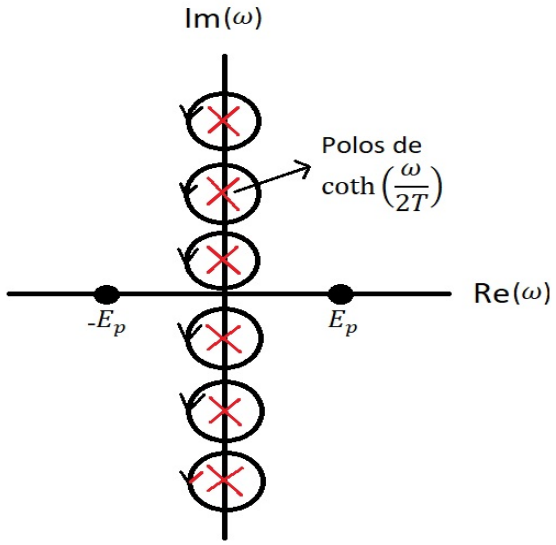

Figure 5.2: Integration contour used for (5.31).

What follows is to modify the integration contour used in (5.31) by the one shown in Fig.5.3 to obtain,

$$T\sum_n = \frac{1}{\omega_n^2+E_p^2} = -\frac{1}{2\pi i}\int_{-i\infty+\eta}^{+i\infty+\eta} \frac{d\omega}{\omega^2-E_p^2}\cdot \coth\left(\frac{\omega}{2T}\right). \tag{5.33}$$

In (5.33) we made a change of variable $\omega\to-\omega$, which gives us the same integral twice. This will be a general result that involves a chemical potential, which will be used later. We use the residue theorem again to collect the poles $\omega=\pm E_p$ through a closed contour that only takes into

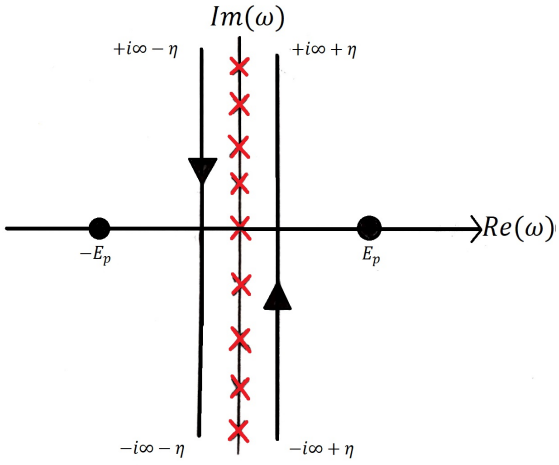

Figure 5.3: Modified integration contour from Eq. (5.33)

consideration the positive half plane,

$$T\sum_n \frac{1}{\omega_n^2+E_p^2} = \frac{\Theta(E_p)}{2E_p}\coth\left(\frac{E_p}{2T}\right) - \frac{\Theta(-E_p)}{2E_p}\coth\left(\frac{-E_p}{2T}\right) = \frac{1}{2E_p}\left[1+2f_B(E_p)\right], \tag{5.34}$$

where $f_B(\epsilon)$ is the Bose distribution function,

$$f_B(\epsilon) \equiv \frac{1}{e^{\epsilon\beta}-1}. \tag{5.35}$$

Something important to mention is that the result obtained (5.34) can also be obtained by using partial fractions in $\frac{1}{\omega_n^2+E_p^2}$ and using (5.26). We might think that the procedure with partial fractions is faster than performing the pole analysis on the weight functions; however, it is not always that effective, and we found this out when trying to compute the second sum in (5.21).

Returning to the main calculation, we have found,

$$\begin{aligned}\frac{1}{T}\sum_n \frac{1}{(2\pi n)^2+x^2} &= T\sum_n \frac{1}{\omega_n^2+E_p^2} \\ &= \frac{1}{2E_p}\left[1+2f_B(E_p)\right] \\ &= \frac{1}{Tx}\left(\frac{1}{2}+\frac{1}{e^x-1}\right).\end{aligned} \tag{5.36}$$

With all this, we can now calculate our original expression, we are going to introduce (5.36) in (5.29) and we are going to integrate concerning x^2 to obtain,

$$\begin{aligned}\sum_n \ln\left\{\beta^2(\omega_n^2+E_p^2)\right\} &= \int_1^{(\beta E_p)^2} dx^2\,\frac{1}{x}\left(\frac{1}{2}+\frac{1}{e^x-1}\right)+\text{const.}\\ &= \beta E_p + 2\ln\left\{1-e^{-\beta E_p}\right\}+\text{const.}\end{aligned} \tag{5.37}$$

where the term constant is independent of β. Now, we need to evaluate the second sum in (5.21), where we have a logarithmic function with a negative argument. If we try to follow the above procedure for this sum, we will run into problems from the start; to solve this, we take the derivative trick which is discussed in more detail in [18] and [21]. This has to do with the non-convergence of the function in the sums of the form of (5.21); however, the derivative of this function does converge.

We take the second sum of (5.21) and differentiating concerning the energy,

$$\frac{\partial}{\partial E_p}\sum_n \ln\left\{-\beta^2\left[(\omega_n-\nu)^2+E_p^2\right]\right\} = \beta\sum_n \frac{2E_p}{-(\nu+i\omega_n)^2+E_p^2}; \tag{5.38}$$

and we know what is equal to (5.38) (*see Eq.(3.21) in* [18]),

$$\beta\sum_n \frac{2E_p}{-(\nu+i\omega_n)^2+E_p^2} = \beta\left\{\frac{1}{2}\left[\coth\left(\frac{E_p-\nu}{2T}\right)+\coth\left(\frac{E_p+\nu}{2T}\right)\right]\right\}, \tag{5.39}$$

integrating and ignoring the terms that depend on temperature,

$$\frac{\beta}{2}\int dE_p\left[\coth\left(\frac{E_p-\nu}{2T}\right)+\coth\left(\frac{E_p+\nu}{2T}\right)\right] \simeq \ln\left\{\sinh\left(\frac{E_p-\nu}{2T}\right)\right\}+\ln\left\{\sinh\left(\frac{E_p+\nu}{2T}\right)\right\}; \tag{5.40}$$

now we have,

$$\begin{aligned}\sinh\left(\frac{E_p+\nu}{2T}\right) &= \frac{1}{2}\,e^{\frac{(E_p+\nu)}{2}\beta}\left(1-e^{-\beta(E_p+\nu)}\right),\\ \sinh\left(\frac{E_p-\nu}{2T}\right) &= \frac{1}{2}\,e^{\frac{(E_p-\nu)}{2}\beta}\left(1-e^{-\beta(E_p-\nu)}\right),\end{aligned} \tag{5.41}$$

thus (5.40) takes the form,

$$\ln\left\{\sinh\left(\frac{E_p-\nu}{2T}\right)\right\}+\ln\left\{\sinh\left(\frac{E_p+\nu}{2T}\right)\right\} \simeq \beta E_p+\ln\left\{1-e^{-\beta(E_p-\nu)}\right\}+\ln\left\{1-e^{-\beta(E_p+\nu)}\right\}. \tag{5.42}$$

