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# Higher Derivative Supersymmetric FRW Cosmology

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# Contents

# Chapter 1

# Introduction and outlook

Current consensus is that quantum effects are not restricted to microscopic systems, e.g., subatomic particles or molecules distributed over small spatial regions, but in fact, the universe itself is governed by the laws of quantum theory. For all practical purposes, we are conceiving the universe as in cosmology, i.e., taking into account only large-scale degrees of freedom. The quantum properties of the universe are thought to be more important for the very early Universe. This is due, on one hand, to its being much smaller relative to the late-time universe, according to the hot big bang model and the inflationary paradigm, and, on the other hand, due to the high energy densities involved, which are comparable to the Planck scale. The quantum character applies to both the geometry of space-time and the matter content.

Then, one is confronted with the problem of determining the set of cosmological states or wave functions. Further, one has to single out the actual quantum state on the basis of correspondence with observations. This brings about another crucial ingredient, namely, boundary conditions [1]. It is hoped that some features of the presently observed universe that remain unexplained, find explanation in the cosmological wavefunction. For example, cosmic inflation provides us with a mechanism to explain the observed flatness and absence of horizons that were not explained by the hot big bang model. However, one then must consider the plausibility of the inflationary solution itself. For a simple classical model of a closed universe with a scalar field, the inflationary phase is not a generic solution, but depends on initial values of the field. The preference of the inflationary initial conditions over the non-inflationary ones is an example of the information one expects to extract from the cosmological wave function.

Now, gravity is the dominant force at astronomical and cosmological scales, and its classical description is provided by General Relativity, and sometimes by one of its numerous modifications such as f(R)-theories. Although several approaches have been put forward and developed to such extent that they constitute robust theoretical frameworks for not only a theory of quantum gravity, but a theory of all interactions, it is still not clear what the final theory of quantum gravity is. In this sense, quantum cosmology, that is, the application of quantum gravity ideas to the cosmological problem, allows one to check their consistency and viability. Another setting for probing quantum gravity theories is that of quantum black holes [5].

The canonical quantization of general relativity in the metric formulation [2] constitutes the earliest attempt for a theory of quantum gravity. As this approach is based on two theoretical frameworks lying at the foundations of modern physics, it is sensible to expect that the results obtained with such ultimate theory of quantum gravity are not too different from those of quantum general relativity, at least in some limiting regime. The main equation of quantum general relativity is the Wheeler-DeWitt equation, which is a functional differential equation on an infinite dimensional configuration space called superspace. This renders obtaining a solution, a highly non-trivial problem, if not impossible in the general case. Thus, one usually has to address simplified versions of problem resulting from the imposition of symmetries.

Averaging over large enough scales and considering our peculiar velocity relative to the average motion of matter, the observed universe is consistent with the Cosmological Principle, which establishes that all observations with respect to a physically preferred frame of reference (determined by the matter content) are independent of orientation and location [3]. In other words, on the large scale, the universe is assumed to be spatially homogeneous and isotropic. The main evidence for homogeneity is provided by the high degree of isotropy of the CMB, because large scale inhomogeneity would lead to important temperature variations. More realistic cosmological models incorporate perturbations of the metric and matter fields, to account for the astronomical structures we observe. Incidentally, the best indirect evidence of cosmic inflation comes from the fact that it provides a coherent account of structure formation based on quantum fluctuations of a matter field, which are frozen (not evolving) and amplified by the rapidly expanding universe [4].

Our aim in this work is quantum cosmology and we do not deal with perturbations, but instead consider a variant of homogeneous and isotropic, or FRW, universe. Under these symmetries, the gravitational field is represented by a single physical degree of freedom, the scale factor $a(t)$. Considering only a cosmological constant, the dynamics is governed by the Friedmann and acceleration (or Raychaudhuri) equations. These can be derived from an effective one-dimensional (time) action that results from evaluating the four-dimensional, say Einstein-Hilbert, action at the FRW metric. As a remnant of general covariance, we still have time-reparameterization invariance, made manifest by the gauge field, $N(t)$, called lapse. This gauge freedom gives rise to a constrained dynamics; since there is not preferred time parameter, the Hamiltonian is zero (which is equivalent to the Friedmann equation in Lagrangian terms). Canonical quantization leads to a greatly simplified, although nontrivial, version of the Wheeler-DeWitt equation mentioned above. Configuration space is given by the possible values of the scale factor and some other (homogeneous) matter fields. It is, therefore, a space with a small number of dimensions, called generically mini-superspace. Since the wave function does not depend on time, it sets the problem of how to recover the notion of an evolving universe. Some schemes have been put forward to solve this by, e.g., using a scalar field as time parameter [53]. On the other hand, semi-classical considerations provide considerable insight on the information contained in the cosmological amplitude.

The main objective of this work was to construct, and study classical and quantum aspects, of a supersymmetric extension of the FRW model with quadratic scalar curvature. For the FRW case, this is the most general modification with quadratic curvature terms [15]. This term brings about a higher-derivative scalar degree of freedom that can drive inflation in the high curvature regime. It is also known as the model of Starobinsky and is still a viable inflationary model, for example, the spectral index and tensor-to-scalar ratio are consistent with the constraints posed by cosmological observations[26]. This model of inflation has the appealing feature that it is not necessary to postulate the existence of a primeval matter scalar field, instead inflation arises as a higher-derivative excitation of the gravitational field itself. Besides this practical application in cosmology, quadratic and higher curvature corrections in the effective gravitational action are also motivated by, e.g, the problem of divergences when applying the perturbation method to a system of interacting gravitational and matter fields [9].

The other ingredient we consider is supersymmetry. Inflation can easily start at an energy comparable the Planck scale[27]. Even if we consider only the physically relevant part of the inflationary stage, taking place after horizon exit [28], with an energy density several orders of magnitude below $M_P$, it is still sufficiently high to be considered in a context of supersymmetry or, given that it is a gravitational effect, supergravity [29, 30]. It is also known that supergravity naturally provides a sort of square root for the Hamiltonian. This property has a practical application in the context of quantum cosmology. Speaking of the FRW case, because the gauge field $N$ acquires a supersymmetric partner $\psi$, extra constraints arise, and they form an algebra. In the quantum theory, we have additional equations for the wave function, which are generally simpler than the Wheeler-DeWitt equation. Sometimes one can find exact and even unique solutions, which is not the case even with the simplest models in quantum cosmology. Putting aside the computational advantage, this could have profound implications concerning the problem of boundary conditions [6].

The problem of building a model compatible with inflation has been addressed in the context of scalar-tensor supergravity theories. The main problem for the embedding of inflation into supergravity has been finding a suitable scalar potential generating inflation consistent way observations and with the delicate picture provided by the Standard Model of Cosmology on the early universe [35, 36]. The generic superpotential in N=1 4D supergravity gives rise to a scalar potential that is too steep for inflation [33]. Some methods have been put forward to produce a flat potential, or a flat direction in multi-field models [37, 38].

Regarding the model of Starobinsky as an $f(R)$ action, one might expect $f(\mathcal{R})$, with $\mathcal{R}$ the four-dimensional chiral curvature superfield, to provide an adequate supersymetrization [31, 32]. The connection between $f(\mathcal{R})$ and $f(R)$ is, however, not straightforward due to auxiliary fields satisfying rather involved algebraic equations of motion [33]. More general actions depending not only on $\mathcal{R}$, but its supersymmetric covariant derivatives, have been used to properly embed the model of Starobinsky into N=1, 2 4D supergravity [34, 16, 26].

On the other hand, the homogeneity at the beginning of inflation has led to the consideration of homogeneous supergravity formulations [39], see also [6]. Due to the dimensional reduction to one dimension, these formulations lack Lorentz constraints, which have been implemented by hand in [40, 41], otherwise the theory has a quite complicated constraint algebra [42]. In view of that, in [43] a supersymmetric extension of the FRW model has been proposed, considering, as usual, the re-scaled scale factor with length dimension. As well as in 4D supergravity, the basic superfield in the Lagrangian does not follow from geometric considerations, but is ad-hoc [25, 43].

In this work, we construct supersymmetric generalizations for the FRW model of Starobinsky in its modified gravity form, using the 'new' superspace formulation [25, 51] for homogeneous supergravity [45], i.e., depending only on time. As FRW models involves gauge fixing, the supersymmetric extension must keep it. This formulation is fully geometric and can be seen as a minimal dimensional reduction from 4D that keeps only the minimum elements for supersymmetry. In fact, for a dimensional reduction to one dimension, to each fermionic component of the supersymmetric charge corresponds, in one dimension, one supersymmetry. In this view, we consider N=1 and N=2 one-dimensional supergravity (as a complex representation, N=2 can be taken as N=1).

Actions based on four-dimensional supergravity and more fundamental theories may contain a vast number of additional fields both dynamical and auxiliary, which usually renders the sole identification of the scalar potential a highly nontrivial problem. In our case, the small number of degrees of freedom involved allows us to write both bosonic and fermionic sectors of the lagrangians and hamiltonians in full detail, not just the leading terms of certain Taylor expansion.

In Chapter 2, after a brief introduction to canonical general relativity, we describe the quantization of an FRW universe with a conformally invariant scalar field, which is the simplest nontrivial model in quantum cosmology. The Hartle-Hawking no-boundary proposal is also applied to obtain a semi-classical approximation to wavefunction. In Chapter 3, we give a detailed derivation of the 1D supergravity formalism we will use for writing our supersymmetric actions. This formalism is then put to work in Chapter 4 with the example of the supersymmetric conformally invariant scalar field, with which we show the basic features, both classical and quantum mechanical, of ordinary supersymmetric cosmological models, that is, not higher derivative yet. Then, in Chapter 5, the classical and quantum dynamics of the purely bosonic model of Starobinsky is described. We discuss the Ostrogradsky Hamiltonian and give canonical transformations relating it to the hamiltonians of scalar–tensor formulations of the Starobinsky model. Chapter 6 is dedicated to the supersymmetric theory following from the $F(\mathcal{R})$ action, where $\mathcal{R}$ is the superfield generalization of the FRW curvature. This chapter originated as a byproduct of our quest for a proper supersymmetric extension of the quadratic curvature. This action corresponds to the straightforward generalization of the bosonic $f(R)$ and, although it turned out to be not enough for that purpose, it sits in between the linear and the quadratic curvature supersymmetric cosmological actions. Some interesting aspects of higher-derivative theories can be illustrated with this action, and even some applications are discussed. Finally, in Chapter 7, we describe two supersymmetric extensions of the model of Starobinsky. The first possess N=1 supersymmetry, which is very simple, the (super)multiplets only contain two real components. Nonetheless, a supersymmetric Lagrangian whose bosonic part contains exactly the FRW model of Starobinsky was found. The second model possesses N=2 supersymmetry. In this case, superfields contain twice as many components. Usually, one of the bosons is an auxiliary supersymmetric field ensuring the off-shell closure of the supersymmetry algebra, but with higher-derivatives it becomes a dynamical field. We present numerical solutions to the pure bosonic equations of motion, which exhibit inflation driven by the quadratic curvature term, whereas the additional scalar field is pushed to the minimum energy state. Finally, we describe the quantization of both models. We obtained numerically wave functions for the N=1 model, and simple exact solutions for the N=1 model. Finally, Chapter 8 is dedicated to conclusions.

# Chapter 2

# Canonical gravity and quantum cosmology

In this chapter we give a brief review of the Hamiltonian formulation of Einstein gravity in the metric representation following reference [2]. Then, we discuss the quantization of a dimensionally reduced model corresponding to FRW cosmology [7].

### 2.0.1 3+1 decomposition

For the Hamiltonian formulation of the gravitational field, one reinterprets spacetime as the evolution of a three-dimensional spacelike hypersurface (3-surface). For simplicity we assume a spacetime $\mathcal{M}$ of topology $\mathbb{R} \times \Sigma$, so that there exists a real function $t$ defined on all of $M$ (global hiperbolicity). The 3-surfaces of constant time define a foliation of $M$. We still need a rule for relating a point P on $\Sigma_t$ with another point Q on $\Sigma_{t'}$ at a later time $t' > t$. Thus, a time-like vector field $t^a$, called evolution, is introduced. The integral curves of $t^a$ will establish a bijective relation between points of different 3-surfaces.

The restriction of the 4D Lorentzian metric $g_{ab}$ to $\Sigma$, naturally induces a 3D spatial metric $h_{ab}$ given by,

$$h_{ab} = g_{ab} + n_a n_b. \tag{2.1}$$

where $n^a$ is the unitary vector field $n^a = \partial^a t / \sqrt{-(\partial_b t)\partial^b t}$, which is everywhere orthogonal to the constant time 3-surfaces, and chosen pointing to the direction of growth of $t$, that is, $t^a \partial_a t \geq 0$. With the spatial metric and the orthogonal vector, one can decompose all other tensors into their time-time, spatial-spatial, and mixed components. For example, $t^a = N n^a + N^a$, where $N \equiv -n_a t^a$ is called lapse function, and $N^a = h^{ab} t_b$ is the shift vector.

This 3+1 decomposition gives rise to intrinsic geometry that only refers to the spatial 3-surface, with no knowledge of what happens off the surface. Intrinsic geometry tensors are the 3D spatial curvature, ${}^3R_{abc}{}^d$ and its contractions. On the other hand, the extrinsic curvature does know about the bending of the 3-surface in the 4D space. This information is contained in the extrinsic curvature tensor, given by the spatial covariant derivative of the orthogonal vector field, $K_{ab} \equiv D_a n_b = h_{ac} h_{bd} \nabla^c n^d$.

Time derivatives are defined as the full spatial projection of the Lie derivative with respect to the evolution vector. For example, $\dot{h}_{ab} = h_a^{\ c} h_b^{\ d} \mathcal{L}_{t^a} h_{cd}$. It can be shown that $K_{ab} = (2N)^{-1}(\dot{h}_{ab} - D_a N_b - D_b N_a)$; extrinsic curvature components are basically the velocities of the spatial metric.

The 4D curvature can be expressed in terms of the extrinsic and intrinsic curvature tensors and their contraction or products. Also, the determinant of the 4D metric, $g$, is proportional to that of the spatial metric, $h$, $g = -N^2 h$. All in all, the Einstein-Hilbert (E-H) action can be re-written as

$$S = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}R + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x\sqrt{h}K \tag{2.2}$$

$$= \frac{1}{16\pi G}\int d^4x N\sqrt{h}\left({}^3R + K_{ab}K^{ab} - K^2\right) \tag{2.3}$$

where the second term on the r.h.s. of (2.2) is a boundary term, sometimes called Gibbons-Hawking term, required to make the action functionally differentiable.

### 2.0.2 Hamiltonian formulation

The canonical variables are the four components of the lapse and shift vector and the six components of the spatial metric, together with their conjugate momenta. Since the action does not depend on $\dot{N}$ nor $\dot{N}^a$, their related momenta are zero and lead to primary constraints,

$$0 \approx p_N, \qquad\qquad 0 \approx p_{N^a}. \tag{2.4}$$

The velocities of the spatial metric components can be solved for from the definition of the momenta

$$p^{ij} \equiv \frac{\delta L}{\delta \dot{h}_{ij}} = \frac{1}{2N}\frac{\delta L}{\delta K_{ij}}. \tag{2.5}$$

Performing the Legendre transformation one gets

$$0 \approx H = \int d^3x\mathcal{H} = \int d^3x\left[NH_0 + N^aC_a + \lambda p_N + \mu^a p_{N_a}\right]. \tag{2.6}$$

The conservation in time of (2.4) implies secondary constraints,

$$0 \approx H_0 \equiv \frac{2}{\sqrt{h}}\left(p_{ab}p^{ab} - \frac{1}{2}(p^c_{\ c})^2\right) - \frac{1}{2}\sqrt{h}\ {}^3R, \tag{2.7a}$$

$$0 \approx C_a \equiv -2D_b p^b_{\ a}. \tag{2.7b}$$

which are known as the Hamiltonian and momentum constraints, respectively.

The total Hamiltonian 2.6, therefore, vanishes as a constraint. There are eight first class constraints, not just the expected four that generate coordinate transformations. This is due to the extra freedom involved in the choice of frame, that is, the time function and evolution vector.

### 2.0.3 Quantization

Adopting Dirac's quantization, equations (2.7) are promoted to quantum constraints to be satisfied by the physical states of the theory. (An alternative is the ADM approach, in which one solves the constraints prior to quantization [6]).

The quantum state is represented by a wave functional $\psi[h_{ij}, \phi]$, where $\phi$ stands for possible matter fields. Since there is a set of metric components and matter fields per space point, the configuration space, called superspace, is infinite dimensional.

The momentum operators are defined in terms of functional derivatives,

$$p^{ij} \to -i\frac{\delta}{\delta h_{ij}}, \qquad p_\phi \to -i\frac{\delta}{\delta\phi}. \tag{2.8}$$

A physically allowed state must obey the Wheeler-DeWitt (WDW) equation,

$$0 = H_0\psi = \left(-G_{ijkl}\frac{\delta}{\delta h_{ik}}\frac{\delta}{\delta h_{kl}} - \sqrt{h}\,{}^{(3)}R + C^{\text{matter}}\right)\psi = 0, \tag{2.9}$$

where $C^{\text{matter}}$ stands for the matter contribution to the $H_0$ constraint, and the momentum constraint equations

$$C_i\Psi = \left(2ih_{ik}D_j\frac{\delta}{\delta h_{jk}} + C_i^{\text{matter}}\right)\Psi = 0. \tag{2.10}$$

The latter can be solved in full generality and has the implication that 3-geometries related by a diffeomorphism have the same amplitude. in this way, superspace is refined to the space of all 3-geometries and field configurations modulo diffeomorphisms.

#### Path integral approach

We have seen the differential approach to canonical quantum gravity. Now, we consider the integral approach provided by the path integral formulation. This will be important for it provides valuable insight on the wavefunction.

The fundamental amplitude for the occurrence of a given space-time

$$e^{iS[g,\phi]/\hbar} \tag{2.11}$$

where $S$ is the Lorentzian action for a given 4-geometry, that is a 4D spacetime of definite topology, endowed with a metric a matter fields.

The amplitude for the transition from a 3-geometry $(\Sigma', \phi')$ to $(\Sigma'', \phi'')$ is given by the path integral

$$\langle h''_{ij}, \phi'' | h'_{ij}, \phi' \rangle = \int \delta g \delta\phi e^{iS[g,\phi]/\hbar}. \tag{2.12}$$

where the sum is over all 4-geometries and field configurations interpolating between the given 3-geometries. Note that (2.12) does not make any reference to time; the proper time between the 3-surfaces depends on the specific 4-geometry, but its measure requires intermediate observations which is not the case for a transition amplitude.

To improve the convergence of functional integral, one defines the wavefunction in the Euclidean regime by sending $t \to it \equiv t^E$ and adjusting the sign of the action so that $e^{iS} \to e^{-I^E}$. Then the arbitrary wavefunction is given by

$$\psi[h_{ij}, \phi] = N \int_C \delta g \delta \phi e^{-I^E[g^E,\phi]/\hbar} \tag{2.13}$$

The specification of the class of 4-geometries in the sum defines the state. As an example, we consider the Hartle and Hawking proposal for the ground cosmological state [7]. It consists of summing over compact Euclidean 4-geometries with a single boundary consisting of a closed 3-geometry with metric $h_{ij}$, and matter field distribution $\varphi$, and such that the action remains finite to the far past. This prescription is known as the no-boundary state because the 4-geometries involved do not have a boundary in the past, but only the one at which the quantum amplitude is evaluated. In other words, the boundary condition is that the universe has no boundary (to the past). As we shall see, the character of this state as a ground state is related to its being a highly symmetric state rather than a minimum energy state.

**Minisuperspace**

Solving the WDW equation on the full superspace is a formidable task unless one simplifies the problem by imposing symmetries, with which in some cases one reduces the number of dimensions of superspace. For example, spatial homogeneity and isotropy of the FRW universes, discussed below, reduces the dimensionality of superspace to a finite small number. In this case, it receives the name of mini-superspace. This brings about the subject of quantum cosmology, which is the quantization of time reparameterization invariant actions. Quantum cosmology models serve as toy quantum gravity models, with which one hopes to learn about the properties of the full solutions to the WDW equation. They are also useful to study aspects that are independent of the number of dimensions, such as the problem of time [5].

## 2.1 Quantum FRW cosmology

We describe here the canonical quantization of the FRW cosmological action following references [7, 8, 9, 10, 11]. The semiclassical approximation to the wavefunction, computed by the saddle-point method, is emphasized.