Using the results (5.37) and (5.42) and substituting them into the large partition function (5.21) we get,

$$\ln Z = \frac{1}{2}\beta V\varsigma^2\left[\mu(\gamma-\mu)-m^2\right]\left(\cos^2\theta-\sin^2\theta\right) - V\int\frac{d^3p}{(2\pi)^3}\left[\beta E_p + \ln\left\{1-e^{-\beta(E_p-\nu)}\right\}+\ln\left\{1-e^{-\beta(E_p+\nu)}\right\}\right], \tag{5.43}$$

where has it been used,

$$\sum_p \to V\int\frac{d^3p}{(2\pi)^3}. \tag{5.44}$$

We can also write $\ln Z$ in terms of the modified mass and the adjusted potential due to dissipation ν, as

$$\ln Z = \frac{1}{2}\beta V\varsigma^2\left[\nu^2-m_{mod}^2\right] - V\int\frac{d^3p}{(2\pi)^3}\left[\beta E_p + \ln\left\{1-e^{-\beta(E_p-\nu)}\right\}+\ln\left\{1-e^{-\beta(E_p+\nu)}\right\}\right]. \tag{5.45}$$

If we take the limit as $\gamma \to 0$, we obtain the well-known partition function for standard complex fields. The partition function (5.45) was constructed on a pure hyperbolic ring $\mathbb{P}$, however, it is surprising that our system, consisting of a massless, dissipative hyperbolic field with explicit Lorentz breaking terms, leads to a partition function similar to that of a free, massive Lorentz-invariant $U(1)$ field. It should be noted that here the authors of [1] did not use the idempotent bases (J^+, J^-) of the ring $\mathbb{H}$, to be able to separate the subsystems. In the next chapter, we will use these bases and calculate the partition function on the hypercomplex ring $\mathbb{H}$.

Chapter 6

Partition function in $\mathbb{H}$

We now have a quantum field theory built on a hypercomplex ring $\mathbb{H}$. This ring incorporates two complex units: the hyperbolic j and the elliptic i. This latter unit gives us the ability to develop a partition function for a system undergoing dissipation, using the standard approach for calculating partition functions in charged fields. The connection between quantum field theory and statistical quantum mechanics is well-established and widely discussed in the literature. This connection is based on the relationship between an imaginary time and the temperature T, which means that,

$$e^{-\beta\hat{H}} \leftrightarrow e^{i\hat{H}t}; \qquad t = i\beta, \quad \beta = \frac{1}{T}. \tag{6.1}$$

To determine the partition function $Z_{\mathbb{H}}$, we will employ an approach widely documented in the literature (as mentioned in [21, 23]) that is used to compute partition functions in conventional complex fields.

6.1 The Lagrangian and Hamiltonian formalism in $\mathbb{H}$

We will begin with the Lagrangian constructed in (2.11), which is written in terms of $(\Omega, \overline{\Omega})$ (see Eq. (2.13)) but we will rewrite it in terms of its components (Ω^+, Ω^-) as,

$$\begin{aligned}\mathcal{L} = \frac{1}{2}\Big\{ & J^+\left[\dot{\Omega}^+\dot{\overline{\Omega^-}} - \partial_i\Omega^+\partial^i\overline{\Omega^-} + \frac{\gamma}{2}\left(\Omega^+\dot{\overline{\Omega^-}} - \dot{\Omega}^+\overline{\Omega^-}\right) - m^2\Omega^+\overline{\Omega^-}\right] \\ & + J^-\left[\dot{\Omega}^-\dot{\overline{\Omega^+}} - \partial_i\overline{\Omega^+}\partial^i\Omega^- + \frac{\gamma}{2}\left(\dot{\Omega}^-\overline{\Omega^+} - \Omega^-\dot{\overline{\Omega^+}}\right) - m^2\overline{\Omega^+}\Omega^-\right]\Big\}.\end{aligned} \tag{6.2}$$

According to Noether's theorem, a system exhibiting continuous symmetry gives rise to a conserved current. This arises from the invariance of the Lagrangian under the action of the group $U(1) \times SO(1,1)$,

$$\partial_t(\bar{\Omega}\dot{\Omega} - \Omega\dot{\overline{\Omega}} + j\gamma\Omega\overline{\Omega}) + \partial_i\left(\overline{\Omega}\partial^i\Omega - \Omega\partial^i\overline{\Omega}\right) = 0; \tag{6.3}$$

where the current is expressed as $j^i = \left(\bar{\Omega}\partial^i\Omega - \Omega\partial^i\overline{\Omega}\right)$, and the charge density is defined as,

$$j_0 = (\overline{\Omega}\dot{\Omega} - \Omega\dot{\overline{\Omega}} + j\gamma\Omega\overline{\Omega}). \tag{6.4}$$

The conserved charge Q associated with this hypercomplex current is,

$$Q = \int j_0\, d^n x = \int \left(\overline{\Omega}\Pi_{\overline{\Omega}} - \Omega\Pi_\Omega\right) d^n x. \tag{6.5}$$

We recover the usual charge of $U(1)$ in the limit when $\psi_1 = \psi_2 = 0$, see Eq.(3.9). By rewriting this charge density in terms of the bases (J^+, J^-) and the conjugate moments, see (2.31), this new term will be associated with the chemical potential. As a result, we will obtain a modified chemical potential, influenced by the dissipative parameter γ. Conservation of current, as is well known, allows us to integrate a chemical potential related to the corresponding charge; this chemical potential is integrated into the Lagrangian similarly to how the time component of a gauge field is incorporated [21–23].

Using the Lagrangian (6.2), we obtain the conjugate moments,[5]

$$J^+\Pi_\Omega \equiv J^+\frac{\partial\mathcal{L}}{\partial(\partial_0\Omega)} = J^+\partial^0\Omega, \qquad J^-\Pi_\Omega \equiv J^-\frac{\partial\mathcal{L}}{\partial(\partial_0\Omega)} = J^-\partial^0\Omega; \tag{6.6}$$

Note the projection of the bases (J^+, J^-) onto the conjugate moments, this is necessary, the usual way cannot be used, otherwise, we would have to deal with the difficulty of having an inverse element of the ring $\mathbb{H}$.

[5]The conjugate moments are in terms of the hypercomplex field Ω itself; however, from this definition, it is possible to obtain the conjugate moments in terms of the components of the hypercomplex field (Ω^+, Ω^-), which will be used later.

We can now write the charge density (6.4) in terms of the conjugate moments, obtaining,

$$j_0 = 2\left[J^+\left(\overline{\Omega^-}\,\Pi_{\overline{\Omega^-}} - \Omega^+\Pi_{\Omega^+}\right) - J^-\left(\Omega^-\Pi_{\Omega^-} - \overline{\Omega^+}\,\Pi_{\overline{\Omega^+}}\right)\right], \tag{6.7}$$

and the Hamiltonian written in terms of the idempotent bases, is,

$$\begin{aligned} H = {} & J^+\left[2\Pi_{\Omega^+}\Pi_{\overline{\Omega^-}} + \frac{1}{2}\partial_i\Omega^+\partial^i\overline{\Omega^-} + \frac{\gamma}{2}\left(\overline{\Omega^-}\,\Pi_{\overline{\Omega^-}} - \Omega^+\Pi_{\Omega^+}\right) + \frac{1}{2}\left(m^2 - \frac{\gamma^2}{4}\right)\Omega^+\overline{\Omega^-}\right] \\ & + J^-\left[2\,\Pi_{\overline{\Omega^+}}\Pi_{\Omega^-} + \frac{1}{2}\partial_i\overline{\Omega^+}\partial^i\Omega^- + \frac{\gamma}{2}\left(\Omega^-\Pi_{\Omega^-} - \overline{\Omega^+}\,\Pi_{\overline{\Omega^+}}\right) + \frac{1}{2}\left(m^2 - \frac{\gamma^2}{4}\right)\overline{\Omega^+}\Omega^-\right]. \end{aligned} \tag{6.8}$$

As previously noted, this formulation has two complex units: the hyperbolic and the elliptic. We will now develop the partition function ($\mathcal{Z}_{\mathbb{H}}$) in the ring $\mathbb{H}$. To begin, we will take as a starting point the well-known definition of the partition function [21–23] in the context of charged complex fields. This is possible thanks to the formulation since we have the elliptic complex unit. In our partition function $\mathcal{Z}_{\mathbb{H}}$, we will deal with exponentials that will incorporate the hyperbolic unit, either in terms of the bases $(1, j)$ or (J^+, J^-), as well as the elliptic unit i. This means that the exponential in the path integral will be hybrid, this is,

$$e^{i(S+ijS')} = e^{iS}e^{-jS'} = J^+e^{iS-S'} + J^-e^{iS+S'}. \tag{6.9}$$

However, the characteristics of the ring $\mathbb{H}$ will allow us to manipulate these exponentials in such a way that the path integral required to calculate the partition function is completely complex, in the sense of the imaginary unit i. The new partition function $\mathcal{Z}_{\mathbb{H}}$ will be separated into two components, one focused on the subsystem of interest, which will be represented by the basis J^+, and another corresponding to the environment, identified by the basis J^-.

6.2 The partition function $\mathcal{Z}_{\mathbb{H}}$

We begin by using the partition function for charged fields[6], as presented by [23],

$$Z = \int \mathcal{D}\pi_1 \mathcal{D}\pi_2 \int\limits_{periodic} \mathcal{D}\phi_1\, \mathcal{D}\phi_2 \times \exp\left\{ \int_0^\beta d\tau \int d^3x \times \left[i\pi_1 \dot{\phi}_1 + i\pi_2 \dot{\phi}_2 - H(\pi_2, \pi_2, \phi_1, \phi_2) + \mu j_0 \right] \right\}, \tag{6.10}$$

where $\dot{\phi}_{1,2} = \frac{\partial \phi_{1,2}}{\partial \tau}$; $\beta = \frac{1}{T}$ and T is the temperature; in addition, the chemical potential that is associated with its conserved charge Q has been used, and the change to an imaginary time variable $\tau = it$ has been made. The term periodic implies that the integration over the field is constrained such that $\phi(\mathbf{x}, 0) = \phi(\mathbf{x}, \beta)$. This can be deduced from the trace operation, which defines that $\phi_a(\mathbf{x}) = \phi(\mathbf{x}, 0) = \phi(\mathbf{x}, \beta)$. This periodicity requirement, which is necessary for the Matsubara frequencies ω_n, is met by using the fact that $e^{i\omega_n \beta} = 1$, in other words, $\omega_n \beta$ has to be an integer multiple of 2π, this is, $\omega_n = 2\pi n T$, $n \in \mathbb{Z}$. On the other hand, there is no restriction on the integration over conjugate momenta. The partition function (6.10) can be generalized to an arbitrary number of fields and conserved charges [23].

Now, to calculate our $\mathcal{Z}_{\mathbb{H}}$, we make the following identification between the fields of (6.10) and our hypercomplex field Ω,

$$\phi_1 \to \Omega, \qquad \phi_2 \to \overline{\Omega}, \qquad \pi_1 \to \Pi_{\Omega}, \qquad \pi_2 \to \Pi_{\overline{\Omega}}. \tag{6.11}$$

With this identification, we are going to have a mixture between the fields (Ω^+, Ω^-), see Eq.(2.13), and their conjugate complexes. If the identification is done differently,[7] we obtain the complete cancellation of the partition function since the product $J^+ \cdot J^- = 0$ arises. Thus, our function $\mathcal{Z}_{\mathbb{H}}$ is expressed in terms of the hypercomplex field Ω as,

$$\mathcal{Z}_{\mathbb{H}} = \int \mathcal{D}\Pi_{\Omega}\, \mathcal{D}\Pi_{\overline{\Omega}} \int\limits_{periodic} \mathcal{D}\Omega\, \mathcal{D}\overline{\Omega} \times \exp\left\{ \int_0^\beta d\tau \int d^3x \left[i\Pi_{\Omega}\, \dot{\Omega} + i\Pi_{\overline{\Omega}} \dot{\overline{\Omega}} - H\left(\Pi_{\Omega}, \Pi_{\overline{\Omega}}, \Omega, \overline{\Omega}\right) + \mu j_0\left(\Omega, \overline{\Omega}\right) \right] \right\}. \tag{6.12}$$

[6] Other equivalent expressions can also be used to represent the partition function for loaded fields. If other expressions such as the one given in [20, 21] are used, the final result will be the same. The difference lies in the algebra; some representations are more difficult to handle.