### 2.1.1 FRW action

The 4D FRW Lorentzian metric for closed spatial geometry ($k = 1$) can be written as

$$ds^2 = \sigma^2 \left[ -N^2(t)dt^2 + a^2(t)\left(d\psi^2 + \sin^2\psi \left[d\theta^2 + \sin^2\theta d\phi^2\right]\right)\right] \tag{2.14}$$

where $t$, $\psi$, $\theta$, $\phi$ are co-moving coordinates, and $\sigma$ is a constant.

Evaluating the Einstein-Hilbert (E-H) action, considering a positive cosmological constant $\Lambda$, $S = (16\pi G)^{-1}\int d^4x\sqrt{-g}(R-2\Lambda)$, at this geometry, results in an effective 1D action, with manifest time-reparameterization invariance, namely

$$S = \frac{1}{2}\int dt N a^3\left[-\frac{\dot{N}\dot{a}}{N^3 a} + \frac{\ddot{a}}{N^2 a^2} + \frac{\dot{a}^2}{N^2 a^2} + \frac{1}{a^2} - \lambda^2\right], \tag{2.15}$$

where one chooses $\sigma^2 = 2G/3\pi G$, for convenience, and also defines $\lambda \equiv \sigma\sqrt{\Lambda/3}$.

The equations of motion following from the action (2.15) are equivalent to those that one gets by imposing homogeneity and isotropy on the full 4D field equations. This is not always the case, but only for those systems satisfying the so called symmetric criticality principle [5].

Given that the Lagrangian in (2.15), depends not only on $a$, $\dot{a}$, but also on $\ddot{a}$, the variation of the action goes like

$$\frac{\delta S}{\delta a} = \int \delta a\left[\frac{\partial L}{\partial a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{a}} + -\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{a}}\right]dt + \left[\left(\frac{\partial L}{\partial \dot{a}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{a}}\right)\delta a + \frac{\partial L}{\partial \ddot{a}}\delta\dot{a}\right]_{t_i}^{t_f} \tag{2.16}$$

Therefore, to eliminate boundary terms and establish the equation of motion, one would have to fix the value of both coordinate $a$ and momentum $\propto \dot{a}$.

To avoid dealing with a higher-derivative theory, one re-defines the action with the Gibbons-Hawking term [2]. In the FRW case, that amounts to integrate parts the second time-derivative of the scale factor in the action and cancel the boundary term with its additive inverse. Thus, one gets

$$S = \frac{1}{2}\int dt\left[-\frac{a\dot{a}^2}{N} + Na - \lambda^2 N a^3\right]. \tag{2.17}$$

In fact, (2.17) is the form of the action that one gets from the $3+1$ decomposed action (2.3), by using the FRW quantities

$$K = \frac{-3\dot{a}}{\sigma N a}, \qquad K_{ij}K^{ij} = \frac{3\dot{a}^2}{\sigma^2 N^2 a^2}, \qquad {}^3R = \frac{6}{\sigma^2 a^2}. \tag{2.18}$$

### 2.1.2 Conformally invariant scalar field

As a first example with matter, consider the 4D Lorentzian action for a conformally invariant scalar field,

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - \xi R\phi^2\right). \tag{2.19}$$

If $\xi = 1/6$, the action is invariant under the following transformation

$$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}, \qquad \phi \to \Omega^{-1}(x)\phi. \tag{2.20}$$

with $\Omega(x) > 0$ an arbitrary function on space-time.

In the FRW case, one defines the invariant field $\varphi$,

$$\varphi(t) = \sqrt{2}\pi\sigma a(t)\phi(t), \tag{2.21}$$

so that the total gravity-matter action reads

$$S = \frac{1}{2}\int dt \left[-\frac{a\dot{a}^2}{N} + Na - \lambda^2 N a^3 + \frac{a\dot{\varphi}^2}{N} - N\frac{\varphi^2}{a}\right]. \tag{2.22}$$

Since there is no time-derivative of the lapse $N$, its associated momentum will vanish as a constraint, $p_N \approx 0$. Performing the Legendre transformation, one gets a Hamiltonian of the form

$$H = NH_0 + lp_N. \tag{2.23}$$

where $l$ is an arbitrary function, and

$$2H_0 = -\frac{p_a^2}{a} - a + \lambda^2 a^3 + \frac{1}{a}\left(p_\varphi^2 + \varphi^2\right). \tag{2.24}$$

Performing Dirac's standard method for singular systems, we also get

$$H_0 \approx 0 \tag{2.25}$$

which is known as the Hamiltonian constraint. In Lagrange terms, (2.25) amounts to the Friedmann equation.

At the quantum level, we postulate the existence of a wavefunction $\psi(a,\varphi)$, obeying the Wheeler-DeWitt (WDW) equation $H_0\psi = 0$. There is an ordering ambiguity due to the term $a^{-1}p_a^2$. A particular ordering choice corresponds to the Laplace-Beltrami operator [5], namely

$$G^{AB}p_A p_B \to -\hbar^2\nabla_{LB}^2 \equiv -\frac{\hbar^2}{\sqrt{-G}}\partial_A\left(\sqrt{-G}G^{AB}\partial_B\right), \tag{2.26}$$

where $G_{AB}$ is the mini superspace metric. This can be identified from kinetic term in the action, which is given by the time integral of $\frac{1}{2}G_{AB}\dot{q}^A\dot{q}^B N^{-1}$.

For the case of (2.22) we have

$$G_{AB} = \mathrm{diag}(-a, a), \tag{2.27}$$

therefore, with this ordering prescription, the WDW equation reads

$$\frac{1}{2a}\left[\hbar^2\left(\partial_a^2 - \partial_\varphi^2\right) - a^2 + \lambda a^4 + \varphi^2 - 2\epsilon_0\right]\psi(a,\varphi) = 0. \tag{2.28}$$

The additional constant $\epsilon_0$ accounts for a possible redefinition of the zero point energy[7].

Due to the conformally invariant field effectively decoupling from the gravitational field, the wave equation (2.28) is separable. Assuming a well-behaved $\psi(a,\varphi)$, we expand it harmonic oscillator eigenfunctions,

$$\psi[a,\varphi] = \sum_n \psi_n(a)\phi_n(\varphi) \tag{2.29}$$

with $\frac{1}{2}\Big(-\hbar^2\partial_\varphi^2 + \varphi^2\Big)\phi_n(\varphi) = \hbar(n+1/2)\phi_n(\varphi)$, and

$$\left[-\hbar^2\frac{d^2}{da^2} + a^2 - \lambda^2 a^4 + 2\epsilon_0\right]\psi_n(a) = \hbar(2n+1)\psi_n(a). \tag{2.30}$$

This last ordinary differential equation can be solved numerically. Figure 2.1 shows one such solution with boundary condition $\psi_0(0) = 0$.

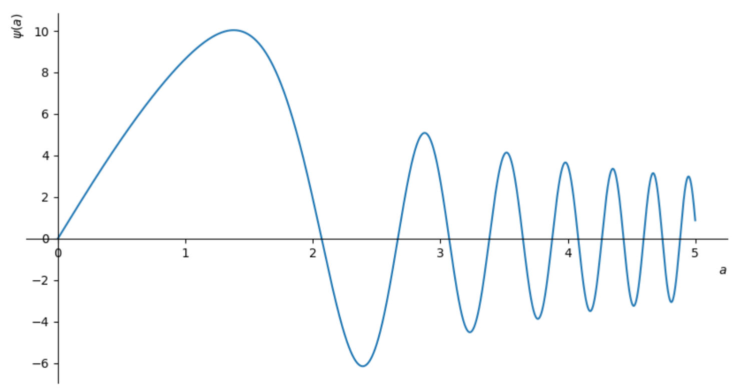



Figure 2.1: Numerical solution to equation (2.28) ($\lambda = 1$, $n = 0$) satisfying the boundary condition that the amplitude for vanishing 3-geometry is zero.

### 2.1.3 No-boundary ground state

We obtained above a cosmological wavefunction for the case of a conformally invariant field. To get a deeper understanding of the cosmological wavefunction, we describe here the complementary approach based on the path integral. We will mainly focus on the semiclassical approximation provided by this method.

For the mini-superspace model, the wavefunction is provided by the functional integral

$$\psi[a,\varphi] = \int \delta a' \delta\varphi' \, e^{-I^E[a,\varphi]/\hbar}. \tag{2.31}$$

The Euclidean action $I^E$ can be obtained from (2.22) by sending $t \to t^E = it$ and multiplying $S$ by $-i$. This yields

$$I_E = \frac{1}{2}\int dt^E \left[-\frac{a}{N}\left(\frac{da}{dt^E}\right)^2 - Na + \lambda^2 N a^3 + \frac{a}{N}\left(\frac{d\varphi}{dt^E}\right)^2 + N\frac{\varphi^2}{a}\right]. \tag{2.32}$$

A simplification like the separability of the WDW equation, originating from the matter field effectively decoupling from the gravitational field, also manifests in this case. The gravitational and matter contributions to the action can be evaluated separately, as can be seen by writing (2.32) in conformal time $\eta$ (i.e., $N(\eta) = a(\eta)$ or, equivalently $ad\eta = d\tau^E$),

$$I^E = \frac{1}{2}\int_{-\infty}^{0} d\eta \left[ -\left(\frac{da}{d\eta}\right)^2 - a^2 + \lambda^2 a^4 + \left(\frac{d\varphi}{d\eta}\right)^2 + \varphi^2 \right]. \tag{2.33}$$

where we have specified the range of integration. Boundary conditions consistent with the no-boundary prescription are: $a(0) = a_0$, $\varphi(0) = \varphi_0$, whereas at $\eta \to -\infty$ both $a$ and $\varphi$ vanish.

The solution to matter equation of motion, $d^2\varphi/d\eta^2 - \varphi = 0$, satisfying the boundary conditions, is $\varphi(\eta) = \varphi_0 e^{\eta}$. Evaluation of the matter part of the action yields $I[\varphi_0] = \varphi_0^2/2$. Therefore, the zeroth order approximation to the wavefunction, that is, the purely classical contribution, is

$$\psi_0[a_0, \varphi_0] = e^{-\varphi_0^2/2}\psi_0[a_0]. \tag{2.34}$$

The fact that the matter component of the wavefunction is that with the minimum energy supports the interpretation of the Hartle-Hawking proposal as defining the cosmological ground state.

On the other hand, the Friedmann equation is $(da/d\eta)^2 = a^2(1-\lambda^2 a^2)$. The solution valid for $\lambda a \leq 1$ is $a(\eta) = \lambda^{-1}\mathrm{sech}(\eta - \eta_0)$, with $a_0 = \lambda^{-1}\mathrm{sech}(\eta_0)$. When $\lambda a_0 < 1$, two different values $\pm|\eta_0|$ yield the same $a_0$, each one is associated to a different geometric configuration. To see this, it is convenient to write the solution in terms of proper Euclidean time $\tau^E$.

Integrating the relation $a(\eta)d\eta = d\tau^E$, we get $\tan[\lambda(\tau^E \pm |\tau_0|)] = \sinh[\eta \pm |\eta_0|]$, where $-\pi/2 \mp |\tau_0| \leq \lambda\tau \leq 0$. Then, $(\lambda a)^{-1} = [1+\sinh^2(\eta - \eta_0)]^{1/2} = \mathrm{sech}[\lambda(\tau^E - \tau_0)]$. That is, $a(\tau) = \lambda^{-1}\cos(\lambda(\tau \pm |\tau_0|))$. Or, by shifting the domain of Euclidean time to $0 \leq \lambda\tau^E \leq \pi/2 \pm |\tau_0|$,

$$a(\tau^E) = \lambda^{-1}\sin(\lambda\tau^E). \tag{2.35}$$

Therefore, the compact 4-geometry extremizing the action is part of a 4-sphere of radius $\lambda^{-1}$. Indeed, defining $\theta = \lambda\tau$, and $\theta_0 = \pi/2 - |\tau_0|$ (so that $\pi/2 + |\tau_0| = \pi - \theta_0$), the line element reads

$$ds^2 = \sigma^2\lambda^{-2}(d\theta^2 + \sin^2\theta d\Omega_3^2) = \sigma^2\lambda^{-2}d\Omega_4^2. \tag{2.36}$$

Evaluation of the gravitational part of the action (2.32) at the classical solutions yields

$$I_E[a_0] = \int_0^{\theta_0, \pi-\theta_0} d\theta L[a(\theta)] = \frac{1}{3\lambda^2}\left[-1 \pm (1-\lambda^2 a_0^2)^{3/2}\right]. \tag{2.37}$$

As shown in Appendix A, the contour integral chooses the $\theta_0$ configuration, despite the fact that it does not provide the dominant term. Thus, to zeroth order approximation one gets

$$\psi_0[a_0] \propto \exp\left[\frac{1}{3\lambda^2}\left[1 - (1-\lambda^2 a_0^2)^{3/2}\right]\right], \tag{2.38}$$

which is valid for $a < \lambda^{-1}$. This wavefunction is shown in Figure 2.3 (a). For very small $a$, the wavefunction decreases exponentially as $a$ approaches zero,

$$\psi_0[a_0] \sim e^{\frac{1}{2}a_0^2}. \tag{2.39}$$

On the other hand, one can address the case $a > 1/\lambda$ by promoting $\tau^E$ to a complex parameter, and analytically continue (2.35), that is

$$a(\tau^E) = \frac{1}{\lambda}\frac{e^{i\lambda\tau^E} - e^{-i\lambda\tau^E}}{2i} \tag{2.40}$$

We need to evaluate the action from $a = 0$ up to $a = a_0 > \lambda^{-1}$. Since the Lagrangian is analytic, we can integrate it along any contour in the complex plane connecting the given endpoints. We choose to integrate over the real axis from 0 a $\pi/2\lambda$, and then continue parallel to the imaginary axis. In this second segment, we make the change of variable $\tau^E = \pi/2\lambda + it$ so that $a(\tau^E) \to a(t) = \lambda^{-1}\cosh(\lambda t)$. The integration goes from $t = 0$ up to either $t_0$ or $-t_0$, where $a_0 = \lambda^{-1}\cosh(\lambda t_0)$. The contribution to the action from these two segments is

$$I_E^{\text{hemi}} = -\frac{1}{3\lambda^2}, \qquad I^{\text{Lorentz}} = \pm\frac{i}{3\lambda^2}(\lambda^2 a_0^2 - 1)^{3/2}. \tag{2.41}$$

The first segment gives the contribution from the Euclidean hemi-(4-sphere), whereas the second part gives the contribution from complex Lorentzian geometry, see Figure 2.2.

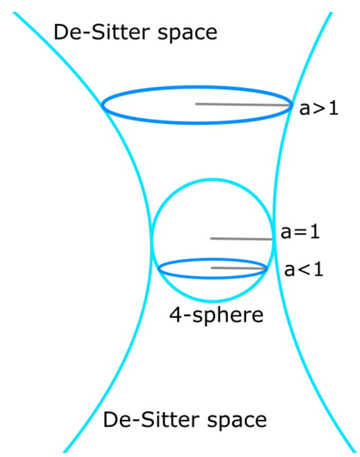



Figure 2.2: Geometry associated to the ground state wavefunction via the semiclassical approximation.

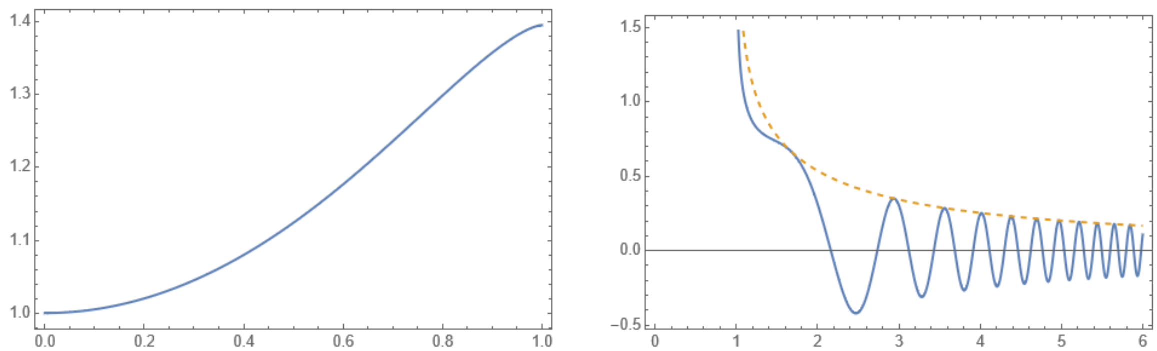
Both Lorentzian configurations contribute in such a way that the ground state wave function is real (see Appendix A), and we have

$$\psi_0[a_0] \propto \exp\left[\frac{1}{3\lambda^2}\right]\cos\left[(\lambda^2 a_0^2 - 1)^{3/2}/3\lambda^2 - \frac{\pi}{4}\right]. \tag{2.42}$$

In this case, we can go further and compute, via the standard WKB method, the first order quantum correction (in accordance with the matter part of the wavefunction, we set $n = 0$ in (2.30) to perform WKB). Then, the semi-classical approximation to the wavefunction is

$$\psi_0[a_0] \approx \left(\lambda^2 a_0^4 - a_0^2 + \hbar - 2\epsilon_0\right)^{-1/4} \cos\left[(\lambda^2 a_0^2 - 1)^{3/2}/3\lambda^2 - \frac{\pi}{4}\right], \tag{2.43}$$

which is depicted in Figure 2.3 (b).



(a) (b)

Figure 2.3: Gravitational component of the wavefunction $\psi_0(a)$ for (a) $a\lambda < 1$, (2.38) and (b) $\lambda a > 1$, (2.43). The dotted line corresponds to the envelope.

**Discussion**

Imaginary Euclidean time is actually real Lorentzian time and the solution for $a > \lambda^{-1}$ is actually De-Sitter spacetime, which can be seen as 4D-hyperboloid embedded in 5D spacetime. In the 3+1 decomposition, De-Sitter spacetime corresponds to the evolution of a 3-sphere, starting with a contracting phase from infinite radius at $t = -\infty$, up to a minimum radius $\lambda^{-1} \propto \Lambda^{-1/2}$, depending on the value of the cosmological constant. Then, it expands back to infinite radius. The two complex conjugate extreme actions are associated with the phases of expansion and contraction of De-Sitter spacetime. The fact that the purely classical contribution (2.42) does not vanish at large radius $a$, is a signature of the expansion without limit of De-Sitter space.

The first-order quantum correction in (2.43) also has some sort of information about the classical spacetime involved. De-Sitter spacetime is conformal to Einstein static universe, $ds^2 = \sigma^2 a^2(-d\eta^2 + d\Omega_3^2)$, and it is intuitively obvious that 3-spheres are distributed uniformly along $\eta$. If $\Xi$ counts the number of 3-spheres per unit of $\eta$ (taking the limit of some discrete distribution), then $d\Xi/d\eta$ is the linear density of 3-spheres over $\eta$ and is a constant. Then, the same distribution of 3-spheres, but along over the scale factor of a Lorentzian De-Sitter spacetime ($a(\tau) = \lambda^{-1}\cosh(\lambda\tau)$) is

$$\frac{d\Xi}{da} = \frac{d\Xi}{d\eta}\frac{d\eta}{da} \propto \frac{d\eta}{d\tau}\frac{d\tau}{da} = \frac{1}{a}\frac{d\tau}{da} = \frac{1}{a}\left(\frac{da}{d\tau}\right)^{-1} = \frac{1}{a\sqrt{\lambda^2 a^2 - 1}}, \tag{2.44}$$

The expression on the r.h.s of (2.44) gives the envelope of $|\psi_0(a)|^2$ for the wavefunction (2.43) (taking $\epsilon_0 = 1/2$, that is, for the ground state energy of the matter field normalized to zero).

We claimed above the $\psi_0[a_0, \varphi_0]$ obtained with the Hartle-Hawking proposal to be a sort of cosmological ground state on the basis of being a state of high symmetry. This has to do with the simplicity of the classical geometry involved. In the case of FRW universes endowed with a cosmological constant, the De-Sitter solutions correspond to the so-called maximally symmetric spaces, since they possess a group of isometries generated by ten independent Killing vector fields. In fact, its isometry group is the five-dimensional Lorentz group. The analogous space in Euclidean geometry is the 4-sphere, which is also the associated classical geometry involved in the wavefunction for small values of $a$.

The solution for $a < \lambda^{-1}$ corresponds a quantum state for a classically forbidden region. Therefore, an exponentially decreasing amplitude as $a$ decreases is to be expected. Being this, a purely quantum state, the zeroth order approximation for Euclidean geometry, (2.38), is really the first quantum correction for Lorentzian geometry. Since, it is not the exact solution of the WDW equation, we must not expect it to vanish at $a = 0$, as the numerical solution in Figure 2.1, nonetheless, its qualitative behavior matches that of the numerical solution.