[7] A different way to do the identification is with the components of Ω, that is, $\phi_1 \to J^+\Omega^+$ and $\phi_2 \to J^-\Omega^-$, $\pi_1 \to J^+\Pi_{\Omega^+}$ and $\pi_2 \to J^-\Pi_{\Omega^-}$. When calculating the integration measure $\int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \to \int \mathcal{D}(J^+\Omega^+)\mathcal{D}(J^-\Omega^-) = 0$, due to the annihilation property (1.6).

We can observe that in the exponential argument in (6.12), the bases (J^+, J^-) are presented together with the imaginary unit i, which gives us as a result a hybrid exponential. Using the properties (1.6) and (1.8) of the ring $\mathbb{H}$, our partition function $\mathcal{Z}_{\mathbb{H}}$ can be decomposed into two parts, namely, one that encapsulates the information of the subsystem of interest, associated with the basis J^+, and another that represents the environment, accompanied by the basis J^-.

Now, we will use the Hamiltonian (6.8) and the charge density (6.6), to obtain,

$$\begin{aligned}\mathcal{Z}_{\mathbb{H}} = J^+ \int \mathcal{D}\Pi_{\Omega^+}\,\mathcal{D}\Pi_{\overline{\Omega^-}} \int\limits_{periodic} \mathcal{D}\Omega^+\mathcal{D}\overline{\Omega^-}\,\exp\left\{\int_0^\beta d\tau \int d^3x \left(i\Pi_{\Omega^+}\,\dot{\Omega}^+ + i\Pi_{\overline{\Omega^-}}\,\dot{\overline{\Omega^-}}\right.\right.\\ \left.\left. -2\,\Pi_{\Omega^+}\Pi_{\overline{\Omega^-}} - \frac{1}{2}\partial_i\Omega^+\partial^i\overline{\Omega^-} + 2\nu\left(\overline{\Omega^-}\,\Pi_{\overline{\Omega^-}} - \Omega^+\Pi_{\Omega^+}\right) - \frac{1}{2}m_{mod}^2\,\Omega^+\overline{\Omega^-}\right)\right\}\\ +J^- \int \mathcal{D}\Pi_{\overline{\Omega^+}}\,\mathcal{D}\Pi_{\Omega^-} \int\limits_{periodic} \mathcal{D}\overline{\Omega^+}\mathcal{D}\Omega^-\,\exp\left\{\int_0^\beta d\tau \int d^4x \left(i\Pi_{\overline{\Omega^+}}\,\dot{\overline{\Omega^+}} + i\Pi_{\Omega^-}\,\dot{\Omega}^-\right.\right.\\ \left.\left. -2\,\Pi_{\overline{\Omega^+}}\,\Pi_{\Omega^-} - \frac{1}{2}\partial_i\overline{\Omega^+}\partial^i\Omega^- - 2\nu\left(\Omega^-\Pi_{\Omega^-} - \overline{\Omega^+}\Pi_{\overline{\Omega^+}}\right) - \frac{1}{2}m_{mod}^2\,\overline{\Omega^+}\Omega^-\right)\right\};\end{aligned} \tag{6.13}$$

where $\nu = \mu - \frac{\gamma}{4}$ is the modified chemical potential, and $m_{mod}^2 = m^2 - \frac{\gamma^2}{4}$ is the modified mass. It is evident that the two fields (Ω^+, Ω^-) correspond to standard complex fields, as given in Eq. (2.10). Moreover, our partition function $\mathcal{Z}_{\mathbb{H}}$ is expressed in terms of these complex fields, which allows us to employ all the formalism available in the literature to perform the corresponding integrations. Integrating out the conjugate momenta in (6.13), we get,

$$\begin{aligned}Z_{\mathbb{H}} = J^+ \int\limits_{periodic} \mathcal{D}\Omega^+\,\mathcal{D}\overline{\Omega^-}\,\exp\left\{\int_0^\beta d\tau \int d^3x \left[-\left(\dot{\Omega}^+ + i2\nu\overline{\Omega^-}\right)^2 - \left(\dot{\overline{\Omega^-}} - i2\nu\Omega^+\right)^2 - \frac{1}{2}\,\nabla\Omega^+\nabla\overline{\Omega^-} - \frac{1}{2}m_{mod}^2\,\Omega^+\,\overline{\Omega^-}\right]\right\}\\ +J^- \int\limits_{periodic} \mathcal{D}\overline{\Omega^+}\,\mathcal{D}\Omega^-\,\exp\left\{\int_0^\beta d\tau \int d^3x \left[-\left(\dot{\overline{\Omega^+}} - i2\nu\Omega^-\right)^2 - \left(\dot{\Omega}^- + i2\nu\overline{\Omega^+}\right)^2 - \frac{1}{2}\,\nabla\overline{\Omega^+}\nabla\Omega^- - \frac{1}{2}m_{mod}^2\,\overline{\Omega^+}\,\Omega^-\right]\right\}.\end{aligned} \tag{6.14}$$

We can see that the formalism of the ring $\mathbb{H}$ is manifested in our function $\mathcal{Z}_{\mathbb{H}}$. We started with a partition function, whose path integral includes both complex units. However, it was possible to "separate" the formalisms associated with each of these units. On the one hand, we have idempotent bases that interlace with the partition functions, meaning that the hyperbolic unit accompanies them.

On the other hand, the partition functions are completely standard complex, thus, we can apply the Matsubara formalism, as we have been doing. We will then perform the corresponding integration on the fields in (6.14). The components of Ω (2.13) can be Fourier-expanded,

$$\begin{aligned}\Omega^{+} &= \zeta^{+}_{n,\mathbf{p}} + \sqrt{\frac{\beta}{V}} \sum_{n} \sum_{\mathbf{p}} e^{i(\mathbf{p}\cdot\mathbf{x}+\omega_n \tau)}\, \Omega^{+}_{n}(\mathbf{p}), \\ \Omega^{-} &= \zeta^{-}_{n,\mathbf{p}} + \sqrt{\frac{\beta}{V}} \sum_{n} \sum_{\mathbf{p}} e^{i(\mathbf{p}\cdot\mathbf{x}+\omega_n \tau)}\, \Omega^{-}_{n}(\mathbf{p});\end{aligned} \tag{6.15}$$