### 2.1.4 Real scalar field

As a second example we take a minimally coupled scalar field with arbitrary potential,

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\Big[-g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - U(\chi)\Big] = \frac{1}{2}\int dt N a^3\Big[\frac{\dot{\phi}^2}{N^2} - V(\phi)\Big], \tag{2.45}$$

where $\phi = \sigma\sqrt{v_0}\chi$ and $V(\phi) = \sigma^4 v_0 U(\chi)$, $v_0 = 2\pi^2$.

The Hamiltonian constraint is

$$0 \approx H_0 = \frac{1}{2}\Big[-\frac{p_a^2}{a} + \frac{p_\phi^2}{a^3} - a + a^3 V(\phi)\Big] \tag{2.46}$$

In this case, the mini superspace metric is $G_{AB} = \mathrm{diag}(-a, a^3)$, $\sqrt{-G} = a^2$. Thus, with the ordering choice (2.26), the WDW equation reads

$$\frac{1}{2}\Big[\hbar^2\Big(a^{-1}\partial_a a\partial_a - a^{-2}\partial_\phi^2\Big) + U(a,\phi)\Big]\psi(a,\phi) = 0 \tag{2.47}$$

where

$$U(a,\phi) = -a^2(1 - a^2 V(\phi)). \tag{2.48}$$

#### Saddle-point approximation

Before considering a numerical solution to equation (2.47), we write the approximation to the wavefunction obtained with the no-boundary prescription. To get an idea of the behavior of the wavefunction, one is required to know the value of the action for different classical configurations. A detailed discussion of this can be found in [8]. We will consider here, a slowly varying potential so that it can be approximated by a positive cosmological constant,

$$\Lambda = \frac{3V}{\sigma^2} \tag{2.49}$$

and ignore derivatives of the wavefunction with respect to $\phi$. Then, from equations (A.12) and (A.15), we get

$$\psi(a,\phi) \approx \begin{cases} \exp\left[\frac{1}{3\hbar V(\phi)}\right]\exp\left[-\frac{(1-a^2V(\phi))^{3/2}}{3\hbar V(\phi)}\right], & \text{if} \quad a^2V < 1 \\ \exp\left[\frac{1}{3\hbar V(\phi)}\right]\cos\left[\frac{(a^2V(\phi)-1)^{3/2}}{3\hbar V(\phi)} - \frac{\pi}{4}\right], & \text{if} \quad a^2V \geq 1 \end{cases} \tag{2.50}$$

where the oscillatory wave-function arises from the sum of complex conjugate components, $\exp[(3\hbar V)^{-1} \pm i[(3\hbar V)^{-1}(a^2V-1)^{3/2} - \frac{\pi}{4}]]$.

For a massive scalar field, $V(\phi) = m^2\phi^2$, the potential of the WDW equation (2.47), is $U(a,\phi) = -a^2(1-a^2m^2\phi^2)$, and is shown in Figure 2.4.

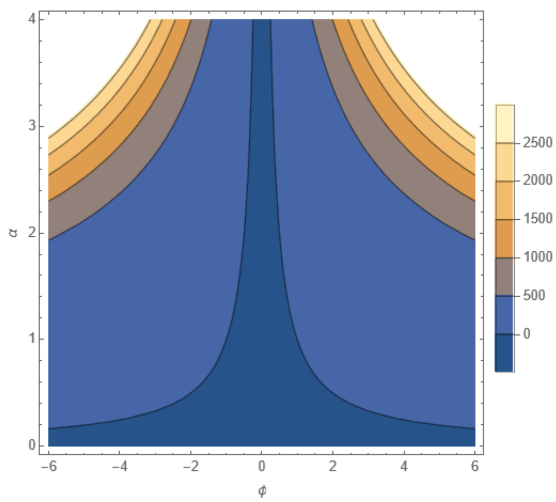



Figure 2.4: Contour plot of $U(a,\phi)$ for the massive scalar field.

The no-boundary approximation to the wavefunction is, from (2.50),

$$\psi(a,\phi) \approx \begin{cases} \exp\left[\frac{1}{3m^2\phi^2}\left(1-(1-m^2a^2\phi^2)^{3/2}\right)\right], & \text{if} \quad U < 0 \\ \exp\left[\frac{1}{3m^2\phi^2}\right]\cos\left[\frac{(m^2\phi^2a^2-1)^{3/2}}{3m^2\phi^2} - \frac{\pi}{4}\right], & \text{if} \quad U \geq 0 \end{cases} \tag{2.51}$$

A numerical solution to the WDW equation of the massive scalar field is shown in Figure 2.5 (a). We choose boundary conditions consistent with the no-boundary proposal: $\psi(0,\phi) = 1$ and $\partial_\phi\psi(a \to 0,\phi) \approx 0$ [6]. As can be observed, the behavior of the wavefunction is pretty close to the considerations of the saddle-point approximation; it oscillates in the region of positive $U$, and increases exponentially in the region of negative $U$.

From (2.51), we know that the wavefunction is diverging as $e^{\frac{1}{2}a^2}$ along $\phi = 0$. Thus, in Figure 2.5 (b) we plot $\psi(a,\phi)$ divided by exponential terms of the semi-classical approximation, in the corresponding regions.

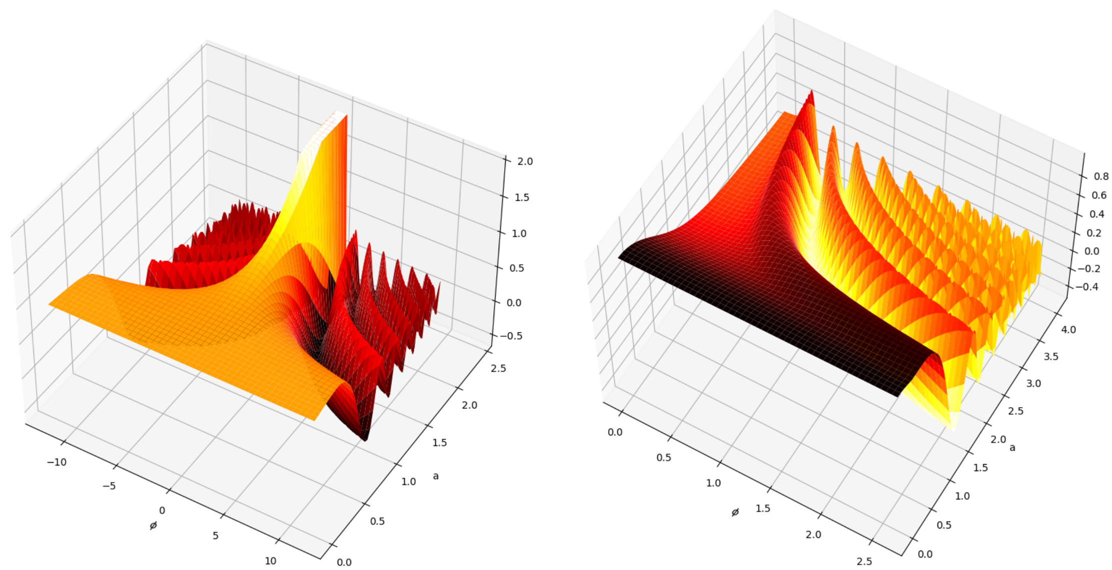



Figure 2.5: (a) $\psi(a,\phi)$ obtained numerically with boundary conditions $\psi(0,\phi) = 1$ with parameters $m = 1 = \hbar$, (b) Res-scaled wavefunction for better visualization.

With boundary condition $\psi(0,\phi) = 0$ and $\partial_a \psi(a \to 0, \phi) \neq 0$, we obtain a very similar wave function.

## 2.2 No-boundary and wormhole wave functions

To finish chapter, we recall here two important wave functions that will usually appear in the supersymmetric models discussed in the coming chapters. They arise from the simplest configuration of an empty flat FRW universe.

The Euclidean action is

$$
\begin{aligned}
I^E &= \frac{1}{2}\int dt \left[ -\frac{a}{N}\left(\frac{da}{dt^E}\right)^2 - Na \right] \\
&= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x \sqrt{g(N,a)} R(N,a) - \frac{1}{\kappa^2}\int_{\partial\mathcal{M}} d^3x \sqrt{h(a)} K^E(N,a). \qquad (2.52)
\end{aligned}
$$

The form of the action in the second line is better for illustrating the basic idea here.

Since there is no matter nor cosmological constant, the Euclidean Ricci scalar vanishes

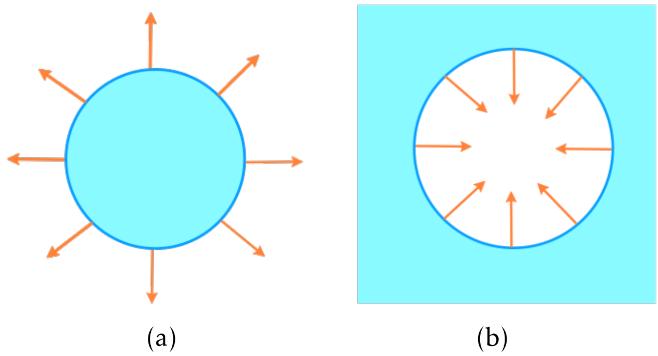



Figure 2.6: 4-geometry associated to the (a) no-boundary and (b) wormhole states. The lighter blue represents the 4-geometry and the circle in darker blue represents the closed 3-geometry, which is the only boundary in the case of (a), and one of the boundaries in the case of (b), since the 4-geometry extends to infinity.

$R = 0$. Thus, the only contribution to the action comes from the surface term,

$$I^E = -\frac{a^2}{2N}\frac{da}{dt^E}\bigg|_{\partial M} \tag{2.53}$$

The Friedmann equation in proper time ($N = 1$) is $(da/dt^E)^2 = 1$. The sign of $da/dt^E$, which is essentially $K^E$ (the trace of $K^E_{ab}$), depends on whether the 3-geometry is increasing or decreasing. For normals diverging it is positive and $I^E = -a^2/2$, whereas for normals converging it is negative and $I^E = a^2/2$. Thus, the wavefunctions $e^{-I^E}$ are,

$$\psi(a) = \begin{cases} e^{\frac{1}{2}a^2}, & \text{No-boundary} \\ e^{-\frac{1}{2}a^2}, & \text{Wormhole} \end{cases} \tag{2.54}$$

The former has already been seen for very small values of $a$, (2.39). The latter takes its name from the geometry involved [11]. Euclidean wormholes are roughly narrow tubes connecting two asymptotically flat regions. The 4-geometries considered in the path integral are, unlike those of the no-boundary proposal, non-compact. The 3-geometries reach infinite radius to the past. The wavefunction is evaluated at the finite boundary which consists of a closed 3-geometry in the tube connecting the asymptotically Euclidean spaces. In the limit of flat ($R = 0$) geometry, depicted in Figure 2.6 (b), the normals to the boundary are converging.

# Chapter 3

# Supersymmetry and superspace

In this chapter, we give a self-contained derivation of the superspace formalism we use to write our supersymmetric actions based on references [50], [25], [51], [48].

## 3.1 N=2 1D supersymmetry algebra

In one (time) dimension there is no Lorentz group and all physical fields, either bosonic or fermionic, are scalars under time transformations. By promoting fermionic scalar fields to anti-commuting quantities already at at the classical level, one can construct (homogeneous) field representations of 1D N-extended supersymmetry algebras [50].

The N=2 supersymmetry algebra reads

$$[Q,\bar{Q}]_{+} = -2P, \tag{3.1a}$$
$$QQ = 0 = \bar{Q}\bar{Q}, \tag{3.1b}$$
$$[P,Q] = 0 = [P,\bar{Q}]. \tag{3.1c}$$

where $\bar{Q} \equiv Q^{\dagger}$, $P \equiv i\partial_t$ and $[,]_{+}$ denotes anti-commutator.

Let $A(t)$ be an element of some component multiplet. Under an infinitesimal supersymmetry transformation parameterized by anti-commuting constants $\xi, \bar{\xi}$, it will transforms as follows

$$\delta_{\xi} A(t) \equiv (\xi Q - \bar{\xi}\bar{Q})A(t). \tag{3.2}$$

The remarkable feature of (3.1) is that the commutator of two supersymmetry transformations results in a time translation,

$$[\delta_{\xi}, \delta_{\eta}]\phi(t) = -2(\xi\bar{\eta} + \bar{\xi}\eta)P\phi(t) = -2i(\xi\bar{\eta} + \bar{\xi}\eta)\dot{\phi}. \tag{3.3}$$

All the elements of the group connected to the identity can be parameterized as follows

$$\mathcal{G}(t,\theta,\bar{\theta}) \equiv e^{\delta_t} e^{\delta_\theta} = e^{-itP} e^{\theta Q - \bar{\theta}\bar{Q}}. \tag{3.4}$$

Multiplying on the left with another element of the group $e^{\delta_\xi}$, we get

$$\mathcal{G}(0,\xi,\bar{\xi})\mathcal{G}(t,\theta,\bar{\theta}) = e^{\delta_\xi}e^{-itP}e^{\delta_\theta} = e^{-itP}e^{\delta_{\xi+\theta}}e^{\frac{1}{2}[\delta_\xi,\delta_\theta]} = e^{-itP}e^{\delta_{\xi+\theta}}e^{-(\xi\bar{\theta}+\bar{\xi}\theta)P}$$
$$= \mathcal{G}(t-i(\xi\bar{\theta}+\bar{\xi}\theta),\theta+\xi,\bar{\theta}+\bar{\xi}). \tag{3.5}$$

Thus, group multiplication amounts to a motion in the space of parameters. This motion can be generated by the differential operators,

$$Q = \partial_\theta - i\bar{\theta}\partial_t, \qquad \bar{Q} = \partial_{\bar{\theta}} + i\theta\partial_t. \tag{3.6}$$

where we used the notation

$$\partial_t = \partial/\partial t, \qquad \partial_\theta = \partial/\partial\theta, \qquad \partial_{\bar{\theta}} = -\partial/\partial\bar{\theta}. \tag{3.7}$$

The differential operators in (3.6) satisfy the algebra except for a sign change in the time-translation operator, $\{Q,\bar{Q}\} = 2i\partial_t$ [25].

## 3.2 Flat superspace

Superspace provides the manifold structure associated to the supersymmetry group. A system of local coordinates in superspace is $z^M = (z^t, z^\theta, z^{\bar{\theta}}) = (t,\theta,\bar{\theta})$; $t$ is the ordinary time, whereas $\theta$, $\bar{\theta} \equiv \theta^*$ are anti-commuting Grassmann numbers. We shall adopt the following summation convention

$$V^M V_M \equiv V^t V_t + V^{\underline{\theta}} U_{\underline{\theta}} \equiv V^t V_t + V^\theta U_\theta - V^{\bar{\theta}} U_{\bar{\theta}}, \tag{3.8}$$

where, in the following, $\underline{\theta}$ denotes either $\theta$ or $\bar{\theta}$. Since $V^M V_M = V^M \delta_M{}^N V_N$, we also have $\delta_M{}^N = \delta_A{}^B = \mathrm{diag}(1,1,-1)$.

An arbitrary function on superspace can be expanded as follows

$$\Phi(z) = \phi(t) + \theta\eta(t) - \bar{\theta}\bar{\eta}(t) + \theta\bar{\theta}G(t). \tag{3.9}$$

The components of (3.9) amount to eight real functions of time. We are only interested in certain functions $F(z)$ called superfields. They are defined by the requirement that its components constitute a supersymmetry multiplet, that is, they provide a linear representation of algebra (3.1). With N=2, the minimal supermultiplets are of (real) dimension $2^2 = 4$. Therefore, superfields are defined by four real constraints on (3.9). For the real superfield one imposes the constraint

$$\Phi(z) = \Phi^*(z), \tag{3.10}$$

that is $\phi$ and $G$ are real, whereas $\bar{\eta} = \eta^*$.

Under a supersymmetry transformation

$$(\delta_\xi\Phi)(z) \equiv (\xi Q - \bar{\xi}\bar{Q})\Phi(z) = \delta_\xi\phi(t) + \theta\delta_\xi\eta(t) - \bar{\theta}\delta_\xi\bar{\eta}(t) + \theta\bar{\theta}\delta_\xi G(t), \tag{3.11}$$

with the representation of the generators given in (3.6). Thus, by equating the corresponding components in the $\theta$-expansion, we get the transformation law of the real multiplet

$$\delta_\xi \phi = \xi\eta - \bar{\xi}\bar{\eta}, \qquad \delta_\xi \eta = \bar{\xi}(G + i\dot{\phi}), \tag{3.12a}$$

$$\delta_\xi G = i(\xi\dot{\eta} + \bar{\xi}\dot{\bar{\eta}}), \qquad \delta_\xi \bar{\eta} = \xi(G - i\dot{\phi}). \tag{3.12b}$$

The components are related by

$$\eta(t) = Q\phi(t), \qquad \bar{\eta}(t) = \bar{Q}\phi(t), \qquad G(t) = \frac{1}{2}(Q\bar{\eta} - \bar{Q}\eta)(t) = \frac{1}{2}[Q,\bar{Q}]\phi(t). \tag{3.13}$$

The real multiplet (3.12) is denoted sometimes as (1,2,1) since it contains one real physical boson, two real fermions (or one complex fermion) and one real auxiliary boson. Other two possible multiplets are (2,2,0) and (0,2,2). The former contains one complex physical boson and no auxiliary field, whereas the latter contains one complex auxiliary boson. Thus, with the (0,2,2) multiplet, the on-shell number of bosonic and fermionic fields does not match, there are no physical bosons. This is only possible with linear multiplets in one dimension [50].

On the other hand, the inverse procedure consists of arranging a given multiplet, say $(\phi, \eta = Q\phi, \ldots)$, into a superfield. The way to proceed is to apply the group element $\mathcal{G}(0,\theta,\bar{\theta})$ to some component, for example,

$$\Phi(z) = e^{\delta_\theta}\phi(t) = \Big(1 + \delta_\theta + (1/2)\delta_\theta\delta_\theta\Big)\phi(t) = \Big(1 + \theta Q - \bar{\theta}\bar{Q} + \frac{1}{2}\theta\bar{\theta}[Q,\bar{Q}]\Big)\phi(t). \tag{3.14}$$

A superfield whose lowest component is $\eta(t)$, denoted $D\Phi$, is obtained in the same way,

$$(D\Phi)(z) \equiv e^{\delta_\theta}\eta(t) = e^{\delta_\theta}Qe^{-\delta_\theta}e^{\delta_\theta}\phi \equiv D\Phi(z). \tag{3.15}$$

This notation suggest the definition of differential operators $D, \bar{D}$ acting on the superfield such that $(D\Phi)(z) = D(\Phi(z))$. They are the supersymmetric covariant derivatives,

$$D \equiv e^{\delta_\theta}Qe^{-\delta_\theta} = \partial_\theta + i\bar{\theta}\partial_t, \qquad \bar{D} \equiv e^{\delta_\theta}\bar{Q}e^{-\delta_\theta} = \partial_{\bar{\theta}} - i\theta\partial_t. \tag{3.16}$$

Of course, $D_\tau = \partial_t$.

### Vielbein and torsion

In the superspace coordinate basis, the exterior derivative is given by $\boldsymbol{d} = dz^M\partial_M = dt\partial_t + d\theta\partial_\theta - d\bar{\theta}\partial_{\bar{\theta}}$. However, the partial derivative of a superfield, $\partial_M\Phi(z)$, is generally not a superfield, i.e., it does not transform as (3.11). Concerning supersymmetry, a more convenient basis for the exterior derivative is provided by the vielbein 1-forms $\boldsymbol{e}^A$,

$$\boldsymbol{d} = dz^M\delta_M^N\partial_N = dz^M e_M^A(z)e_A^N(z)\partial_N = \boldsymbol{e}^A(z)D_A \tag{3.17}$$

where $A = \tau, \Theta, \bar{\Theta}$, is a tangent space index. The operators $D_A = e_A^M\partial_M$ are the supersymmetric covariant derivatives (3.16). They are given by the inverse vielbein fields $e_A^M(z)$.

The flat vielbein and its inverse are, therefore,

$$e_M^{\ A}(z) = \begin{bmatrix} 1 & 0 & 0 \\ -i\bar{\theta} & 1 & 0 \\ i\theta & 0 & -1 \end{bmatrix}, \qquad e_A^{\ M}(z) = \begin{bmatrix} 1 & 0 & 0 \\ i\bar{\theta} & 1 & 0 \\ -i\theta & 0 & -1 \end{bmatrix}. \tag{3.18}$$

The vielbein 1-forms are $e^\tau = dt - id\theta\bar{\theta} - id\bar{\theta}\theta$, $e^\Theta = d\theta$, $e^{\bar{\Theta}} = d\bar{\theta}$. The torsion 2-forms (defined below) are zero except for $de^\tau = -2id\bar{\theta}d\theta$. Thus, we have a non-vanishing torsion component

$$T_{\Theta\bar{\Theta}}^{\ \ \tau}(z) = 2i = T_{\Theta\bar{\Theta}}^{\ \ \tau}(z). \tag{3.19}$$

## 3.3 'Curved' superspace

Now we will consider general coordinate transformations on superspace,

$$z'^M = z^M + \Xi^M(z) \tag{3.20}$$

where the parameters $\Xi^M(z)$ are superfield themselves.

A scalar superfield $F(z)$ will transform as $F'(z') = F(z)$, or

$$\delta_\Xi F(z) \equiv F'(z) - F(z) = -\Xi^M(z)\partial_M F(z) = -\Xi^A(z) D_A F(z) \tag{3.21}$$

If the superfield parameters with tangent space index, $\Xi^A = \Xi^M E_M^{\ A}$, are chosen to be field independent, then the commutator of two successive transformations (3.21) gives

$$\begin{aligned}[][\delta_X, \delta_\Xi]F(z) &= \delta_X(-\Xi^A D_A) - (\ldots) = \Xi^A X^B D_B D_A - X^A \Xi^B D_B D_A \\ &= \Xi^A X^B (D_B D_A - (-)^{ab} D_B D_A) = -\Xi^A X^B T_{BA}^{\ \ C} D_C F \end{aligned} \tag{3.22}$$

where we used the very important result[1]

$$D_B D_C - (-)^{bc} D_C D_B = -T_{BC}^{\ \ D}(z) D_D F(z), \tag{3.23}$$

In flat superspace, defined by (3.18) and (3.19), setting $\Xi^\Theta(z) = \xi(t), \Xi^{\bar{\Theta}}(z) = \bar{\xi}(t), X^\Theta = \chi(t)$, and $X^{\bar{\Theta}}(z) = \bar{\chi}(t)$, one recovers the global supersymmetry transformations (3.3),

$$[\delta_X, \delta_\Xi]F(z) = \left(\Xi^\Theta X^{\bar{\Theta}} T_{\bar{\Theta}\Theta}^{\ \ \tau} + \Xi^{\bar{\Theta}} X^\Theta T_{\Theta\bar{\Theta}}^{\ \ \tau}\right) D_\tau F(z) = -2i(\chi\bar{\xi} + \bar{\chi}\xi)\partial_t F(z). \tag{3.24}$$

#### Constraints on the torsion

Torsion 2-forms are defined as the covariant derivative of the vielbein 1-forms. Since we are not considering any connection (and, therefore, no curvature), they are given by

$$\begin{aligned} \boldsymbol{T}^A = \boldsymbol{d}\boldsymbol{E}^A &= dz^M dz^N \partial_N E_M^{\ A}(z) = \frac{1}{2} dz^M dz^N \left(\partial_N E_M^{\ A}(z) - (-)^{mn} \partial_M E_N^{\ A}(z)\right) \\ &= \frac{1}{2} dz^M dz^N T_{NM}^{\ \ A}(z) = \frac{1}{2} E^B E^C T_{CB}^{\ \ A}(z). \end{aligned} \tag{3.25}$$

[1] Let $F$ be any function (0-form), $0 = \boldsymbol{d}^2 F = \boldsymbol{d}(E^B D_B F) = (\boldsymbol{d}E^B) D_B F + E^B E^A D_A D_B F = \boldsymbol{T}^B D_B F + E^B E^A D_A D_B F = \frac{1}{2} E^B E^A \left(T_{AB}^{\ \ D} D_D F + D_A D_B F - (-)^{ba} D_B D_A F\right)$.

From its definition, $T_{AB}{}^{C} = -(-)^{ab} T_{BA}{}^{C}$. Then,

$$T_{\tau\tau}{}^{A} = 0, \qquad T_{\tau\underline{\Theta}}{}^{A} = -T_{\underline{\Theta}\tau}{}^{A}, \qquad T_{\Theta\bar{\Theta}}{}^{A} = T_{\bar{\Theta}\Theta}{}^{A}. \tag{3.26}$$

where $\underline{\Theta} = \Theta, \bar{\Theta}$.

The generalization to arbitrary superspace coordinate transformations brings about a lot of unnecessary torsion components. To reduce to the minimum number of fields, one is required to impose covariant constraints. Figuring out proper constraints is the price to pay for the enjoying the benefits of manifest supersymmetry provided by the superspace formalism. In this case, they are given by [25]

$$T_{\underline{\Theta}\underline{\Theta}}{}^{\underline{\Theta}} = 0 = T_{\Theta\Theta}{}^{\tau} = T_{\bar{\Theta}\bar{\Theta}}{}^{\tau}, \qquad T_{\Theta\bar{\Theta}}{}^{\tau} = T_{\bar{\Theta}\Theta}{}^{\tau} = 2i, \qquad T_{\underline{\Theta}\tau}{}^{\tau} = 0 = T_{\tau\underline{\Theta}}{}^{\tau}, \qquad T_{\tau\tau}{}^{\tau} = 0. \tag{3.27}$$

The remaining components are constrained to vanish by the Bianchi identities (we only have Bianchi identities of the first kind following from $\boldsymbol{d}^2\boldsymbol{E}^A = \boldsymbol{d}\boldsymbol{T}^A = 0$),

$$T_{\tau\underline{\Theta}}{}^{\underline{\Theta}} = 0 \tag{3.28}$$

**The Wess-Zumino (W-Z) gauge and the 1D supergravity multiplet**

Vielbein fields transform as

$$E'^{A}_{M}(z') = \frac{dz^N}{dz'^M} E_N^A(z). \tag{3.29}$$

Or, in the infinitesimal case,

$$\begin{aligned}\delta E_M^A &= -\Xi^L \partial_L E_M^A - (\partial_M \Xi^L) E_L{}^A = -\Xi^L(\partial_L E_M^A - (-)^{ml}\partial_M E_L{}^A) - \partial_M \Xi^A = -\Xi^L T_{LM}^A - \partial_M \Xi^A \\ &= -\Xi^B T_{BM}^A - \partial_M \Xi^A \end{aligned}\tag{3.30}$$

With suitably chosen superfield transformation parameters $\Xi^A$, the vielbein can be taken to a form with the minimum number of fields,

$$E_M^A(z)| = \begin{bmatrix} N(t) & \frac{1}{2}\Psi(t) & \frac{1}{2}\bar{\Psi}(t) \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{3.31}$$

where $|$ stands for evaluation at $\theta = 0 = \bar{\theta}$.

Time-dependent supersymmetry transformations are parameterized by

$$\Xi^\tau(z) = -2i\theta\bar{\xi}(t) - 2i\bar{\theta}\xi(t), \qquad \Xi^\Theta(z) = \xi(t). \tag{3.32}$$

The higher components of $\xi^\tau(z)$ were chosen such that W-Z gauge (3.31) is preserved under the transformation, that is $\delta_\xi E_{\underline{\theta}}{}^A| = 0$.

$N(t)$ and its supersymmetric partner $\Psi(t)$ constitute the 1-D supergravity multiplet. To find its transformation law under (3.32) we use (3.30),

$$\begin{aligned}\delta E_t{}^\tau &= -\Xi^B T_{Bt}{}^\tau - \partial_t \Xi^\tau = -(-)^{bc}\Xi^B E_t^C T_{BC}{}^\tau - \partial_t\Xi^\tau = -(-)^b \Xi^B E_t^\Theta T_{B\Theta}{}^\tau + (-)^b \Xi^B E_t^{\bar\Theta} T_{B\bar\Theta}{}^\tau - \partial_t \Xi^\tau \\ &= (-)^1 \Xi^{\bar\Theta} E_t^\Theta T_{\bar\Theta\Theta}{}^\tau + (-)^1 \Xi^\Theta E_t^{\bar\Theta} T_{\Theta\bar\Theta}{}^\tau - \partial_t \Xi^\tau, \qquad (3.33a)\\ \delta E_t{}^\Theta &= -\Xi^B T_{Bt}{}^\Theta - \partial_t \Xi^\Theta = -(-)^{bc}\Xi^B E_t^C T_{BC}{}^\Theta - \partial_t \Xi^\Theta. \qquad (3.33b)\end{aligned}$$

Finally, evaluating (3.33) at $\underline{\theta} = 0$, using (3.31) as well as $T_{\Theta\bar\Theta}{}^\tau = 2i$, we get

$$\delta_\xi N = -i(\xi\bar\Psi + \bar\xi\Psi), \qquad \delta_\xi \Psi = -2\dot\xi, \qquad \delta_\xi\bar\Psi = -2\dot{\bar\xi}. \qquad (3.34)$$

**Covariant multiplet**

In the curved superspace it is convenient to define the supermultiplet in a covariant (coordinate independent) way as follows,

$$\begin{aligned}\phi(t) &= F(z)|, &(3.35a)\\ \eta(t) &= D_\Theta F(z)|, &(3.35b)\\ \bar\eta(t) &= D_{\bar\Theta}\Phi(z)|, &(3.35c)\\ 2G(t) &= [D_\Theta, D_{\bar\Theta}]F(z)|. &(3.35d)\end{aligned}$$

We need to make sure that (3.35) is indeed a (super)multiplet. This can be done in a covariant way, that is, keeping the full dependence on the superspace coordinates $z$ and setting $\underline{\theta} = 0$ at the end of the computations (see Appendix B for the full derivation). The result is

$$\begin{aligned}\delta\phi &= -\xi\eta + \bar\xi\bar\eta, &(3.36a)\\ \delta\eta &= \bar\xi\Big(-G - i\dot\phi/N\Big) + i\bar\xi\Big(\Psi\eta - \bar\Psi\bar\eta\Big)/2N, &(3.36b)\\ \delta\bar\eta &= \xi\Big(-G + i\dot\phi/N\Big) - i\xi\Big(\Psi\eta - \bar\Psi\bar\eta\Big)/2N, &(3.36c)\\ \delta G &= -i\Big(\xi\dot\eta + \bar\xi\dot{\bar\eta}\Big)/N + iG\Big(\xi\bar\Psi + \bar\xi\Psi\Big)/2N - \dot\phi\Big(\xi\bar\Psi - \bar\xi\Psi\Big)/2N^2 \\ &\quad - \Psi\bar\Psi\Big(\xi\eta - \bar\xi\bar\eta\Big)/4N^2. &(3.36d)\end{aligned}$$

## 3.4 New superspace

This superspace is specially design for the covariant multiplet. A system of local coordinates is $Z^{\tilde M} = (Z^t, Z^\Theta, Z^{\bar\Theta}) = (t, \Theta, \bar\Theta)$. A new superfield $\Phi(Z)$ will contain (3.35) as follows

$$\Phi(Z) = \phi(t) + \Theta\eta(t) - \bar\Theta\bar\eta(t) + \Theta\bar\Theta G(t). \qquad (3.37)$$

This expression can be obtained in a covariant way, that is, keeping the old superspace coordinates $z$,

$$\Phi(z,\Theta) = e^{Z^{\underline{\Theta}}\mathcal{D}_{\underline{\Theta}}}F(z) = \Big(1 + Z^\Theta D_\Theta - Z^{\bar\Theta} D_{\bar\Theta} + \frac{1}{2} Z^\Theta Z^{\bar\Theta}[D_\Theta, D_{\bar\Theta}]\Big)F(z). \qquad (3.38)$$

The transformation of (3.38) under local supersymmetry is given by (3.21). Taking a series of steps, it can be rewritten as a coordinate transformation in the new superspace: $\delta_\Xi\Phi(z,\Theta) = -\Xi^A(z)D_A\Phi(z,\Theta) = -\Xi^A(z)\hat{\eta}_A^{\tilde{M}}(z,\Theta)\partial_{\tilde{M}}\Phi(z,\Theta)$, where $\hat{\eta}_{\tilde{N}}^{\tilde{M}}(z,\Theta)$ is an intermediate vielbein with the new index $\tilde{M} = t,\Theta,\bar{\Theta}$. At the end, the old $\theta$-coordinates are set to zero. Thus, one obtains the following,

$$\delta_\Gamma\Phi(Z) = -\Gamma^{\tilde{M}}(Z)\partial_{\tilde{M}}\Phi(Z) \tag{3.39}$$

where $\Gamma^{\tilde{M}}(Z)$ are the new superfield parameters. For the local supersymmetry transformation (3.39), the new superfield parameters are given by

$$\Xi^\tau(Z) = i\Big(\Theta\bar{\xi}(t) + \bar{\Theta}\xi(t)\Big)/N(t) + \Theta\bar{\Theta}\Big(\xi(t)\bar{\Psi}(t) - \bar{\xi}(t)\Psi(t)\Big)/2N^2(t), \tag{3.40a}$$

$$\Xi^\Theta(Z) = \xi(t) - i\Big(\Theta\bar{\xi}(t) + \bar{\Theta}\xi(t)\Big)\Psi(t)/2N(t) + \Theta\bar{\Theta}\Psi(t)\bar{\Psi}(t)\xi(t)/4N^2(t), \tag{3.40b}$$

$$\Xi^{\bar{\Theta}}(Z) = \bar{\xi}(t) - i\Big(\Theta\bar{\xi}(t) + \bar{\Theta}\xi(t)\Big)\bar{\Psi}(t)/2N(t) + \Theta\bar{\Theta}\Psi(t)\bar{\Psi}(t)\bar{\xi}(t)/4N^2(t). \tag{3.40c}$$

**New vielbein**

The new superfields require suitable covariant derivatives given by a new vielbein as

$$\nabla_A\Phi(Z) = \nabla_A^{\tilde{M}}(Z)\partial_{\tilde{M}}\Phi(Z). \tag{3.41}$$

One can determine the components of the inverse vielbein by using the basic definition analogous to (3.15),

$$\nabla_A\Phi(z,\Theta) = e^{Z^{\underline{\Theta}}D_{\underline{\Theta}}}D_AF(z) = (1 + Z^\Theta D_\Theta - Z^{\bar{\Theta}}D_{\bar{\Theta}} - Z^\Theta Z^{\bar{\Theta}}D_{[\bar{\Theta}}D_{\Theta]})D_AF(z) \tag{3.42}$$

The detailed derivation is offered in Appendix (B). We obtain the following inverse vielbein,

$$\nabla_A^{\tilde{M}} = \begin{bmatrix} \frac{1}{N} - i\frac{\Theta\bar{\Psi}+\bar{\Theta}\Psi}{2N^2} - \frac{\Theta\bar{\Theta}\Psi\bar{\Psi}}{2N^3} & -\frac{\Psi}{2N} - i\frac{\Theta\Psi\bar{\Psi}}{4N^2} & -\frac{\bar{\Psi}}{2N} + i\frac{\bar{\theta}\Psi\bar{\Psi}}{4N^2} \\ i\frac{\bar{\theta}}{N} - \frac{\Theta\bar{\Theta}\bar{\Psi}}{2N^2} & 1 - i\frac{\bar{\theta}\Psi}{2N} - \frac{\Theta\bar{\Theta}\Psi\bar{\Psi}}{4N^2} & -i\frac{\bar{\Theta}\bar{\Psi}}{2N} \\ -i\frac{\Theta}{N} - \frac{\Theta\bar{\Theta}\Psi}{2N^2} & i\frac{\Theta\Psi}{2N} & -1 + i\frac{\Theta\bar{\Psi}}{2N} + \frac{\Theta\bar{\Theta}\Psi\bar{\Psi}}{4N^2} \end{bmatrix}, \tag{3.43}$$

whereas the vielbein itself is

$$\nabla_{\tilde{M}}^A(Z) = \begin{bmatrix} N + i(\Theta\bar{\Psi} + \bar{\Theta}\Psi) & \frac{1}{2}\Psi & \frac{1}{2}\bar{\Psi} \\ -i\bar{\Theta} & 1 & 0 \\ i\Theta & 0 & -1 \end{bmatrix}. \tag{3.44}$$

Having (3.43) at hand, one can write down immediately the covariant derivatives. One gets to write simpler expressions with the following definition,

$$\psi(t) \equiv \Psi(t)/2N(t). \tag{3.45}$$

Thus, for example, for $\Phi = \phi + \Theta\eta - \bar{\Theta}\bar{\eta} + \Theta\bar{\Theta}G$,

$$\nabla_\Theta\Phi(z) = i\bar{\eta} + \bar{\Theta}(i\dot{\phi}/N + \psi\bar{\eta} + \bar{\psi}\eta + G) + \Theta\bar{\Theta}(\dot{\bar{\eta}}/N - \bar{\psi}\dot{\phi}/N - i\psi\bar{\psi}\bar{\eta} + i\bar{\psi}G), \tag{3.46a}$$

$$\nabla_{\bar{\Theta}}\Phi(z) = -i\eta + \Theta(-i\dot{\phi}/N - \psi\bar{\eta} - \bar{\psi}\eta + G) + \Theta\bar{\Theta}(\dot{\eta}/N - \psi\dot{\phi}/N + i\psi\bar{\psi}\eta - i\psi G). \tag{3.46b}$$

**Scalar density**

A scalar density is a superfield $\mathcal{E}(Z) = e + \Theta\rho - \bar{\Theta}\bar{\rho} + \Theta\bar{\Theta}M$ transforming under supersymmetry as follows

$$\delta_\xi \mathcal{E}(Z) = -\partial_{\tilde{M}}(\Gamma^{\tilde{M}}(-)^m \mathcal{E}(Z)). \tag{3.47}$$

Or, in component form, from (3.40) and (3.47),

$$\delta_\xi e = -(\xi\rho - \bar{\xi}\bar{\rho}) - \frac{i}{2}N^{-1}e(\bar{\xi}\Psi + \xi\bar{\Psi}), \tag{3.48a}$$

$$\delta_\xi \rho = -i\frac{d}{dt}\left(N^{-1}e\bar{\xi}\right) - \bar{\xi}M + \frac{i}{2}N^{-1}(\bar{\xi}\bar{\rho} + \xi\rho)\bar{\Psi} - \frac{1}{4}N^{-2}e\Psi\bar{\Psi}\bar{\xi}, \tag{3.48b}$$

$$\delta_\xi \bar{\rho} = i\frac{d}{dt}(N^{-1}\xi e) - \xi M + \frac{i}{2}N^{-1}(\bar{\xi}\bar{\rho} + \xi\rho)\Psi - \frac{1}{4}N^{-2}e\Psi\bar{\Psi}\xi, \tag{3.48c}$$

$$\delta_\xi M = -\frac{d}{dt}\Big(iN^{-1}(\bar{\xi}\bar{\rho} + \xi\rho) + \frac{1}{2}N^{-2}e(\xi\bar{\Psi} - \bar{\xi}\Psi)\Big). \tag{3.48d}$$

Note that the transformation of the highest component is a total time derivative (a spacetime derivative in more dimensions).

The product of a scalar superfield $\mathcal{L}$ and a density is also a density, and can be used to define invariant actions (up to surface terms) under time dependent supersymmetry transformations,

$$\delta \int dt d\Theta d\bar{\Theta}\mathcal{E}\mathcal{L} = -\int dt d\Theta d\bar{\Theta}\partial_{\tilde{M}}(\eta^{\tilde{M}}(-)^m \mathcal{E}\mathcal{L}), \tag{3.49}$$

For example, choosing $e = N$, then, consistency of (3.32) and (3.48), yields also $\rho = \frac{i}{2}\bar{\Psi}$ and $M = 0$. This scalar density can be obtained as the super-determinant of the vielbein:

$$\mathcal{E} \equiv \mathrm{Sdet}\nabla_{\tilde{M}}^{\ A} = N + \frac{i}{2}\Theta\bar{\Psi} + \frac{i}{2}\bar{\Theta}\Psi. \tag{3.50}$$

**Superfield equation of motion**

A supersymmetric action depending on a single superfield $\Phi$ is of the form,

$$S = \int dt L \equiv \int dt d\Theta d\bar{\Theta}\mathcal{L} = \int dt d\Theta d\bar{\Theta}\mathcal{E}\mathcal{J}. \tag{3.51}$$

The equation of motion for an even parity superfield reads

$$\nabla_\Theta \frac{\partial \mathcal{J}}{\partial \nabla_\Theta \Phi} + \nabla_{\bar{\Theta}} \frac{\partial \mathcal{J}}{\partial \nabla_{\bar{\Theta}} \Phi} - \frac{\partial \mathcal{J}}{\partial \Phi} = 0. \tag{3.52}$$

All integrals above are defined by the following rule

$$\int dt d\Theta d\bar{\Theta}\Big[\phi(t) + \Theta\eta(t) - \bar{\Theta}\bar{\eta}(t) + \Theta\bar{\Theta}G(t)\Big] = \int dt\ G(t). \tag{3.53}$$

## 3.5 N=1 supersymmetry

Here we collect the analogous results for N=1 supersymmetry. They can be obtain following the same steps as in the N=2 case, just described.