where V is the volume of the entire system, in addition, anticipating Bose-Einstein condensation, we have separated the zero-momentum mode, $\zeta^{+;-}_{n,\mathbf{p}} \equiv \zeta_{+,-} = \Omega(n = \mathbf{p} = 0) = 0$. This allows for the possibility of condensation of the bosons in the zero-momentum state. Condensation implies that, in the limit of an infinite volume, a finite proportion of the particles are in the state $n = 0$, $\mathbf{p} = 0$ [23]; furthermore, we assume that ζ remains constant in space-time. We will proceed to substitute (6.15) in (6.14) after performing an integration by parts, which will lead us to obtain the following expression,

$$Z_{\mathbb{H}} = J^{+} \left(\prod_{n} \prod_{\mathbf{p}} \int \mathcal{D}\Omega^{+}_{n}(\mathbf{p})\, \mathcal{D}\overline{\Omega^{-}_{n}}(\mathbf{p}) \right) e^{S_{+}} + J^{-} \left(\prod_{n} \prod_{\mathbf{p}} \int \mathcal{D}\overline{\Omega^{+}_{n}}(-\mathbf{p})\, \mathcal{D}\Omega^{-}_{n}(-\mathbf{p}) \right) e^{S_{-}} \tag{6.16}$$

where,

$$\begin{aligned} S^{+} &= \frac{1}{2}\beta V \zeta_{+}^{2} \left(8\nu^{2} - m^{2}_{mod}\right) - \frac{1}{2} \sum_{n} \sum_{\mathbf{p}} \left(\Omega^{+}_{-n}(-\mathbf{p}), \;\; \overline{\Omega^{-}_{-n}}(-\mathbf{p}) \right) D_{+} \begin{pmatrix} \Omega^{+}_{n}(\mathbf{p}) \\ \overline{\Omega^{-}_{n}}(\mathbf{p}) \end{pmatrix}, \\ S^{-} &= \frac{1}{2}\beta V \zeta_{-}^{2} \left(8\nu^{2} - m^{2}_{mod}\right) - \frac{1}{2} \sum_{n} \sum_{\mathbf{p}} \left(\overline{\Omega^{+}_{n}}(\mathbf{p}), \;\; \Omega^{-}_{n}(\mathbf{p}) \right) D_{-} \begin{pmatrix} \overline{\Omega^{+}_{-n}}(-\mathbf{p}) \\ \Omega^{-}_{-n}(\mathbf{\text{-}p}) \end{pmatrix} \end{aligned} \tag{6.17}$$

and,

$$D_{+} = D_{-} = 2\beta^{2} \begin{pmatrix} \omega_{n}^{2} - 4\nu^{2} & 4\nu\omega_{n} + \frac{1}{2}E^{2}_{\mathbf{p}} \\ -4\nu\omega_{n} - \frac{1}{2}E^{2}_{\mathbf{p}} & \omega_{n}^{2} - 4\nu^{2} \end{pmatrix}, \tag{6.18}$$

with: $E^{2}_{\mathbf{p}} = \mathbf{p}^{2} + m^{2}_{mod}$.

Now we carry out the integrations and then apply the logarithm property (5) for a hypercomplex number [64, 65], obtaining,

$$\ln \mathcal{Z}_{\mathbb{H}} = J^{+}\left[\frac{1}{2}\beta V\zeta_{+}^{2}\left(8\nu^{2}-m_{mod}^{2}\right)+\ln\{\det(D_{+})\}^{1/2}\right]+J^{-}\left[\frac{1}{2}\beta V\zeta_{-}^{2}\left(8\nu^{2}-m_{mod}^{2}\right)+\ln\{\det(D_{-})\}^{1/2}\right]. \tag{6.19}$$

After performing certain algebraic manipulations with the determinants of D_{+} and D_{-} in (6.19), we obtain,

$$\begin{aligned}\ln \mathcal{Z}_{\mathbb{H}} = J^{+}\Bigg(&\frac{1}{2}\beta V\zeta_{+}^{2}\left(8\nu^{2}-m_{mod}^{2}\right)-\frac{1}{2}\sum_{n}\sum_{\mathbf{p}}\ln\left\{2\beta^{2}\left[(\omega_{n}+i2\nu)^{2}+\frac{i}{2}E_{p}^{2}\right]\right\}\\ &-\frac{1}{2}\sum_{n}\sum_{\mathbf{p}}\ln\left\{2\beta^{2}\left[(\omega_{n}-i2\nu)^{2}-\frac{i}{2}E_{p}^{2}\right]\right\}\Bigg)\\ +J^{-}\Bigg(&\frac{1}{2}\beta V\zeta_{-}^{2}\left(8\nu^{2}-m_{mod}^{2}\right)-\frac{1}{2}\sum_{n}\sum_{\mathbf{p}}\ln\left\{\beta^{2}\left[-2i\left(\omega_{n}+i2\nu\right)^{2}+E_{p}^{2}\right]\right\}\\ &-\frac{1}{2}\sum_{n}\sum_{\mathbf{p}}\ln\left\{\beta^{2}\left[2i\left(\omega_{n}-2i\nu\right)^{2}+E_{p}^{2}\right]\right\}\Bigg)\end{aligned} \tag{6.20}$$

In expression (6.20), we observe a sharp split in the partition function of the entire system, one part corresponding to the system of interest, projected with J^{+}, and the other referring to the environment, whose projection is J^{-}. Now, it remains to solve the Matsubara sums in (6.20). To address these sums, we can resort to the calculus of complex variables, as discussed in section 5.3, or apply the ingenious method of the derivative with respect to energy. However, it is essential to perform a careful analysis to identify the location of the poles in these new Matsubara sums, since this could present us with certain difficulties.

On the other hand, it is worth mentioning that for each subsystem, the two Matsubara sums corresponding to the number of complex fields are obtained, noting that for the subsystem of interest, the energy appears as imaginary, which tells us that there is a non-stationary structure, with an imaginary component that is related to the dissipation or loss of energy. Regarding the environment, we observe the presence of imaginary frequencies, which suggests a non-resonant response or a tendency towards relaxation towards thermal equilibrium.

The structure of the hypercomplex ring $\mathbb{H}$ can allow the appearance of imaginary energies and complex frequencies; however, the presence of dissipation and non-equilibrium in the system can generate non-stationary structures and imaginary frequencies. Also, the interaction between the subsystem of interest and the environment can influence the appearance of these imaginary energies and complex frequencies. Therefore, further analysis of this obtained partition function is required to obtain more precise details.