The N=1 superspace has local coordinates $z^M = (t,\Theta)$, where $\Theta$ is a real Grassmann parameter, $\Theta^* = \Theta$, $\Theta\Theta = 0$. Superfields have only two components (3.54), one real boson and one real fermion. Thus, a real superfield is of the form

$$\Phi(t,\Theta) = \phi(t) + i\Theta\eta(t). \tag{3.54}$$

Complex conjugation reverses order, so the product of two real odd parity Grassmann numbers is purely imaginary.

The N=1 (new) vielbein reads

$$\nabla_M^{\ A} = \begin{bmatrix} N - i\Theta\Psi & \frac{1}{2}\Psi \\ i\Theta & 1 \end{bmatrix}, \tag{3.55}$$

Covariant derivatives are defined with the inverse vielbein fields $\nabla_A^{\ M}$ as $\nabla_A\Phi = \nabla_A^{\ M}\partial_M\Phi$. They are given by

$$\nabla_\tau\Phi = N^{-1}\dot{\phi} - i\psi\eta + i\Theta N^{-1}(\dot{\eta} + \dot{\phi}\psi), \tag{3.56a}$$

$$\nabla_\Theta\Phi = i\eta - \Theta(iN^{-1}\dot{\phi} + \psi\eta). \tag{3.56b}$$

where, as in the N=2 case, we define $\psi \equiv \Psi/2N$.

Under a local supersymmetry transformation parameterized by $\zeta(t)$, the one-dimensional supergravity multiplet transforms as

$$\delta_\zeta N = -i\zeta\Psi, \qquad \delta_\zeta\Psi = -2\dot{\zeta}, \tag{3.57}$$

whereas, for the real multiplet contained in (3.54), we have,

$$\delta_\zeta\phi = -i\zeta\eta, \qquad \delta_\zeta\eta = \zeta(N^{-1}\dot{\phi} - i\psi\eta). \tag{3.58}$$

Supersymmetric actions are of the form $S = \int dtL \equiv \int dtd\Theta\mathcal{L} = \int dtd\Theta\mathcal{E}\mathcal{J}$ where $\mathcal{E}$ is the scalar density given the super-determinant of (3.55),

$$\mathcal{E} = N(1 - i\Theta\psi). \tag{3.59}$$

Finally, formal superfield equations of motion can be readily derived. For an odd parity superfield $\Gamma$ (used in Section 7.1), we have

$$\nabla_\theta\frac{\partial\mathcal{J}}{\partial\nabla_\theta\Gamma} + \frac{\partial\mathcal{J}}{\partial\Gamma} = 0 \tag{3.60}$$

Compared to the case of even parity, there is an extra minus sign in the first term of (3.60) which arises from $\nabla_\theta(\delta\Gamma...) = -\delta\Gamma\nabla_\theta....$

# Chapter 4

# Supersymmetric cosmology

In this chief chapter we use the formalism just develop to write a supersymmetric extension of the FRW action with linear curvature only and consider non-minimally coupled scalar matter.

## 4.1 Supersymmetric FRW action

The scale factor is embedded in a real superfield

$$\mathcal{A}(t,\Theta,\bar{\Theta}) = a(t) + i\Theta\bar{\lambda}(t) + i\bar{\Theta}\lambda(t) + \Theta\bar{\Theta}B(t). \tag{4.1}$$

(This parameterization is pure convention, we can dispense with the imaginary unit by redefining the $\lambda$'s, e.g, $i(\Theta\bar{\lambda}+\bar{\Theta}\lambda) \to \Theta\lambda - \bar{\Theta}\bar{\lambda}$.)

It will be convenient to express the superfield action in terms of the FRW-curvature superfield, defined by the following,

$$\mathcal{R}[\mathcal{A}] = \frac{1}{2}\mathcal{A}^{-1}(\nabla_{\bar{\Theta}}\nabla_{\Theta} - \nabla_{\bar{\Theta}}\nabla_{\Theta})\mathcal{A} + \mathcal{A}^{-2}\nabla_{\bar{\Theta}}\mathcal{A}\nabla_{\Theta}\mathcal{A} + \sqrt{k}\mathcal{A}^{-1}. \tag{4.2}$$

Substituting (4.1) and expanding, using the covariant derivatives (3.46), we get the expansion in $\Theta$-variables

$$\begin{aligned}\mathcal{R}(t,\Theta,\bar{\Theta}) = {} & \frac{\sqrt{k}}{a} - \frac{B}{a} + \frac{\lambda\bar{\lambda}}{a^2} + \Theta\Big(\frac{\dot{\bar{\lambda}}}{Na} - \frac{\dot{a}\bar{\psi}}{Na} - i\frac{\bar{\lambda}}{a}\psi\bar{\psi} + i\frac{B}{a}\bar{\psi} + \frac{\dot{a}\bar{\lambda}}{Na^2} + 2i\frac{B\bar{\lambda}}{a^2} - i\frac{\lambda\bar{\lambda}}{a^2}\bar{\psi} - i\frac{\sqrt{k}\bar{\lambda}}{a^2}\Big) \\ & -\bar{\Theta}\Big(\frac{\dot{\lambda}}{Na} - \frac{\dot{a}\psi}{Na} + i\frac{\lambda}{a}\psi\bar{\psi} - i\frac{B}{a}\psi + \frac{\dot{a}\lambda}{Na^2} - 2i\frac{B\lambda}{a^2} + i\frac{\lambda\bar{\lambda}}{a^2}\psi + i\frac{\sqrt{k}\lambda}{a^2}\Big) + \Theta\bar{\Theta}\Big(\frac{\ddot{a}}{N^2a} - \frac{\dot{N}\dot{a}}{N^3a} \\ & + \frac{\dot{a}^2}{N^2a^2} + 2\frac{B^2}{a^2} - \sqrt{k}\frac{B}{a^2} - 2i\frac{\psi\dot{\bar{\lambda}} + \bar{\psi}\dot{\lambda}}{Na} - i\frac{\dot{\psi}\bar{\lambda} + \dot{\bar{\psi}}\lambda}{Na} - 6\frac{\lambda\bar{\lambda}}{a^2}\psi\bar{\psi} - 2i\frac{\lambda\dot{\bar{\lambda}} + \bar{\lambda}\dot{\lambda}}{Na^2} + 4B\frac{\lambda\bar{\lambda}}{a^3} \\ & -4i\dot{a}\frac{\psi\bar{\lambda} + \bar{\psi}\lambda}{Na^2} + 2B\frac{\psi\bar{\lambda} - \bar{\psi}\lambda}{a^2} + 2\frac{B}{a}\psi\bar{\psi} - 2\sqrt{k}\frac{\lambda\bar{\lambda}}{a^3}\Big)\end{aligned} \tag{4.3}$$

The superfield Lagrangian for the supersymmetric (vacuum) FRW model is

$$\mathcal{L} = j\mathcal{E}\mathcal{A}^3\mathcal{R}. \tag{4.4}$$

where $j = 3/\kappa^2 = 3/8\pi G$.

Performing the $\Theta$-integration, using the rule (3.53), and integrating out the auxiliary field $B$ (its equation of motion yields $B = \sqrt{k} + \lambda^2/2a$), we get

$$L = jNa^3\Big(\frac{\ddot{a}}{N^2 a} - \frac{\dot{N}\dot{a}}{N^3 a} + \frac{\dot{a}^2}{N^2 a^2} + \frac{k}{a^2} - i\frac{\dot{\psi}\bar{\lambda} + \dot{\bar{\psi}}\lambda}{Na} - i\frac{\psi\dot{\bar{\lambda}} + \bar{\psi}\dot{\lambda}}{Na} + i\frac{\lambda\dot{\bar{\lambda}} + \bar{\lambda}\dot{\lambda}}{Na^2} + 2\frac{\lambda\bar{\lambda}}{a^2}\psi\bar{\psi} - \sqrt{k}\frac{\lambda\bar{\lambda}}{a^3} - 2\sqrt{k}\frac{\psi\bar{\lambda} - \bar{\psi}\lambda}{a^2}\Big). \tag{4.5}$$

Off-shell we had two real boson $a, B$ and one complex fermion $\lambda$. With $B$ the numbers of bosonic and fermionic variables match and the supersymmetry algebra closes off-shell [50, 25]. On-shell, we are left with one real boson and one complex fermion; the equation of motion for $a$ is of second-order in derivatives while those of $\lambda, \bar{\lambda}$ are first-order.

The classical dynamics can be studied, for example, by promoting dynamical variables to elements of a Grassmann algebra, that is, writing them in a superfield fashion; the simplest choice is that with two real anti-commuting elements $\epsilon_1$, $\epsilon_2$, besides the identity. Then, $a = a_0(t) + \epsilon_1\epsilon_2 a_{21}(t)$, $\lambda = \epsilon_1 l_1(t) + i\epsilon_2 l_2(t)$, etc. The equations or motion are solved order by order in the $\epsilon$−expansion [47]. Since the theory in question involves fermions, we prefer to pass directly to its quantization, promoting the classical Grassmann algebra to a Clifford algebra of quantum operators. Also, starting at the quantum level, semi-classical equations of motion can be derived through a WKB approach, yielding supersymmetric Einstein-Klein-Gordon equations, with extra terms that dominate at very early times [54].

## 4.2 Conformally invariant scalar field

This example is the supersymmetric extension of the model reviewed in Section 2.1.2. The straightforward generalization to superfields of the bosonic action is

$$\mathcal{L} = \mathcal{E}\mathcal{A}^3\Big[j\mathcal{R} + \frac{1}{2}\nabla_\theta\Phi\nabla_{\bar{\theta}}\Phi - \frac{1}{2}\mathcal{R}\Phi^2\Big]. \tag{4.6}$$

where $\Phi = \phi + i\Theta\bar{\eta} + i\bar{\Theta}\eta + \Theta\bar{\Theta}G$.

The component Lagrangian, after eliminating the auxiliary fields $B$ of $\mathcal{A}$ and $G$ of $\Phi$, and integrating by parts, is

$$\begin{aligned}L = \frac{3N}{\kappa^2}\Big(&-\frac{a\dot{a}^2}{N^2} + ka + 2ia\dot{a}\frac{\psi\bar{\lambda} + \bar{\psi}\lambda}{N} + ia\frac{\lambda\dot{\bar{\lambda}} + \bar{\lambda}\dot{\lambda}}{N} + 2a\psi\bar{\psi}\lambda\bar{\lambda} - 2\sqrt{k}a(\psi\bar{\lambda} - \bar{\psi}\lambda)\\ &-\sqrt{k}\lambda\bar{\lambda}\Big) + \frac{N}{2}\Big(a^3\frac{\dot{\phi}^2}{N^2} + \frac{a\dot{a}^2}{N^2}\phi^2 + 2a^2\frac{\dot{a}\dot{\phi}}{N^2}\phi - ka\phi^2 - ia^3\frac{\eta\dot{\bar{\eta}} + \bar{\eta}\dot{\eta}}{N} - 2a^3\psi\bar{\psi}\eta\bar{\eta}\\ &+2\sqrt{k}a\phi^2(\psi\bar{\lambda} - \bar{\psi}\lambda) + 2ia^2\phi\frac{\dot{\lambda}\bar{\eta} + \dot{\bar{\lambda}}\eta}{N} + 2ia\dot{a}\phi\frac{\lambda\bar{\eta} + \bar{\lambda}\eta}{N} - 2ia^2\dot{a}\phi\frac{\psi\bar{\eta} + \bar{\psi}\eta}{N}\\ &+2\sqrt{k}a^2\phi(\psi\bar{\eta} - \bar{\psi}\eta) + 3ia^2\dot{\phi}\frac{\lambda\bar{\eta} + \bar{\lambda}\eta}{N} - ia\phi^2\frac{\lambda\dot{\bar{\lambda}} + \bar{\lambda}\dot{\lambda}}{N} + 2\sqrt{k}a\phi(\lambda\bar{\eta} - \bar{\lambda}\eta)\end{aligned}$$

$$-2ia\dot{a}\phi^2\frac{\psi\bar{\lambda}+\bar{\psi}\lambda}{N}+3\sqrt{k}a^2\eta\bar{\eta}-2ia^2\phi\dot{\phi}\frac{\psi\bar{\lambda}+\bar{\psi}\lambda}{N}-2a^2\phi\psi\bar{\psi}(\lambda\bar{\eta}-\bar{\lambda}\eta)$$
$$-2ia^3\dot{\phi}\frac{\psi\bar{\eta}+\bar{\psi}\eta}{N}-a\lambda\bar{\lambda}\eta\bar{\eta}-2a\phi^2\psi\bar{\psi}\lambda\bar{\lambda}+\sqrt{k}\phi^2\lambda\bar{\lambda}\Big). \tag{4.7}$$

The superfield analogue of the FRW conformal transformation and field redefinition (2.20) is

$$\mathcal{A}\to\Omega\mathcal{A}, \qquad \Phi\to\Omega^{-1}\Phi \tag{4.8}$$

where $\Omega=\omega+i\Theta\bar{\rho}+...$, and $\Omega^{-1}=\omega^{-1}-i\Theta\omega^{-2}\bar{\rho}+....$ Therefore, the transformation of the matter superfield amounts to

$$\phi\to\omega^{-1}\phi, \qquad \eta\to\omega^{-1}\eta-\omega^{-2}\phi\rho. \tag{4.9}$$

Thus, we re-rewrite the Lagrangian (4.7) in terms of the invariant matter multiplet

$$\varphi=a\phi, \qquad \chi=a\eta+\lambda\phi. \tag{4.10}$$

Additionally, we make a re-scaling $\lambda\to a^{\frac{1}{2}}\lambda$, $\chi\to a^{\frac{1}{2}}\chi$, so as to get standard fermionic kinetic terms. With all these manipulations, we get the following Lagrangian

$$L=\frac{3N}{\kappa^2}\Big(-\frac{a\dot{a}^2}{N^2}+ka+2i\sqrt{a}\dot{a}\frac{\psi\bar{\lambda}+\bar{\psi}\lambda}{N}+i\frac{\lambda\dot{\bar{\lambda}}+\bar{\lambda}\dot{\lambda}}{N}+2\psi\bar{\psi}\lambda\bar{\lambda}-\sqrt{k}\frac{\lambda\bar{\lambda}}{a}$$
$$-2\sqrt{k}\sqrt{a}(\psi\bar{\lambda}-\bar{\psi}\lambda)\Big)+\frac{N}{2}\Big(\frac{a\dot{\varphi}^2}{N^2}-k\frac{\varphi^2}{a}-i\frac{\chi\dot{\bar{\chi}}+\bar{\chi}\dot{\chi}}{N}-\sqrt{k}\varphi\frac{\lambda\bar{\chi}-\bar{\lambda}\chi}{a^2}$$
$$+2\sqrt{k}\varphi\frac{\psi\bar{\chi}-\bar{\psi}\chi}{\sqrt{a}}-i\dot{a}\varphi\frac{\lambda\bar{\chi}+\bar{\lambda}\chi}{Na^2}+i\varphi\frac{\dot{\lambda}\bar{\chi}+\dot{\bar{\lambda}}\chi}{Na}-2i\sqrt{a}\dot{\varphi}\frac{\psi\bar{\chi}+\bar{\psi}\chi}{N}$$
$$+2i\dot{\varphi}\frac{\lambda\bar{\chi}+\bar{\lambda}\chi}{Na}+3\sqrt{k}\frac{\chi\bar{\chi}}{a}+i\varphi\frac{\lambda\dot{\bar{\chi}}+\bar{\lambda}\dot{\chi}}{Na}-2\chi\bar{\chi}\psi\bar{\psi}+\frac{\lambda\bar{\lambda}}{a^3}\chi\bar{\chi}\Big). \tag{4.11}$$

### 4.2.1 Hamiltonian formulation

Conjugate momenta are defined as usual, but for fermionic variables, we use the following notation

$$\pi_\lambda\equiv\frac{\partial L}{\partial\dot{\bar{\lambda}}}, \qquad \pi_{\bar{\lambda}}\equiv-\frac{\partial L}{\partial\dot{\lambda}}, \tag{4.12}$$
$$\pi_\eta\equiv\frac{\partial L}{\partial\dot{\bar{\eta}}}, \qquad \pi_{\bar{\eta}}\equiv-\frac{\partial L}{\partial\dot{\eta}}. \tag{4.13}$$

With this we keep our expressions congruent with the superfield parameterization (4.1) and $\pi^*_\lambda=\pi_{\bar{\lambda}}$, etc.

The Hamiltonian is given by the Legendre transformation $H=\dot{a}p_a+\dot{\varphi}p_\varphi-\dot{\lambda}\pi_{\bar{\lambda}}+\dot{\bar{\lambda}}\pi_\lambda-\dot{\eta}\pi_{\bar{\eta}}+\dot{\bar{\eta}}\pi_\eta-L$. The resulting Hamiltonian is of the form

$$H=NH_0+\frac{1}{2}\Psi\bar{S}-\frac{1}{2}\bar{\Psi}S, \tag{4.14}$$

where

$$0 \approx H_0 = -\frac{1}{4j}\frac{p_a^2}{a} + \frac{p_\varphi^2}{2a} - jka + k\frac{\varphi^2}{2a} - \frac{i}{4j}\varphi p_a \frac{\lambda\bar{\chi} + \bar{\lambda}\chi}{a^3} + \frac{1}{8j}\frac{\varphi^2}{a^5}\lambda\bar{\lambda}\chi\bar{\chi}$$
$$- ip_\varphi \frac{\lambda\bar{\chi} + \bar{\lambda}\chi}{a^2} + j\frac{\sqrt{k}}{a}\lambda\bar{\lambda} - \frac{3}{2}\frac{\lambda\bar{\lambda}}{a^3}\chi\bar{\chi} + \frac{\sqrt{k}}{2a}\varphi\frac{\lambda\bar{\chi} - \bar{\lambda}\chi}{a} - \frac{3}{2}\frac{\sqrt{k}}{a}\chi\bar{\chi}, \tag{4.15}$$

$$0 \approx \bar{S} = \frac{ip_a}{\sqrt{a}}\bar{\lambda} + \frac{ip_\varphi}{\sqrt{a}}\bar{\chi} + \frac{\varphi\lambda\bar{\lambda}\bar{\chi}}{2a^2\sqrt{a}} + 2j\sqrt{k}\sqrt{a}\bar{\lambda} + \frac{\bar{\lambda}\chi\bar{\chi}}{a\sqrt{a}} - \bar{\chi}\varphi\frac{\sqrt{k}}{\sqrt{a}}. \tag{4.16}$$

as well as the complex conjugate $S$, are secondary constraints arising from the conservation of $p_N \approx 0$, $p_\psi \approx 0$, $p_{\bar{\psi}} \approx 0$. They all are first-class.

We also have (primary) second class constraints,

$$0 \approx C_\lambda \equiv \pi_\lambda + ji\lambda - \frac{i\varphi}{2a}\chi, \qquad 0 \approx C_{\bar{\lambda}} \equiv \pi_{\bar{\lambda}} - ji\bar{\lambda} + \frac{i\varphi}{2a}\bar{\chi}, \tag{4.17a}$$

$$0 \approx C_\chi \equiv \pi_\chi - \frac{i}{2}\chi + \frac{i\varphi}{2a}\lambda, \qquad 0 \approx C_{\bar{\chi}} \equiv \pi_{\bar{\chi}} + \frac{i}{2}\bar{\chi} - \frac{i\varphi}{2a}\bar{\lambda}. \tag{4.17b}$$

Basic Poisson brackets, defined in (C.2), are

$$\{a, p_a\} = 1, \qquad \{\varphi, p_\varphi\} = 1, \tag{4.18}$$

$$\{\lambda, \pi_{\bar{\lambda}}\} = 1 \qquad \{\bar{\lambda}, \pi_\lambda\} = -1, \tag{4.19}$$

$$\{\chi, \pi_{\bar{\chi}}\} = 1 \qquad \{\bar{\chi}, \pi_\chi\} = -1, \tag{4.20}$$

whereas the matrix of Poisson brackets between the second-class constraints is

$$C = \begin{Bmatrix} \{C_\lambda, C_\lambda\} & \{C_\lambda, C_{\bar{\lambda}}\} & \{C_\lambda, C_\chi\} & \{C_\lambda, C_{\bar{\chi}}\} \\ \{C_{\bar{\lambda}}, C_\lambda\} & \{C_{\bar{\lambda}}, C_{\bar{\lambda}}\} & \{C_{\bar{\lambda}}, C_\chi\} & \{C_{\bar{\lambda}}, C_{\bar{\chi}}\} \\ \{C_\chi, C_\lambda\} & \{C_\chi, C_{\bar{\lambda}}\} & \{C_\chi, C_\chi\} & \{C_\chi, C_{\bar{\chi}}\} \\ \{C_{\bar{\chi}}, C_\lambda\} & \{C_{\bar{\chi}}, C_{\bar{\lambda}}\} & \{C_{\bar{\chi}}, C_\chi\} & \{C_{\bar{\chi}}, C_{\bar{\chi}}\} \end{Bmatrix} = \begin{Bmatrix} 0 & 2ij & 0 & 0 \\ 2ij & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{Bmatrix} \tag{4.21}$$

Computing the basic Dirac brackets, we get

$$\{a, p_a\}_D = 1, \qquad \{\varphi, p_\varphi\}_D = 1, \tag{4.22}$$

$$\{\lambda, \bar{\lambda}\}_D = \frac{-i}{2j}, \qquad \{\chi, \bar{\chi}\}_D = i, \tag{4.23}$$

and also,

$$\{p_a, \lambda\}_D = \frac{1}{4j}\frac{\varphi}{a^2}\chi, \qquad \{p_a, \chi\}_D = \frac{\varphi}{2a^2}\lambda, \tag{4.24}$$

$$\{p_\varphi, \lambda\}_D = -\frac{1}{4j}\frac{\chi}{a}, \qquad \{p_\varphi, \chi\}_D = -\frac{\lambda}{2a}, \tag{4.25}$$

together with their complex conjugate relations. Therefore, we define new bosonic momenta

$$P_a = p_a + \frac{i}{2}\frac{\varphi}{a^2}(\lambda\bar{\chi} + \bar{\lambda}\chi), \qquad P_\varphi = p_\varphi - \frac{i}{2a}(\lambda\bar{\chi} + \bar{\lambda}\chi), \tag{4.26}$$

such that the only non-vanishing Dirac brackets are

$$\{a, P_a\}_D = 1, \qquad \{\varphi, P_\varphi\}_D = 1, \tag{4.27}$$

$$\{\lambda, \bar{\lambda}\}_D = \frac{-i}{2j}, \qquad \{\chi, \bar{\chi}\}_D = i. \tag{4.28}$$

Re-writing the supersymmetric constraint (4.16) in terms of (4.26), yields

$$\bar{S} = \left(ia^{-\frac{1}{2}}P_a + 2j\sqrt{k}a^{\frac{1}{2}}\right)\bar{\lambda} + \left(ia^{-\frac{1}{2}}P_\varphi - \varphi\sqrt{k}a^{-\frac{1}{2}}\right)\bar{\eta} + \frac{1}{2}a^{-\frac{3}{2}}\bar{\lambda}\eta\bar{\eta}. \tag{4.29}$$

### 4.2.2 Quantization

For simplicity, we set $\hbar = 1$ and $j = 3$ ($\kappa^2 = 1$), the basic (anti)commutation relations are

$$[a, P_a] = i, \qquad [\varphi, P_\varphi] = i, \tag{4.30}$$

$$[\lambda, \bar{\lambda}]_+ = \frac{1}{6}, \qquad [\chi, \bar{\chi}]_+ = -1. \tag{4.31}$$

To handle the fermionic sector a ground state is defined to be annihilated by operators $\lambda$ and $\chi$. Three more independent states can then be created by acting on the vacuum with $\bar{\lambda}, \bar{\chi}$, namely

$$|1\rangle = \sqrt{6}\bar{\lambda}|0\rangle, \qquad |2\rangle = \bar{\chi}|0\rangle, \qquad |3\rangle = \sqrt{6}\bar{\lambda}\bar{\chi}|0\rangle. \tag{4.32}$$

Their norms are related by $\langle 0|0\rangle = \langle 1|1\rangle - \langle 2|2\rangle = -\langle 3|3\rangle$. Thus, only half the states will be physical. The nonphysical ones can be discarded setting their wave functions to zero.

Acting on the states with the fermionic operators yields,

$$\lambda|0\rangle = 0, \qquad \bar{\lambda}|0\rangle = |1\rangle/\sqrt{6}, \qquad \chi|0\rangle = 0, \qquad \bar{\chi}|0\rangle = |2\rangle, \tag{4.33a}$$

$$\lambda|1\rangle = |0\rangle/\sqrt{6}, \qquad \bar{\lambda}|1\rangle = 0, \qquad \chi|1\rangle = 0, \qquad \bar{\chi}|1\rangle = -|3\rangle, \tag{4.33b}$$

$$\lambda|2\rangle = 0, \qquad \bar{\lambda}|2\rangle = |3\rangle/\sqrt{6}, \qquad \chi|2\rangle = -|0\rangle, \qquad \bar{\chi}|2\rangle = 0, \tag{4.33c}$$

$$\lambda|3\rangle = |2\rangle/\sqrt{6}, \qquad \bar{\lambda}|3\rangle = 0, \qquad \chi|3\rangle = |1\rangle, \qquad \bar{\chi}|3\rangle = 0. \tag{4.33d}$$

as well as,

$$[\chi, \bar{\chi}]|\psi\rangle = -\psi_0|0\rangle - \psi_1|1\rangle + \psi_2|2\rangle + \psi_3|3\rangle, \tag{4.34a}$$

$$\lambda[\chi, \bar{\chi}]|\psi\rangle = -\psi_1|0\rangle/\sqrt{6} + \psi_3|2\rangle/\sqrt{6}, \tag{4.34b}$$

$$\bar{\lambda}[\chi, \bar{\chi}]|\psi\rangle = -\psi_0|1\rangle/\sqrt{6} + \psi_2|3\rangle/\sqrt{6}. \tag{4.34c}$$

A generic state is of the form

$$|\psi(a,\varphi)\rangle = \psi_0(a,\phi)|0\rangle + \psi_1(a,\phi)|1\rangle + \psi_2(a,\phi)|2\rangle + \psi_3(a,\phi)|3\rangle, \tag{4.35}$$

but physical states are required to satisfy the Wheeler-DeWitt equation,

$$H_0|\psi\rangle = 0, \tag{4.36}$$

as well as the supersymmetric constraints,

$$S|\psi\rangle = 0, \qquad \bar{S}|\psi\rangle = 0. \tag{4.37}$$

When dealing with the purely bosonic model, we fixed the ordering ambiguity in the WDW equation by choose the Laplace-Beltrami operator. In this case, do not have such a rule at hand, since we are considering first order equations[2]. Instead, we use Weyl normal ordering, which assigns the anti-commutator to a product of bosonic operators, and commutator to a product of fermionic operators. Other ordering choices might be available, but they do not lead to fundamentally different results; the exponential components of the wavefunction, in the sense of Chapter 2, are the left without change, the difference arises in the quantum pre-factor [7].

For comparison with the result of Section 2.1.2, in the following we choose closed spatial geometry and set $k = 1$. Choosing the usual representation for the bosonic momenta, $P_a = -i\partial_a$, $P_\varphi = -i\partial_\varphi$, the supersymmetric constraint operators read

$$-S = \lambda\left(\frac{1}{2}(a^{-\frac{1}{2}}\partial_a + \partial_a a^{-\frac{1}{2}}) - 2j\sqrt{a}\right) + \chi\left(a^{-\frac{1}{2}}\partial_\varphi + \varphi a^{-\frac{1}{2}}\right) - \frac{1}{4}\lambda[\chi,\bar{\chi}]a^{-\frac{3}{2}}, \tag{4.38}$$

$$\bar{S} = \bar{\lambda}\left(\frac{1}{2}(a^{-\frac{1}{2}}\partial_a + \partial_a a^{-\frac{1}{2}}) + 2j\sqrt{a}\right) + \bar{\chi}\left(a^{-\frac{1}{2}}\partial_\varphi - \varphi a^{-\frac{1}{2}}\right) + \frac{1}{4}\bar{\lambda}[\chi,\bar{\chi}]a^{-\frac{3}{2}} \tag{4.39}$$

Applying them to the arbitrary state vector (4.35), using (4.33-4.34), yields

$$0 = \left(\partial_a - \frac{1}{4a} - 2ja\right)\frac{\psi_1|0\rangle + \psi_3|2\rangle}{\sqrt{6}} + \left(\partial_\varphi + \varphi\right)\left(-\psi_2|0\rangle + \psi_3|1\rangle\right) - \frac{1}{4a}\frac{-\psi_1|0\rangle + \psi_3|2\rangle}{\sqrt{6}}, \tag{4.40a}$$

$$0 = \left(\partial_a - \frac{1}{4a} + 2ja\right)\frac{\psi_0|1\rangle + \psi_2|3\rangle}{\sqrt{6}} + \left(\partial_\varphi - \varphi\right)\left(\psi_0|2\rangle - \psi_1|3\rangle\right) + \frac{1}{4a}\frac{-\psi_0|1\rangle + \psi_2|3\rangle}{\sqrt{6}}. \tag{4.40b}$$

Using the orthogonality of states $|i\rangle$ we get partial differential equations for the coefficient functions $\psi_i(a,\phi)$. The lowest and highest components are determined uniquely by the two first order PDEs following from (4.40),

$$\psi_0 = c_0\sqrt{a}e^{-3a^2+\frac{1}{2}\varphi^2}, \tag{4.41a}$$

$$\psi_3 = c_3\sqrt{a}e^{3a^2-\frac{1}{2}\varphi^2}. \tag{4.41b}$$

Thus, we obtain in the gravitational part, the no-boundary (Hartle-Hawking) and vacuum (wormhole) wave functions described in Section 2.2, respectively. They are exact and unique solutions to wavefunction coefficients of the empty (vacuum) and completely filled states, $|0\rangle$, $\bar{\lambda}\bar{\chi}|0\rangle$, which also happen to be bosonic states.

On the other hand, the intermediate or fermionic states satisfy a system of coupled equations. Re-scaling $a \to \sqrt{6}a$, they read

$$0 = (\partial_a - a)\psi_1 - \left(\partial_\varphi + \varphi\right)\psi_2, \tag{4.42}$$

$$0 = (\partial_a + a)\psi_2 - \left(\partial_\varphi - \varphi\right)\psi_1 \tag{4.43}$$

[2]We leave the investigation of an analogous rule, suitable for the fermionic and supersymmetric case, for a future work

Adding and subtracting the equations, we get

$$0 = \left(\partial_a - \partial_\varphi\right)(\psi_1 + \psi_2) + (-a + \varphi)(\psi_1 - \psi_2), \tag{4.44}$$

$$0 = \left(\partial_a + \partial_\varphi\right)(\psi_1 - \psi_2) + (-a - \varphi)(\psi_1 + \psi_2), \tag{4.45}$$

Assuming $a \neq \pm\varphi$, we can solve for $\psi_1 + \psi_2$ ($\psi_1 - \psi_2$) and substitute in the remaining equation to get

$$\left[\partial_a^2 - \partial_\varphi^2 - a^2 + \varphi^2\right]\psi_\pm(a, \varphi) = 0 \tag{4.46}$$

Since $\psi_1 = \frac{1}{2}(\psi_+ + \psi_-)$ and $\psi_2 = \frac{1}{2}(\psi_+ - \psi_-)$, we can choose, for example, $\psi_+ = \psi = \psi_-$, so that $\psi_1 = \psi$ and $\psi_2 = 0$, to eliminate the negative norm state, recalling $\langle 1|1\rangle = -\langle 2|2\rangle$. Therefore, we find in the intermediate states, a more complex dynamics that is on par with that of the purely bosonic model.

For other models with a minimally coupled scalar field with arbitrary potential see [53]. Also, for an application of the exact solutions to the problem of time see [55].

# Chapter 5

# Higher-derivatives in cosmology

Cosmic inflation refers to an epoch of accelerated growth of the spatial (physical) volume of the universe. Considering only a small spatial region, its volume is given by $dv = \sqrt{h}d^3x = a^3(t)v_0$, with $a(t)$ a locally defined scale factor and $v_0$ a co-moving volume. The conditions for inflation are therefore[3]: $\dot{a} > 0$ and $\ddot{a} > 0$. During inflation, the co-moving radius of the Hubble sphere (roughly the 2-sphere of points receding from the center at the speed of light) shrinks, $(d/dt)(aH)^{-1} < 0$ (which is equivalent $\ddot{a} > 0$). Expanding this expression, one obtains the usual condition for inflation

$$\epsilon(t) \equiv -\frac{\dot{H}(t)}{H^2(t)} < 1, \tag{5.1}$$

Inflation is quantified by the number $n$ of e-folds. It can be defined such that, starting with a given value $a_0$ of the scale factor, at time $t$ it will be given by $a(t) = e^{n(t)}a_0$. It also can be expressed as follows

$$n(t_f) = \int_{t_i}^{t_f} H(t)dt = \ln(a_{t_f}/a_{t_i}), \tag{5.2}$$

To solve the problems for which it was invented, e.g., the horizon and flatness problems[4], one requires $n \approx 60$ at the end of inflation. To achieve this, the condition (5.1) must be sustained for sufficiently long. Therefore, one also imposes the condition $|\eta(t)| < 1$ where

$$\eta(t) \equiv -\frac{\ddot{H}}{2H\dot{H}} = \epsilon(t) - \frac{1}{2H}\frac{\dot{\epsilon}(t)}{\epsilon(t)} \tag{5.3}$$

Considering Einstein gravity, the Raychaudhuri equation with a barotropic fluid reads, $3\ddot{a} = -4\pi G(\rho + 3p)$.

Thus, an inflationary stage can take place if the pressure is sufficiently negative (the strong energy condition does not hold during inflation [3]).

A simple way to generate inflation[4] is with a scalar field $\phi$. For the homogeneous field, $\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ and $p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$, therefore, we have accelerated expansion if the potential energy dominates over the kinetic energy, $V(\phi) > \dot{\phi}^2$ (the limit at which the scalar field behaves like a barotropic fluid).

[3] A universe contracting down to a bouncing radius would have negative velocity and positive acceleration

[4] A positive cosmological constant is another possibility.

An alternative to inflation driven by matter is that produced by a higher-derivative excitation of the gravitational field due to higher-order curvature terms[26]. This is known as the model Starobinsky, who formulated an effective action with quadratic curvature terms arising from quantum corrections of conformally covariant matter fields [13, 14].

For FRW-cosmology, the most general action including quadratic curvature terms is [15],

$$L_S = \sqrt{-g}\Big(R + \frac{\alpha}{6}R^2\Big), \tag{5.4}$$

A cosmological constant can also be added. Inflation takes place in the large curvature regime $R \gg M^2 = \alpha^{-1}$. Despite being known for a long time now, it continues to be a viable inflationary model. For example, the predicted spectral index and tensor-to-scalar ratio, $n_s \approx 0.96$, $r \approx 0.004$, respectively, for $M \approx 10^{13}$ GeV, are in good agreement with current observational data [17, 26].

(5.4) is simplest example of the $f(R)$-theories of modified gravity, which are of interest for they can give an account, not only of inflation and late-time accelerated cosmic expansion [18, 17, 19], but practically any type of evolution with a suitable $f$ [21]. This is possible due to a higher-derivative scalar degree of freedom so-called scalaron, $\phi = f'(R)$.

Analogously to the ordinary FRW case, the equations for a spatially homogeneous and isotropic $f(R)$ universe can be obtained directly from the 4D action evaluated at the FRW metric which, from now on, we write as $ds^2 = -N^2(t)dt^2 + a^2(t)\Big((1-kr^2)^{-1}dr^2 + r^2 d\Omega_2^2\Big)$.

Therefore, the Lagrangian reads

$$L_f = \frac{1}{2\kappa^2} N a^3 f[R(N,a)]. \tag{5.5}$$

with the time-dependent FRW curvature

$$R(N,a) = 6\Big(-\frac{\dot{N}\dot{a}}{N^3 a} + \frac{\ddot{a}}{N^2 a} + \frac{\dot{a}^2}{N^2 a^2} + \frac{k}{a^2}\Big). \tag{5.6}$$

From (5.5), one gets the following fourth-order equation of motion in the scale factor

$$\frac{1}{N}\frac{d}{dt}\left(\frac{\dot{f}'}{N}\right) + 2\frac{\dot{a}\dot{f}'}{aN^2} - \left(\frac{1}{aN}\frac{d}{dt}\left(\frac{\dot{a}}{N}\right) + 2\frac{\dot{a}^2}{a^2N^2} + 2\frac{k}{a^2}\right)f' + \frac{1}{2}f = 0, \tag{5.7}$$

where $\dot{f}' = \frac{d}{dt}[f'(R)]]$, $\ddot{f}' = \frac{d^2}{dt^2}[f'(R)]$.

The subset of three diagonal spatial $f(R)$-field equations (see e.g., [17]) reduce to (5.7) in the FRW case. When $\dot{a} \neq 0$, a first integral is available,

$$\frac{\dot{a}\dot{f}'}{aN^2} - \frac{1}{aN}\frac{d}{dt}\left(\frac{\dot{a}}{N}\right)f' + \frac{1}{6}f = 0, \tag{5.8}$$

which corresponds to the (00)-constraint or Friedmann equation. Also, a simple linear combination of (5.7) and (5.8) gives us the trace,

$$\frac{1}{N}\frac{d}{dt}\left(\frac{\dot{f}'}{N}\right) + 3\frac{\dot{a}}{Na}\frac{\dot{f}'}{N} - \frac{1}{3}Rf' + \frac{2}{3}f = 0. \tag{5.9}$$

With ordinary gravity this last equation reduces to $R = 0$ (vacuum); for nonlinear functions it is seen as the evolution equation of the extra scalar degree of freedom $f'(R)$.

For the model of Starobinsky (5.4) with $k = 0$, equations (5.7-5.9) read, in proper time gauge $N = 1$,

$$0 = \dddot{H} + 6H\ddot{H} + \frac{9}{2}\dot{H}^2 + 9H^2\dot{H} + M^2\Big(\frac{3}{2}H^2 + \dot{H}\Big), \tag{5.10a}$$

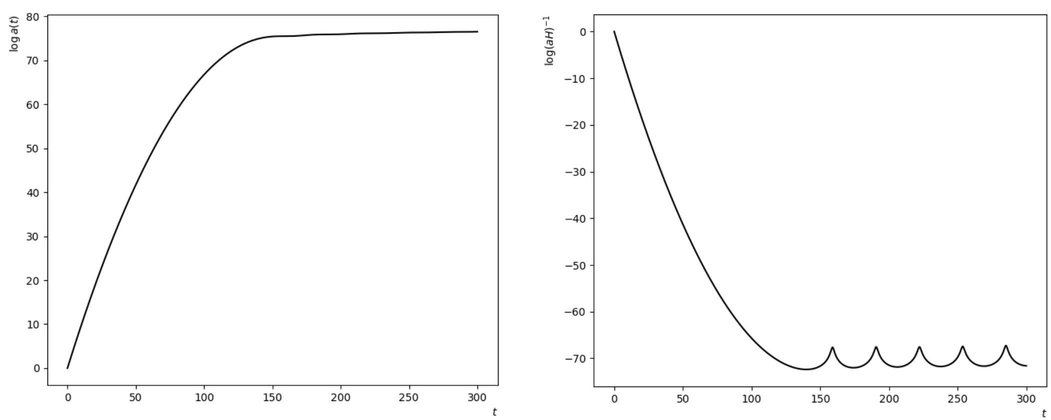
$$0 = \ddot{H} - \frac{\dot{H}^2}{2H} + 3H\dot{H} + \frac{1}{2}M^2 H, \tag{5.10b}$$

$$0 = \ddot{R} + 3H\dot{R} + M^2 R, \tag{5.10c}$$

respectively. (5.10c) shows that the scalar curvature behaves like a massive scalar field.

For a taste of inflation, consider equation (5.10b) under slow-roll conditions (5.1) and (5.3). Then, $\dot{H} \approx -M^2/6$, whose solution is given by $H(t) = H_0 - \frac{1}{6}M^2(t - t_0)$. Consequently, $a(t) = a_0 \exp\Big(H_0(t - t_0) - \frac{M^2}{12}(t - t_0)^2\Big)$ and $R(t) = 12H^2(t) - M^2$. The slow-roll parameter is $\epsilon \approx M^2/6H^2$; inflation starts with $H \gg M$ and ends when $H \approx M/\sqrt{6}$ at time $t_f \approx t_i + 6H_0/M^2$. The number of e-folds (5.2) is, therefore, $N \approx 3H_0^2/M^2$.

Equation (5.10a) can also be solved numerically. Figure 5.1 shows one such stable inflationary solution. The scale factor increases from the order of $10^0$ up to $10^{32}$, which corresponds to, roughly, 73 e-folds. From that point, we get a (positive) oscillating Hubble factor corresponding to expansion (the flattened part of the curve) without inflation. At the final time displayed, we have in total 76 e-folds, approximately.




(a) (b)

Figure 5.1: Numerical solution to (5.10a) with initial values $a = 1$, $H = 5M$, $\dot{H} = -\frac{1}{6}M^2$, $\ddot{H} = 0$ and $M = 0.2$. (a) shows the logarithm of $a(t)$ and (b) of the co-moving Hubble length.