Conclusion

In this work, we have made an alternative formulation to study open systems whose construction is based on a hyper-complex ring, which contains two imaginary units, namely, the standard unit i and the hyperbolic unit j; generalizing the formalism used in [1]. The algebraic structure of the ring in the formulation in hand has led us to a non-canonical theory which, in turn, is committed to the emergence of a new $SO(1,1)$ symmetry. The consequences that this non-canonical theory throws at us are: the quantization scheme at hand allows us to cure the pathologies that come from the standard quantization since we have two complex units, the system does not leave the original Hilbert space, independently of the volume of the system. This leads to having field commutators with different characteristics from standard field commutators since our field commutators have an explicit dependence on the dissipative parameter and the background dimension.

On the other hand, we have shown that the vacuum state temporally evolves as an entangled state, independent of whether dissipation is on or off. This result is motivation for future works, some of which are already under development. One of these works consists of using these entangled states to calculate the time-dependent entanglement entropy, with which we can have a tool to measure the entanglement in dissipative systems. We also review the asymptotic entangled states for finite and infinite systems and, due to the structure of the Hamiltonian operator that contains the geometric information of the system, the ergodic theory naturally arises. Here we have an element that is important when studying the ergodicity of a system, the geometry. With this element we begin to introduce ourselves to the study of the ergodic behavior of these asymptotic entangled states; however, it is not enough to be able to say if a system reaches thermal equilibrium or remains in a cyclo-steady state. This result is motivation to continue investigating how ergodicity is involved in our formulation; a work that we are currently developing.

The hypercomplex partition function is a mathematical tool to describe dissipative systems, where energy is dissipated due to interactions with its environment. To build it, we have started from the known, from the known way to build a partition function with charged fields, where a chemical potential appears, due to the symmetry of the Lagrangian, in which case, for our hypercomplex partition function, this chemical potential is imaginary. This imaginary potential can be interpreted as a rate of dissipation or decoherence in the system. The inclusion of a complex chemical potential can modify the thermodynamic properties of the system, such as free energy, heat capacity and entropy. In particular, an imaginary chemical potential can induce an asymmetry in the partition function, which can be relevant to describe quantum systems in the presence of dissipative effects. An imaginary chemical potential in a quantum field theory at finite temperature also has implications and relevance. It may be relevant for describing condensed matter systems such as superconductors, superfluids and spin systems. On the other hand, it may have implications for quantum information theory, particularly for describing decoherence and dissipation in quantum systems. Importantly, research on this topic is still ongoing, and a deep understanding of thermodynamic properties and quantum field theory is required to fully explore the implications of a complex chemical potential in finite-temperature QFT.

The hypercomplex partition function developed in this study presents two crucial aspects. The first, and most prominent, is the ability to decompose the partition function for each subsystem independently. In addition, the emergence of an imaginary chemical potential is observed, which could have a significant impact on the behavior of the Bose-Einstein condensate as well as on the low and high temperature limits (this project is currently under development). The obtained hypercomplex partition function offers a powerful tool to study complex dissipative and non-equilibrium systems. Its unique structure can reveal fundamental properties of these systems and open new perspectives on quantum processes and non-equilibrium thermodynamics.

Appendix A. Algebraic Ring

A ring $\mathfrak{R}$ is defined as a nonempty set having two internal operations, addition, and multiplication, denoted by $(a+b)$ and (ab) respectively, and satisfying specific properties [**?**]. It is required that $\mathfrak{R}$ be an abelian group under the operation $(a+)$, where the identity element is denoted by (0), and the inverse is denoted by $(-a)$. In the context of multiplication, $\mathfrak{R}$ is defined as a commutative semigroup, which implies that it is not required to contain an identity element or inverses. So $\mathfrak{R}$ is defined to be a ring if the following axioms are satisfied:

1. $(a+b)+c = a+(b+a)$, *addition is associative.*
2. $a+b = b+a$, *addition is commutative.*
3. *There is an element* 0 *such that* $(a+0=a)$, $\forall a \in \mathfrak{R}$.
4. *For every* $a \in \mathfrak{R}$ *there exists an element* $-a$ *such that* $(a-a) \equiv a+(-a) = 0$.
5. $(ab)c = a(bc)$, *multiplication is associative.*
6. These laws are related by the distributive laws:
   $(\forall a, b, c \in \mathfrak{R})$, $(a+b)c = ac+bc$ and $c(a+b) = ca+cb$.
7. $(ab) = (ba)$, *multiplication is commutative.*
8. The distributive law holds that $a(b+c) = ab+ac$, and by point 6, this implies $(a+b)c = ac+bc$. Establishing a connection between both internal operations.

A ring is classified as an identity ring, or simply a ring with $\mathbf{1}$, when it has a multiplicative identity. A ring is considered commutative if the multiplication operation satisfies the commutative property, which implies that for any pair of elements a and b in $\mathfrak{R}$, it holds that $a \cdot b = b \cdot a$. The additive identity element, represented as $\mathbf{0}$, is known as the zero of the ring. In a ring, it is possible for

the product $ab = 0$ to occur even when both a and b are nonzero. Such elements are known as zero divisors. A commutative ring that possesses an identity element $\mathbf{1}$ and has no zero divisors is called an *integral domain* [17]. If the ring $\mathfrak{R}$ has an element 1, and for an element $a \in \mathfrak{R}$ there exists an element $b \in \mathfrak{R}$ such that $ab = 1 = ba$, then b is said to act as the inverse of a in $\mathfrak{R}$. A commutative ring $\mathfrak{R}$ with neutral element $\mathbf{1}$ is considered a field if every nonzero element has an inverse. Therefore, we can conclude that $\mathfrak{R}$ constitutes a field if the following conditions hold:

- $\mathfrak{R}$ is an additive abelian group,
- $\mathfrak{R} \setminus \{0\}$ is a multiplicative abelian group,
- the distributive laws hold.

The noncommutative ring $Mat_{n\times n}(\mathbb{R})$ of matrices $n \times n$ over $\mathbb{R}$ exhibits the property of containing zero divisors, especially when $n > 1$. The number systems $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are classified as fields. The number system $\mathbb{Z}$ is an integral domain that is not a field. The ring $\mathbb{Z}_n$ of integers modulo n, is an integral domain (in fact, a field) if and only if n is prime [25].