## 5.1 Dual matter-action

The 4D model of Starobinsky is higher-derivative, nonetheless, it is well known that it has no ghosts, no tachyons, and does not suffer from the Ostrogradskian instability [21, 22].

That its one-dimensional homogeneous version is also a physically acceptable model can be seen from its dual form corresponding to a minimally coupled scalar field on an FRW background, as shown next.

A standard procedure to reduce the order of a lagrangian is by means of additional fields that keep track of the higher derivative terms. Using the method of Lagrange multipliers, we write

$$L = \frac{1}{2\kappa^2} N a^3 \Big[ f(\phi) + l\big(\phi - R(N,a)\big)\Big] \tag{5.11}$$

The equation of motion for $l$ enforces the constraint $\phi = R(N,a)$. Further, variation of the action with respect to $\phi$ yields $l = -f'(\phi)$. Substituting this value, we eliminate the multiplier and get the following Lagrangian [52]

$$L_f^\phi = \frac{1}{2\kappa^2} N a^3 \Big[ f'(\phi)\big(R(N,a) - \phi\big) + f(\phi)\Big], \tag{5.12}$$

which leads to the same classically dynamics as (5.5). (5.12) already corresponds to a second-order theory as shown later (see 5.26).

The next step is to perform a Weyl re-scaling of the gravitational field $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = f'(\phi) g_{\mu\nu}$ [23, 24], which in the FRW case amounts to

$$N \to n \equiv \sqrt{f'(\phi)}N, \qquad a \to A \equiv \sqrt{f'(\phi)}a. \tag{5.13}$$

The result is

$$L_f^\phi \to \frac{1}{2\kappa^2} n A^3 \Big[ R(n,A) + \frac{3f''^2}{2f'^2}\frac{\dot{\phi}^2}{n^2} + \frac{f - \phi f'}{f'^2}\Big]. \tag{5.14}$$

For the model of Starobinsky (5.4), $f'(\phi) = 1 + \alpha\phi/3$, (5.14) yields

$$L_S^\phi = \frac{1}{2\kappa^2} n A^3 \Big[ R(n,A) + \frac{1}{6(1+\alpha\phi/3)^2}\Big(\alpha^2 \frac{\dot{\phi}^2}{n^2} - \alpha\phi^2\Big)\Big], \tag{5.15}$$

from which we see that, for large values of $\phi$, the scalar potential approximates a cosmological constant $\Lambda = 3/4\alpha$.

Further, the matter part of (5.14) can be written in canonically normalized form with the Legendre transformation

$$\phi \to \varphi = c^{-1} \ln f'(\phi) \tag{5.16}$$

where $c^2 = \frac{2}{3}\kappa^2$. Indeed, performing the transformation, we get

$$L_f^\phi \to L_f^\varphi = \frac{1}{2\kappa^2} n A^3 R(n,A) + \frac{A^3}{2n}\dot{\varphi}^2 - nA^3 V(\varphi) \tag{5.17}$$

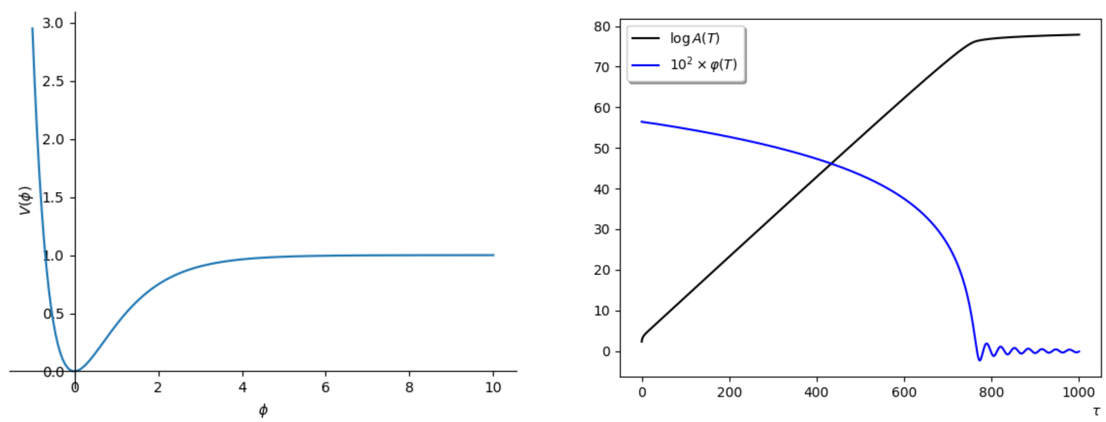
with the following scalar potential

$$V(\varphi) = \frac{1}{2\kappa^2} e^{-2\beta\varphi}(\phi e^{\beta\varphi} - f(\phi))|_{\phi=\phi(\varphi)}. \tag{5.18}$$

For the model of Starobinsky, the scalar potential is

$$V(\varphi) = \frac{3M^2}{4\kappa^2}(1 - e^{-c\varphi})^2, \tag{5.19}$$

and it is shown in Fig. 5.2 (a). No doubt (5.4) is an inflationary model, (5.19) has the archetypal profile of a slow-roll inflation potential, possessing a long plateau where the almost constant potential energy dominates (provided the initial kinetic energy is sufficiently small). However, the evolution of the scale factor, although inflationary in both cases, does not look quite the same. A numerical solution $(T, A(T))$ is shown in Fig. 5.2 (b); here $T$ is proper-time in the Einstein frame ($n = 1$) and we set initial conditions equivalent to those used for the solution in the $f(R)$-modified gravity frame (Fig. 5.1). The connection between the variables is $t(T) = \int^T \exp(-\frac{1}{2}c\varphi)$, $a(T) = e^{-c\varphi(T)/2}A(T)$.




(a) (b)

Figure 5.2: (a) Starobinsky potential (5.19). (b) Solution in the Einstein frame with initial values $A = 10.03$, $h \equiv A^{-1}dA/dT = 0.99$, $\varphi = 5.64$ $\dot{\varphi} = -0.01$. We re-scaled $\varphi$ for clarity.

## 5.2 Hamiltonian formulation

We derive here the Ostrogradsky Hamiltonian formulation, and write the canonical transformations that relate it to the other two scalar-tensor formulations (5.12) and (5.17).

### Ostrogradsky Hamiltonian formulation

The canonical formulation of the FRW model of Starobinsky can be obtained directly from the higher derivative action following Ostrogradsky's approach (see e.g., [21] for a review). For simplicity, we integrate by parts the linear second-order derivative of $a$ and take $k = 0$, so that the Lagrangian reads

$$L_S = jNa^3\left[-\frac{\dot{a}^2}{N^2a^2} + \alpha\left(-\frac{\dot{N}\dot{a}}{N^3a} + \frac{\ddot{a}}{N^2a} + \frac{\dot{a}^2}{N^2a^2}\right)^2\right]. \tag{5.20}$$

where $j = 3/\kappa^2$.

In this formalism, there are eight canonical variables, namely

$$a, \tag{5.21a}$$

$$p_a \equiv \frac{\partial L_S}{\partial \dot{a}} - \frac{d}{dt}\frac{\partial L_S}{\partial \ddot{a}} = -2j\left[\frac{a\dot{a}}{N} + \frac{\alpha a^2}{N}\frac{d}{dt}\left(\frac{\ddot{a}}{N^2a} - \frac{\dot{N}\dot{a}}{N^3a} + \frac{\dot{a}^2}{N^2a^2}\right)\right], \tag{5.21b}$$

$$A \equiv \dot{a}, \tag{5.21c}$$

$$p_A \equiv \frac{\partial L_S}{\partial \ddot{a}} = 2j\alpha\frac{a^2}{N}\left(\frac{\ddot{a}}{N^2a} - \frac{\dot{N}\dot{a}}{N^3a} + \frac{\dot{a}^2}{N^2a^2}\right), \tag{5.21d}$$

$$N, \tag{5.21e}$$

$$p_N \equiv \frac{\partial L_S}{\partial \dot{N}} - \frac{d}{dt}\frac{\partial L_S}{\partial \ddot{N}} = -2j\alpha\frac{a^2\dot{a}}{N^2}\left(\frac{\ddot{a}}{N^2a} - \frac{\dot{N}\dot{a}}{N^3a} + \frac{\dot{a}^2}{N^2a^2}\right), \tag{5.21f}$$

$$n \equiv \dot{N}, \tag{5.21g}$$

$$p_n \equiv \frac{\partial L_S}{\partial \ddot{N}} = 0. \tag{5.21h}$$

Since (5.20) does not depend on $\ddot{N}$, we get the primary constraint (5.21h). It can already be seen that $p_N$ can be written in terms of $A$ and $P_A$, which would yield another constraint.

The Hamiltonian of Ostrogradsky is defined by the Legendre transformation

$$H_{\text{Ost}} \equiv Ap_a + \dot{A}p_A + np_N + \dot{n}p_n - L_S \tag{5.22a}$$

$$= Ap_a + j\frac{aA^2}{N} + \frac{1}{4j\alpha}\frac{N^3p_A^2}{a} - \frac{A^2p_A}{a} + n\left(\frac{Ap_A}{N} + p_N\right) + \dot{n}p_n. \tag{5.22b}$$

By construction, the Ostrogradsky Hamiltonian (5.22a) is linear in $p_a$ and $p_N$, since the velocities $A$ and $n$ are canonical variables themselves. In general, higher-derivative theories suffer from the Ostrogradskian instability leading to a problematic dynamics which can manifest in several ways [21].

However, this is not the end of the story and the model of Starobinsky, as most $f(R)$ theories, avoid the instability. This is due to the system (5.20) being degenerate in the sense that the matrix $(\partial^2 L_S/\partial\ddot{q}_i\partial\ddot{q}_j)$ is not invertible. Otherwise, we would get fourth-order equations of motion for both $a$ and $N$, but only $a$ acquires a higher-derivative degree of freedom.

As we get a primary constraint, we proceed following Dirac's standard procedure. The result is that there are three first-class constraints

$$0 \approx C \equiv p_n, \tag{5.23a}$$

$$0 \approx D \equiv \frac{Ap_A}{N} + p_N, \tag{5.23b}$$

$$0 \approx H_0 \equiv \frac{Ap_a}{N} + j\frac{aA^2}{N^2} + \frac{1}{4j\alpha}\frac{N^2p_A^2}{a} - \frac{A^2p_A}{Na}. \tag{5.23c}$$

(5.23b) follows from conservation in time of (5.23a), whereas (5.23c) follows from conservation of (5.23b). The time derivative of (5.23c) vanishes identically and no more constraints arise. Therefore, the total Hamiltonian vanishes as a constraint

$$0 \approx H = NH_0 + nD + \nu C, \tag{5.24}$$

where $N$, $n$ and $\nu$ remain arbitrary.

**Hamiltonian of the scalar-'tensor' formulation**

Now we obtain the Hamiltonian corresponding to action (5.12) specialized to Starobinsky. To avoid unnecessary factors of 6, we re-scale $\phi \to 6\phi$, so that the Lagrangian reads

$$L_S^\phi = \frac{1}{2\kappa^2} N a^3 \Big[(1 + 2\alpha\phi)R - 6\alpha\phi^2\Big], \tag{5.25}$$

which is already a second-order theory, but not the full dual Einstein gravity-matter yet, due to the non-minimal coupling. Integrating by parts, we get,

$$L_S^\phi = jNa^3 \left[ -\frac{\dot{a}^2}{a^2N^2}(1 + 2\alpha\phi) - 2\alpha\frac{\dot{a}\dot{\phi}}{aN^2} - \alpha\phi^2 \right]. \tag{5.26}$$

with $j = 3/\kappa^2$.

The Hamiltonian is of the form $H^\phi = NH_0 + \mu p_N$, with the Hamiltonian constraint

$$0 \approx H_0^\phi \equiv -\frac{1}{2j\alpha}\frac{p_a p_\phi}{a^2} + \frac{1}{4j\alpha^2}(1 + 2\alpha\phi)\frac{p_\phi^2}{a^3} + j\alpha a^3\phi^2. \tag{5.27}$$

This approach and that of Ostrogradsky give equivalent Hamiltonian constraints; (5.23c) and (5.27) transform into each other by means of the following canonical transformation

$$a = a, \qquad p_a^{\mathrm{Ost}} = p_a - 2\frac{\phi p_\phi}{a}, \tag{5.28a}$$

$$A = -\frac{1}{2j\alpha}\frac{Np_\phi}{a^2}, \qquad p_A = -2j\alpha\frac{a^2}{N}\phi, \tag{5.28b}$$

$$N = N, \qquad p_N^{\mathrm{Ost}} = p_N + \frac{\phi p_\phi}{N}, \tag{5.28c}$$

Also (5.23b) is mapped into a constraint: $0 \approx p_N = N^{-1}Ap_A + p_N^{\mathrm{Ost}} = D$. Here $p_a^{\mathrm{Ost}}$ and $p_N^{\mathrm{Ost}}$ denote the corresponding momenta in the Ostrogradsky formalism.

Both versions of the Hamiltonian constraint, (5.27) or (5.23c), yield the same nontrivial relation that, expressed in configuration space variables, can be recognized as the generalized Friedmann equation for the model of Starobinsky (5.10b).

**Hamiltonian of the full scalar-'tensor' dual**

A further canonical transformation corresponding to the Weyl-Legendre transformation described in 5.1, is readily derived,

$$\phi = (2\alpha)^{-1}(e^{c\varphi}-1), \qquad p_\phi = \alpha e^{-c\varphi}(\tilde{a}p_{\tilde{a}} + \tilde{N}p_{\tilde{N}}) + \frac{2\alpha}{c}e^{-c\varphi}p_\varphi, \tag{5.29a}$$

$$a = e^{-\frac{1}{2}c\varphi}\tilde{a}, \qquad p_a = e^{\frac{c}{2}\varphi}p_{\tilde{a}}, \tag{5.29b}$$

$$N = e^{-\frac{1}{2}c\varphi}\tilde{N}, \qquad p_N = e^{\frac{c}{2}\varphi}p_{\tilde{N}} \tag{5.29c}$$

with $c^2 = \frac{2}{3}\kappa^2$. Indeed, (5.29) transforms (5.27) into

$$0 \approx H_0^\varphi = -\frac{\kappa^2}{12}\frac{p_{\tilde{a}}^2}{\tilde{a}} + \frac{p_\varphi^2}{2\tilde{a}^3} + \frac{3M^2}{4\kappa^2}\tilde{a}^3(1-e^{-c\varphi})^2. \tag{5.30}$$

Therefore, by combining the transformations (5.28) and (5.29), one can pass directly from the Ostrogradsky Hamiltonian (5.23c) to (5.30).

## 5.3 Quantization

Now we study the canonical quantization of the model of Starobinsky in its modified gravity form but written as a second-order theory.

To implement the ordering choice of Chapter 2, involving the mini-superspace metric, it would be convenient to diagonalize the kinetic term of (5.27). For the moment, we write the WDW equation with the following ordering

$$\left[\frac{1}{2j\alpha}a^{-2}\partial_a\partial_\phi - \frac{1}{4j\alpha^2}a^{-3}\partial_\phi(1+2\alpha\phi)\partial_\phi - jka(1+2\alpha\phi) + j\alpha a^3\phi^2\right]\psi(a,\phi) = 0 \tag{5.31}$$

Note that we allowed for the possibility of non-vanishing spatial curvature.

A convenient coordinate transformation is

$$a' = a, \qquad Q = a(1+2\alpha\phi), \tag{5.32}$$

with which we are replacing $\phi = R/6$ with $Q = af'(R)$. The differential operators transform accordingly,

$$\partial_a = \partial_{a'} + \frac{Q}{a'}\partial_Q, \qquad \partial_\phi = 2\alpha a'\partial_Q. \tag{5.33}$$

In terms of these new variables, equation (5.31) reads

$$\left[\hbar^2\partial_a\partial_Q + U(a,Q)\right]\psi(a,Q) = 0 \tag{5.34}$$

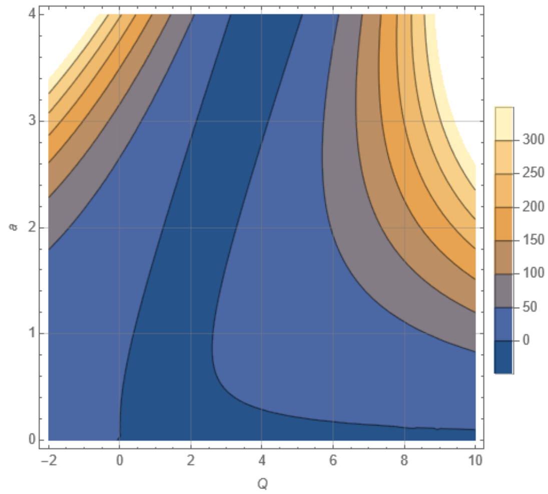



Figure 5.3: Contour plot of potential $U(a,Q)$ for $k=1$, $\alpha = 1/4$.

with the following potential

$$U(a,Q) = j^2\Big[-kaQ + \frac{a^2}{4\alpha}(Q-a)^2\Big] \tag{5.35}$$

Fig. 5.3 shows this potential in the case of $k=1$. The qualitative behavior of the wavefunction is generally correlated with the sign of $U$. In this case we have two branches $U=0$, and a central region of negative potential.

With this form of the WDW equation, the kinetic term can be diagonalized immediately with the following coordinates

$$x = Q - a, \qquad\qquad y = Q + a. \tag{5.36}$$

In term of these variables, (5.34)

$$\Big[\hbar^2(\partial_y^2 - \partial_x^2) + U(x,y)\Big]\psi(x,y) = 0 \tag{5.37}$$

with

$$U(x,y) = \frac{j^2}{4}\Big[-k(y^2 - x^4) + \frac{(y-x)^2}{4\alpha}x^2\Big] \tag{5.38}$$

**Approximate solution in the Einstein frame**

We are going to consider the saddle-point approximate solution the WDW equation, for the model of Starobinsky in its Einstein gravity-matter form (5.30). To maintain the same ordering, we just perform on (5.34) the following transformation

$$\varphi = c^{-1}\ln(1+2\alpha\phi), \qquad\qquad A = \sqrt{1+2\alpha\phi}\,a, \tag{5.39}$$

Note that the transformation of $N$ is no longer needed. The differential operators transform as follows

$$\partial_a = e^{c\varphi/2}\partial_A \qquad \partial_\phi = \alpha e^{-c\varphi}\Big(A\partial_A + \frac{2}{c}\partial_\varphi\Big). \tag{5.40}$$

Thus, re-writing (5.31) in terms of $A$ and $\varphi$, we get

$$e^{c\varphi/2}\left[A^{-1}\partial_A A\partial_A - 2jA^{-2}\partial_\varphi^2 + 4j^2A^2\left(-1+\frac{1}{4\alpha}A^2\left(1-e^{-c\varphi}\right)^2\right)\psi(A,\varphi)\right] = 0. \tag{5.41}$$

With a final re-scaling[5] $\tilde{a} = \sqrt{2j}A$, $\chi = (2j)^{-1/2}\varphi$, we get

$$\tilde{a}^{-1}\partial_{\tilde{a}}\tilde{a}\partial_{\tilde{a}} - \tilde{a}^{-2}\partial_\chi^2 - \tilde{a}^2\left(1-\frac{1}{8j\alpha}\tilde{a}^2\left(1-e^{-2\chi}\right)^2\right)\psi(A,\varphi) = 0 \tag{5.42}$$

Thus, comparing (5.42) with (2.47), we identify

$$V(\chi) = \frac{1}{8j\alpha}\left(1-e^{-2\chi}\right)^2 = \frac{1}{8j\alpha}\frac{(Q-a)^2}{Q^2}, \tag{5.43}$$

$$\tilde{a}^2 = 2jQa \tag{5.44}$$

Thus, the saddle-point wavefunction (2.50), written in terms of the $a, Q$ variables is

$$\psi(a,Q) \approx \begin{cases} \exp\Big[8j\alpha Q^2\Big(1-[1-\frac{1}{4\alpha}\frac{a}{Q}(Q-a)^2]^{3/2}\Big)/3(Q-a)^2\Big], & \text{if} \quad 1 > \tilde{a}^2V \\ \exp\Big[8j\alpha Q^2/3(Q-a)^2\Big], & \text{if} \quad 1 \le \tilde{a}^2V \end{cases} \tag{5.45}$$

We took all these steps to obtain an approximate solution to the WDW equation (5.34) in the frame of modified gravity, using the results in the Einstein frame (see Section 2.1.4). Our motivations for this was to use (5.45) to extract the divergent component of the numerical solution, as in Section 2.1.4.