Appendix B. The partition function of a scalar field

We will initiate our analysis by examining the space coordinate, its canonical moment, and the corresponding operators that satisfy the commutation relation; we will also address the relationship of completeness and mutual projection, respectively. We will also use natural units to facilitate our discussion ($\hbar = 1$),

$$[\hat{q}, \hat{p}] = -i, \tag{6.21}$$

$$\int dq \, \langle q|q\rangle = \mathbb{1}, \quad \int \frac{dp}{2\pi} \, \langle p|p\rangle = \mathbb{1}, \tag{6.22}$$

$$\langle q\rangle \, p = e^{ipq}. \tag{6.23}$$

The partition function can be represented in space as follows,

$$Z = Tr[e^{-\beta \hat{H}}] = \int dq \, \langle q| \, e^{\beta \hat{H}} \, |q\rangle \, . \tag{6.24}$$

where $\beta = 1/T$, and T is the temperature. Now we divide the β in n-intervals ($\beta = n\epsilon$), and in each piece, a completion relationship for the right operator is introduced and for the left side we introduce the conjugated moment of (6.22),

$$Z = \int \frac{dp_n}{2\pi} \cdots \int \frac{dp_1}{2\pi} \int dq_n \cdots \int dq_1 \int dq \, \langle q|p_n\rangle \, \langle p_n| \, e^{-\epsilon \hat{H}} \, |q_n\rangle \cdot \langle q_n|p_{n-1}\rangle \cdots \langle p_1| \, e^{-\epsilon \hat{H}} \, |q_1\rangle \, \langle q_1|q\rangle \, . \tag{6.25}$$

Performing a series expansion of the exponential, we obtain,

$$Z \approx \int \prod_{i=1}^{n} \frac{\mathrm{d}p_i \, \mathrm{d}q_i}{2\pi} e^{ip_i(q_{i+1}-q_i)} \left(1 - \epsilon H\left(q_i, p_i\right) + \mathcal{O}\left(\epsilon^2\right)\right) \Big|_{x_{N+1}=x_1} \tag{6.26}$$

where the limits have been considered,

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{x}{m}\right)^m &= \mathrm{e}^x,\\ \lim_{n\to\infty}\prod_{m=1}^{n}\left(1+\frac{x_n}{m}\right) &= \exp\left[\lim_{m\to\infty}\frac{1}{m}\sum_{n=1}^{m}x_m\right].\end{aligned} \tag{6.27}$$

Thus, the partition function is now,

$$Z=\lim_{n\to\infty}\int\left[\prod_{i=1}^{n}\frac{\mathrm{d}q_i\ \mathrm{d}p_i}{2\pi}\right]\exp\left\{-\sum_{j=1}^{n}\epsilon\left[\frac{p_j^2}{2m}-ip_j\frac{q_{j+1}-q_j}{\epsilon}+V\left(q_j\right)\right]\right\}\Bigg|_{x_{n+1}=x_1}, \tag{6.28}$$

and making the following identifications,

$$\begin{aligned}\int\prod_{i=1}^{n}dq_i &\to \int\mathcal{D}q,\\ \int\prod_{i=1}^{n}\frac{dp_i}{2\pi} &\to \int\mathcal{D}p,\end{aligned} \tag{6.29}$$

and

$$\begin{aligned}\frac{q_{i+1}-q_i}{\epsilon} &\to \dot{q}(t_i)\\ \epsilon\sum_{m=0}^{n-1}f(t_m) &\to \int_0^\beta d\tau f(\tau),\end{aligned} \tag{6.30}$$

which results in,

$$Z=\int\limits_{q(\beta)=q(0)}\frac{\mathcal{D}q\mathcal{D}p}{2\pi}\exp\left\{-\int_0^\beta\mathrm{d}\tau\left[\frac{p(\tau)^2}{2m}-ip(\tau)\dot{q}(\tau)+V(q(\tau))\right]\right\}. \tag{6.31}$$

We notice that the integral about the conjugate moments is Gaussian, thus, it can be done explicitly,

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}p_i}{2\pi}\exp\left\{-\epsilon\left[\frac{p_i^2}{2m}-ip_i\frac{q_{i+1}-q_i}{\epsilon}\right]\right\}=\sqrt{\frac{m}{2\pi\epsilon}}\exp\left[-\frac{m\left(q_{i+1}-q_i\right)^2}{2\epsilon}\right], \tag{6.32}$$

resulting in,

$$Z = C \int\limits_{q(\beta)=q(0)} \mathcal{D}q \exp\left\{ -\int_0^\beta \mathrm{d}\tau \left[\frac{m}{2}\left(\frac{\mathrm{d}q(\tau)}{\mathrm{d}\tau}\right)^2 + V(q(\tau)) \right] \right\}; \qquad C \equiv \left(\frac{m}{2\pi\epsilon}\right)^{n/2}. \tag{6.33}$$

However, this factor is independent of the properties of potential $V(q)$ and, therefore, does not contain dynamic information, thus we should not worry too much about the apparent divergence, it is divergent in the limit $\epsilon \to 0$ and $n \to \infty$. Thus, we can simply write a continuous "functional integral",

$$Z = C \int\limits_{q(\beta)=q(0)} \mathcal{D}q \exp\left\{ -\int_0^\beta \mathrm{d}\tau \left[\frac{m}{2}\left(\frac{\mathrm{d}q(\tau)}{\mathrm{d}\tau}\right)^2 + \frac{m\omega^2}{2} q(\tau)^2 \right] \right\}. \tag{6.34}$$

Representing $q(\tau)$ in Fourier Terms,

$$q(\tau) = T \sum_{n=-\infty}^{\infty} (a_n + i b_n)\, e^{i\omega_n \tau}, \tag{6.35}$$

and ignoring the non-zero Matsubara modes (since the coordinate $q(\tau)$ is real), we have,

$$\begin{aligned} Z &= C' \int_{-\infty}^{\infty} \mathrm{d}a_0 \int_{-\infty}^{\infty} \prod_{n\geq 1} \mathrm{d}a_n\, \mathrm{d}b_n \exp\left[-\frac{1}{2} m T \omega^2 a_0^2 - mT \sum_{n\geq 1} \left(\omega_n^2 + \omega^2\right)\left(a_n^2 + b_n^2\right) \right] \\ &= C' \sqrt{\frac{2\pi}{mT\omega^2}} \prod_{n=1}^{\infty} \frac{\pi}{mT\left(\omega_n^2 + \omega^2\right)}, \quad C' = C \left| \det\left[\frac{\delta x(\tau)}{\delta x_n} \right] \right|. \end{aligned} \tag{6.36}$$

We can calculate the factor C' in the limit $(\omega \to 0)$, since it does not depend on ω. On the other hand, there is a divergence when evaluating the integral in the zero mode, for the moment a regularization will be used and it will be evaluated in a finite interval. It is possible to calculate the factor C' using the expression (6.24) and performing the calculations in a finite volume, thus we have,

$$C' = \frac{T}{2\pi} \sqrt{2\pi m T} \prod_{n=1}^{\infty} \frac{m T \omega_n^2}{\pi}. \tag{6.37}$$

Obtaining the partition function for the harmonic oscillator,

$$Z = \frac{T}{\omega} \prod_{n=1}^{\infty} \frac{\omega_n^2}{\omega_n^2 + \omega^2}. \tag{6.38}$$

When dealing with the partition function of a scalar field, it is important to note that the key difference with conventional statistical mechanics is that the trace must be interpreted as an integration over all the degrees of freedom, which are, in principle, infinite, corresponding to the fields. This also establishes a connection with quantum field theory under zero-temperature conditions. It is feasible to apply the result obtained previously, provided that the following observations are considered. First, let us consider the spatial coordinate (q) as an internal variable of the scalar field $\phi(t, q)$. Therefore, the field observed at a specific position, denoted as $\phi(t, 0)$, behaves similarly to $q(t)$. The most effective way to establish the partition function of a scalar field is to use the Lagrangian density corresponding to that field instead of the previously mentioned Lagrangian. Second, if we have not yet determined the number of spatial coordinates, it will be necessary to carry out an additional integration in d-dimensions. The Lagrangian for a scalar field now includes an additional term: the derivative concerning the spatial coordinates, represented as $\partial_i \phi \partial_i \phi$. This concept arises from the interaction with the nearest neighbor, although it does not influence the calculation of the partition function (a fundamental aspect for the calculation of the partition function is the quadratic term involving the time derivative of the field, that is, $\partial_t \phi \partial_t \phi$.). Third, the path integral must be carried out not only in the time direction but also in the additional dimensions ($q \to x$). The result is therefore the following,

$$Z = \int_{\phi(\beta,\mathbf{x})=\phi(0,\mathbf{x})} \prod_{\mathbf{x}} [C\mathcal{D}\phi(\tau, \mathbf{x})] \exp\left[- \int_0^{\beta} \mathrm{d}\tau \int \mathrm{d}^d\mathbf{x} \mathcal{L}_E \right], \tag{6.39}$$

where,

$$\mathcal{L}_E = -\mathcal{L}_M(t \to -i\tau) = \frac{1}{2} \left(\frac{\partial \phi}{\partial \tau} \right)^2 + \sum_{i=1}^{d} \frac{1}{2} \left(\frac{\partial \phi}{\partial x^i} \right)^2 + V(\phi). \tag{6.40}$$

Generalities. One small spacetime dimension

The partition function on a flat Euclidean space-time manifold $\mathbb{R}^{d-q} \times \mathbb{T}^{q+1}$ can also be computed in terms of a $SL(q+1,\mathbb{Z})$ Eisenstein series with $s = \frac{d+1}{2}$ [24]. Where the oscillator contribution can be calculated. For oscillator modes quantized in Fock space, the trace can be evaluated directly using the normal ordered Hamiltonian: $:\hat{H}':$ such that no divergences occur. Letting N_{ki} denote the occupation number of the oscillator associated with k_i, we have,

$$Z'_d(\beta) = Tr(e^{-\beta:\hat{H}':}) = \prod_{n_i\in\mathbb{Z}^d} \sum_{N_{k_i}\in\mathbb{N}} e^{-\beta\omega_{k_i}N_{k_i}} = \prod_{n_i\in\mathbb{Z}^d}{}' \frac{1}{1-e^{-\beta\omega_{k_i}}}, \tag{6.41}$$

obtaining,

$$\ln\{Z'_d(\beta)\} = -\sum_{n_i\in\mathbb{Z}^d}{}' \ln(1-e^{-\beta\omega_{k_i}}). \tag{6.42}$$

For high values of L_i, the sums can be approximated by integrals using the expression $dk_i = \frac{2\pi}{L_i}dn_i$. Carrying out the integrals with respect to angles in hyperspherical coordinates, we obtain the following,

$$\begin{aligned} \ln Z'_d(\beta) &= -\frac{V_d}{(2\pi)^d}\int d^dk\, \ln(1-e^{-\beta\omega_{k_i}}) \\ &= -\frac{V_d}{(2\pi)^d}\mathrm{Vol}(\mathbb{S}^{d-1})\int_0^\infty dk k^{d-1}\ln(1-e^{-\beta k}). \end{aligned} \tag{6.43}$$

The finite contribution of the zero mode $\ln Z_0(\beta) \sim \ln V_d$ is subdominant in the large volume limit, thus it can be mostly neglected unless otherwise noted. To perform the integration in (6.43), we have that $\mathrm{Vol}(\mathbb{S}^{d-1}) = 2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$, while for the other part, a change of variable can be performed, this is $x = \beta k$ and using integration by parts,

$$\int_0^\infty \mathbf{d}k k^{d-1}\ln(1-e^{\beta k}) = -\frac{1}{d\beta^d}\int_0^\infty \mathbf{d}x\frac{x^d}{e^x-1} = -\frac{1}{d\beta^d}\Gamma(d+1)\zeta(d+1). \tag{6.44}$$

Arriving at the partition function for a scalar blackbody,

$$\ln Z_d(\beta) = \frac{\Gamma(d)\zeta(d+1)V_d}{2^{d-1}\pi^{d/2}\Gamma(\frac{d}{2})\beta^d}. \tag{6.45}$$

When $z = d$,

$$\xi(z) = \frac{\Gamma(\frac{z}{2})\zeta(z)}{\pi^{\frac{z}{2}}}, \tag{6.46}$$

and at $z = d + 1$,

$$\ln\{Z_d(\beta)\} = \xi(d+1)\frac{V_d}{\beta^d} = \begin{cases} \frac{\pi}{6}\frac{L_1}{\beta} & \text{for } d = 1 \\ \frac{\zeta(3)}{2\pi}\frac{L_1 L_2}{\beta^2} & \text{for } d = 2 \\ \frac{\pi^2}{90}\frac{L_1 L_2 L_3}{\beta^3} & \text{for } d = 3 \\ \vdots & \vdots \end{cases} \tag{6.47}$$

For more extensive and precise details on the calculation of partition functions in d-dimensions, using Einstein series, see [24]. These results are very useful for calculating integrals of the type (6.43), since we can easily place the dimension in which we want to perform the integration. These integrals do not only appear in statistical mechanics, they also appear in string theory, quantum loop gravity and other areas.

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