A numerical solution to (5.34) for $k = 1$, is shown in Figure 5.4. The wavefunction oscillates in the region of positive (and sufficiently small) $U(a,Q)$, whereas it increases exponentially in the region of negative potential. We identified two regions of oscillatory behavior, above the right-hand branch of $U = 0$, already described in [10], and another above the left-hand branch with $Q > 0$, the oscillations in this region are much smaller in amplitude. On the other hand, the trick to extract the diverging component worked to a certain extent. There is still a divergent component close to the right-hand branch of the curve $U = 0$.

[5]This is required only because we used a different normalization from that in Section 2.1.4.

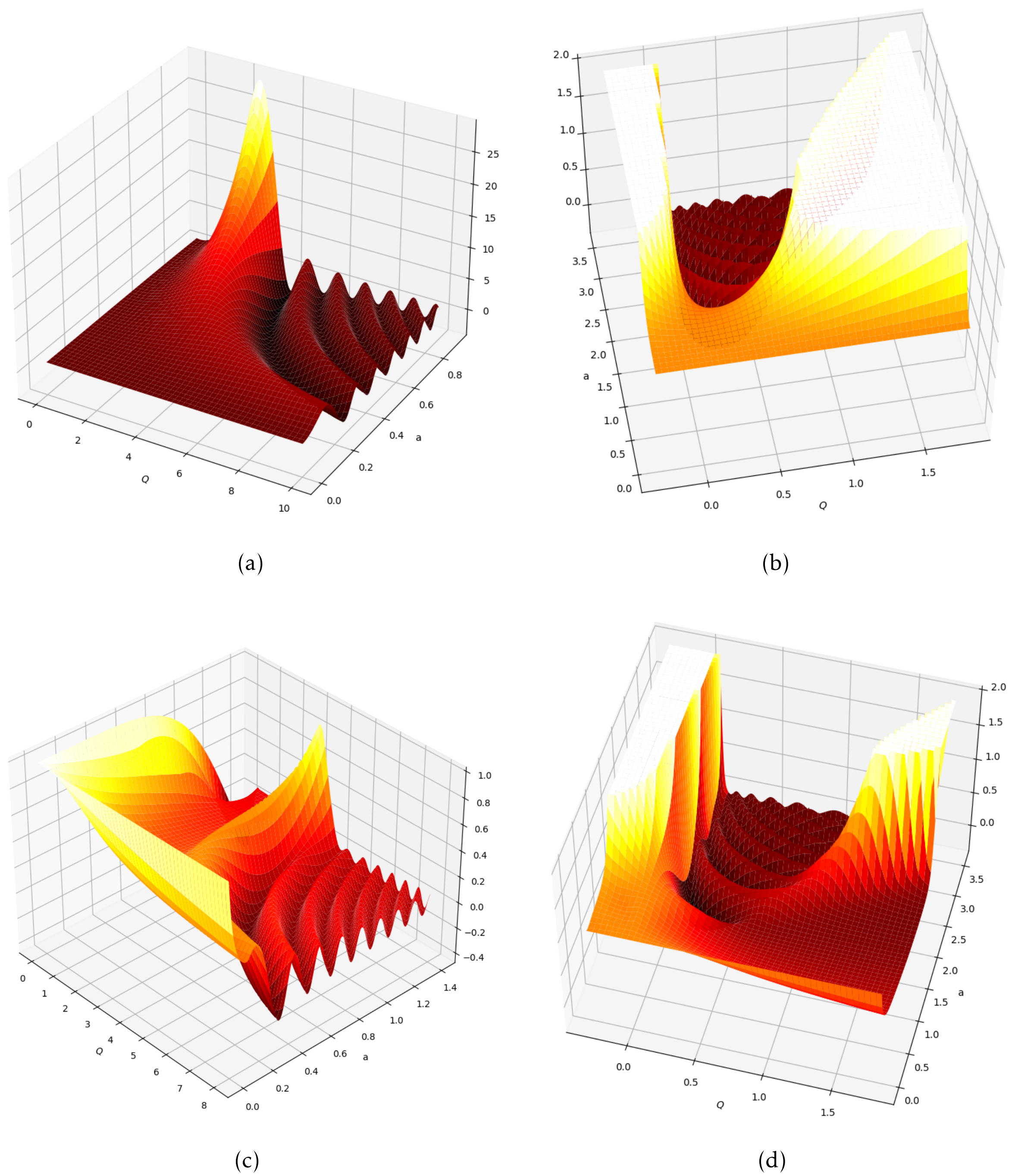



Figure 5.4: Numerical solution to the WDW equation (5.34) with boundary condition $\psi(0, Q) = 1$. A symmetry condition is supplemented along the $Q = 0$ axis of the type $\psi(a, 0) = \psi(a - \delta a, \delta Q)$ [12]. In (a) one can see the wavefunction starts to oscillate above the right-hand branch of the curve $U(a, Q) = 0$. Another region where $\psi$ oscillates is above the left-hand branch with positive $Q$, shown in (b). On the other hand, (c) and (d) show the wavefunction divided by the saddle-point approximation given in (5.45) for (a) and (b), respectively, by which we extract part, but not the totality of the divergent component.

# Chapter 6

# Higher-derivative supersymmetric theories: F(R)

In this chapter we study the straightforward superfield generalization of the bosonic $f(R)$ action, namely, an arbitrary function of $\mathcal{R}$, the FRW curvature superfield (4.2).

For convenience, from now on, we parameterize the scale factor superfield as

$$\mathcal{A}(t,\Theta,\bar{\Theta}) = a(t)\Big[1 + i\Theta\bar{\lambda}(t) + i\bar{\Theta}\lambda(t) - \Theta\bar{\Theta}\Big(s(t) - \sqrt{k}a^{-1} - \lambda(t)\bar{\lambda}(t)\Big)\Big]. \tag{6.1}$$

With this definition, the lowest component of the curvature superfield $\mathcal{R}$ simplifies to just the field $s(t)$. With parameterization (6.1), the multiplet $(a,\lambda,s)$ will transform in a non-linear fashion under local supersymmetry. Indeed, from (3.36),

$$\delta_\xi a = ia(\bar{\lambda}\xi + \lambda\bar{\xi}), \tag{6.2a}$$

$$\delta_\xi \lambda = \Big(is - \frac{\dot{a}}{Na}\Big)\xi - 2i\lambda\bar{\lambda}\xi + i(\psi\bar{\lambda} + \bar{\psi}\lambda)\xi, \tag{6.2b}$$

$$\begin{aligned}\delta_\xi s = &-2\dot{a}\frac{\xi\bar{\lambda} - \bar{\xi}\lambda}{Na} - \frac{\xi\dot{\bar{\lambda}} - \bar{\xi}\dot{\lambda}}{N} + \dot{a}\frac{\psi\bar{\xi} - \bar{\psi}\xi}{Na} - is(\psi\bar{\xi} + \bar{\psi}\xi)\\ &- i\psi\bar{\psi}(\lambda\bar{\xi} + \bar{\lambda}\xi) + 2is(\xi\bar{\lambda} + \bar{\xi}\lambda).\end{aligned} \tag{6.2c}$$

The curvature superfield is defined as in (4.2), but now it reads

$$\begin{aligned}\mathcal{R} = s &+ \Theta\Big(\frac{\dot{\bar{\lambda}}}{N} + 2\frac{\dot{a}\bar{\lambda}}{Na} - \frac{\dot{a}\bar{\psi}}{Na} - i\psi\bar{\psi}\bar{\lambda} - is\bar{\psi} - 2is\bar{\lambda} + i\sqrt{k}\frac{\bar{\psi}}{a} + i\sqrt{k}\frac{\bar{\lambda}}{a}\Big) - \bar{\Theta}\Big(\frac{\dot{\lambda}}{N} + 2\frac{\dot{a}\lambda}{Na} - \frac{\dot{a}\psi}{Na}\\ &+ i\psi\bar{\psi}\lambda + is\psi + 2is\lambda - i\sqrt{k}\frac{\psi}{a} - i\sqrt{k}\frac{\lambda}{a}\Big) + \Theta\bar{\Theta}\Big(\frac{\ddot{a}}{N^2a} - \frac{\dot{N}\dot{a}}{N^3a} + \frac{\dot{a}^2}{N^2a^2} + \frac{k}{a^2} + 2s^2 - 3\sqrt{k}\frac{s}{a}\\ &+ 2\sqrt{k}\frac{\psi\bar{\lambda} - \bar{\psi}\lambda}{a} + 5\sqrt{k}\frac{\lambda\bar{\lambda}}{a} + 2\sqrt{k}\frac{\psi\bar{\psi}}{a} - i\frac{\dot{\psi}\bar{\lambda} + \dot{\bar{\psi}}\lambda}{N} - 6i\dot{a}\frac{\psi\bar{\lambda} + \bar{\psi}\lambda}{Na} - 2s\psi\bar{\psi} - 2i\frac{\psi\dot{\bar{\lambda}} + \bar{\psi}\dot{\lambda}}{N}\\ &- 8\lambda\bar{\lambda}s - 4\psi\bar{\psi}\lambda\bar{\lambda} - 2s(\psi\bar{\lambda} - \bar{\psi}\lambda) - 2i\frac{\lambda\dot{\bar{\lambda}} + \bar{\lambda}\dot{\lambda}}{N}\Big).\end{aligned} \tag{6.3}$$

The superfield Lagrangian is

$$\mathcal{L}_F = j\mathcal{E}\mathcal{A}^3F(\mathcal{R}), \tag{6.4}$$

leading to the following component Lagrangian[6]

$$
\begin{aligned}
L_F = jNa^3\Bigg[F'(s)\Bigg(&\frac{\ddot a}{N^2a}-\frac{\dot N\dot a}{N^3a}+\frac{\dot a^2}{N^2a^2}+\frac{k}{a^2}+2s^2-3\sqrt{k}\frac{s}{a}-i\frac{\dot\psi\bar\lambda+\dot{\bar\psi}\lambda}{N}+i\frac{\lambda\dot{\bar\lambda}+\bar\lambda\dot\lambda}{N}+4s\lambda\bar\lambda \\
&-i\frac{\psi\dot{\bar\lambda}+\bar\psi\dot\lambda}{N}-i\dot a\frac{\psi\bar\lambda+\bar\psi\lambda}{Na}+2\psi\bar\psi\lambda\bar\lambda+3s(\psi\bar\lambda-\bar\psi\lambda)-2\sqrt{k}\frac{\psi\bar\lambda-\bar\psi\lambda}{a}-\sqrt{k}\frac{\lambda\bar\lambda}{a}\Bigg)+F''(s)\times \\
&\Bigg(-\frac{\dot\lambda\dot{\bar\lambda}}{N^2}-4\frac{\dot a^2\lambda\bar\lambda}{N^2a^2}-2\dot a\frac{\lambda\dot{\bar\lambda}-\bar\lambda\dot\lambda}{N^2a}+2\dot a^2\frac{\psi\bar\lambda-\bar\psi\lambda}{N^2a^2}+2\sqrt{k}\frac{\lambda\bar\lambda}{a}\psi\bar\psi-4s\psi\bar\psi\lambda\bar\lambda+4\sqrt{k}\frac{s}{a}\lambda\bar\lambda \\
&-s^2\psi\bar\psi-2s^2(\psi\bar\lambda-\bar\psi\lambda)+3\sqrt{k}s\frac{\psi\bar\lambda-\bar\psi\lambda}{a}+2\sqrt{k}\frac{s}{a}\psi\bar\psi-\frac{\dot a^2\psi\bar\psi}{N^2a^2}-k\frac{\psi\bar\psi}{a^2}-4is\dot a\frac{\psi\bar\lambda+\bar\psi\lambda}{Na} \\
&+\dot a\frac{\psi\dot{\bar\lambda}-\bar\psi\dot\lambda}{N^2a}-is\frac{\psi\dot{\bar\lambda}+\bar\psi\dot\lambda}{N}+i\sqrt{k}\frac{\psi\dot{\bar\lambda}+\bar\psi\dot\lambda}{Na}-4\lambda\bar\lambda s^2-k\frac{\psi\bar\lambda-\bar\psi\lambda}{a^2}-2is\frac{\lambda\dot{\bar\lambda}+\bar\lambda\dot\lambda}{N}-k\frac{\lambda\bar\lambda}{a^2} \\
&+i\sqrt{k}\frac{\lambda\dot{\bar\lambda}+\bar\lambda\dot\lambda}{Na}+3i\sqrt{k}\dot a\frac{\psi\bar\lambda+\bar\psi\lambda}{Na^2}-i\psi\bar\psi\frac{\lambda\dot{\bar\lambda}+\bar\lambda\dot\lambda}{N}\Bigg)-3F(s)\Bigg(s-\frac{\sqrt{k}}{a}+\lambda\bar\lambda+\psi\bar\lambda-\bar\psi\lambda\Bigg)\Bigg]
\end{aligned}
\tag{6.5}
$$

If $F'' \neq 0$, we get a higher-derivative theory in fermions, since $\dot\lambda\dot{\bar\lambda}$ yields second-order equations of motion, not first-order as in the linear case (4.4). Accordingly, $s$ is no longer an auxiliary field, it becomes a higher-derivative degree of freedom. In principle, one could integrate it out, but that results in a higher-order equation of motion for the scale factor.

To investigate the associated bosonic $f(R)$, it suffices to consider only the bosonic sector of (6.5), which is of the form $L_F^{\text{bos}} = (2\kappa^2)^{-1}Na^3 f(R,s)$, with

$$
f(R,s) = RF'(s) + 12s^2F'(s) - 18sF(s) \tag{6.6}
$$

We have set $k = 0$, for it is the only case where eliminating $s$ yields a pure function $f$ of $R$. This $f(R)$ will be generally different from the "input" function $F(R)$. Let's consider two simple examples involving linear combinations. First, $F_1(\mathcal{R}) = \mathcal{R} + \alpha^{\frac{1}{2}}\mathcal{R}^2$, then we have $f_1(R,s) = R(1+2\alpha^{1/2}s) - 6s^2 + 6\alpha^{1/2}s^3$; $s$ satisfies a quadratic equation with real solutions $s_\pm$ yielding $f_1^\pm(R) = \frac{5}{3}R - \frac{4}{9\alpha} \pm \frac{4}{9}R\sqrt{1-\alpha R} \mp \frac{4}{9\alpha}\sqrt{1-\alpha R}$. Therefore, to have a real $f$, $R$ must be sufficiently smaller that $M^2 = \alpha^{-1}$. Further, only $f_1^-$ reduces to linear $R$ in the vanishing-$\alpha$ limit. In fact, it can be expanded in powers of the curvature, $f_1^-(R) = R + \frac{\alpha}{6}R^2 + \frac{\alpha^2}{36}R^3 + \frac{\alpha^3}{96}R^4 + \ldots$. However, this is valid for small $R$, which is not the regime of Starobinsky inflation.

For a second example, we take $F_2(\mathcal{R}) = \frac{2}{3}(\mathcal{R}+2\alpha\mathcal{R}^3)$, then $f_2(R,s) = \frac{2}{3}(R+6\alpha s^2) - 4s^2 + 24\alpha s^4$ and $s$ satisfies a cubic equation with trivial solution returning the linear action, and non-trivial solutions yielding $f_2(R) = R - \frac{1}{6\alpha} - \frac{\alpha}{6}R^2$. Thus, we get a positive cosmological constant and negative sign quadratic curvature term. However, it is known that the 4D version of this, fails to satisfy stability conditions $f''(R) > 0$, $f'(R) > 0$, which prevent a negative mass squared for the scalaron and the appearance of ghosts, respectively [17].

[6] Any function of the generic superfield $\Phi = \phi + \Theta\eta - \bar\Theta\bar\eta + \Theta\bar\Theta G$, can be Taylor expanded around the lowest component $\Phi_{\underline{\Theta}} = 0$, leading to $F(\Phi) = F(\phi) + F'(\phi)(\phi)\left(\Theta\eta - \bar\Theta\bar\eta\right) + \Theta\bar\Theta\left(F'(\phi)G + F''(\phi)\eta\bar\eta\right)$.

Therefore, at least with the most straightforward conceivable generalizations for the model of Starobinsky, we get component lagrangians whose bosonic part have little to do with large-curvature inflation. In the next chapter, it will be clear that the $F(\mathcal{R})$ action does not have enough degrees of freedom to provide a suitable supersymmetric model of Starobinsky. Nonetheless, it provides a pretty manageable model to study properties of higher-derivative supersymmetric theories, and novel actions of the $F(R)$ type could even find application in quantum cosmology, as illustrated below.

### Hamiltonian formulation

The component Lagrangian is, after integrating by parts,

$$\begin{aligned}L_F = jNa^3\Bigg[&F'(s)\Big(-\frac{\dot a^2}{N^2a^2}+\frac{k}{a^2}-3\sqrt{k}\frac{s}{a}+2s^2+2i\dot a\frac{\psi\bar\lambda+\bar\psi\lambda}{Na}+2\psi\bar\psi\lambda\bar\lambda+i\frac{\lambda\dot{\bar\lambda}+\bar\lambda\dot\lambda}{N}+4s\lambda\bar\lambda\\
&-2\sqrt{k}\frac{\psi\bar\lambda-\bar\psi\lambda}{a}-\sqrt{k}\frac{\lambda\bar\lambda}{a}+3s(\psi\bar\lambda-\bar\psi\lambda)\Big)+F''(s)\Big(-\frac{\dot a\dot s}{N^2a}-\frac{\dot\lambda\dot{\bar\lambda}}{N^2}-4\frac{\dot a^2\lambda\bar\lambda}{N^2a^2}-2\dot a\frac{\lambda\dot{\bar\lambda}-\bar\lambda\dot\lambda}{N^2a}\\
&+2\dot a^2\frac{\psi\bar\lambda-\bar\psi\lambda}{N^2a^2}+\dot a\frac{\psi\dot{\bar\lambda}-\bar\psi\dot\lambda}{N^2a}+3i\sqrt{k}\dot a\frac{\psi\bar\lambda+\bar\psi\lambda}{Na^2}-4is\dot a\frac{\psi\bar\lambda+\bar\psi\lambda}{Na}-i\psi\bar\psi\frac{\lambda\dot{\bar\lambda}+\bar\lambda\dot\lambda}{N}-k\frac{\lambda\bar\lambda}{a^2}\\
&+i\sqrt{k}\frac{\psi\dot{\bar\lambda}+\bar\psi\dot\lambda}{Na}-2is\frac{\lambda\dot{\bar\lambda}+\bar\lambda\dot\lambda}{N}+i\sqrt{k}\frac{\lambda\dot{\bar\lambda}+\bar\lambda\dot\lambda}{Na}-is\frac{\psi\dot{\bar\lambda}+\bar\psi\dot\lambda}{N}-\frac{\dot a^2\psi\bar\psi}{N^2a^2}-4s\psi\bar\psi\lambda\bar\lambda-k\frac{\psi\bar\psi}{a^2}\\
&+2\sqrt{k}\frac{\lambda\bar\lambda}{a}\psi\bar\psi+3\sqrt{k}s\frac{\psi\bar\lambda-\bar\psi\lambda}{a}-k\frac{\psi\bar\lambda-\bar\psi\lambda}{a^2}+2\sqrt{k}s\frac{\psi\bar\psi}{a}-4s^2\lambda\bar\lambda+4\sqrt{k}\frac{s}{a}\lambda\bar\lambda-s^2\psi\bar\psi\\
&-2s^2(\psi\bar\lambda-\bar\psi\lambda)+i\dot s\frac{\psi\bar\lambda+\bar\psi\lambda}{N}\Big)-3F(s)\Big(s-\frac{\sqrt{k}}{a}+\lambda\bar\lambda+\psi\bar\lambda-\bar\psi\lambda\Big)\Bigg]\end{aligned}\tag{6.7}$$

Note that it has a quadratic term in fermionic velocities. Typical fermionic lagrangians contain at most linear terms in the velocities, which ultimately lead to second-class constraints as in Section 4.2. In this case, however, we can solve for the (physical) velocities in terms of coordinates and momenta, and the Hamiltonian formulation can be performed as usual. Later in Section 6.2, we compare this Hamiltonian formulation with that resulting from the equivalent action that is not higher derivative.

Conjugate bosonic momenta are defined as usual whereas fermionic ones are defined as in Chapter 4,

$$\pi \equiv \frac{\partial L}{\partial\dot{\bar\lambda}}, \qquad \bar\pi \equiv -\frac{\partial L}{\partial\dot\lambda}. \tag{6.8}$$

The Hamiltonian is given by the Legendre transformation $H_F = \dot a p_a + \dot s p_s - \dot\lambda\bar\pi + \dot{\bar\lambda}\pi - L_F$. As with the previous examples, we obtain a Hamiltonian vanishing as a constraint,

$$0 \approx H_F = NH_0 + \frac{1}{2}\Psi\bar S - \frac{1}{2}\bar\Psi S, \tag{6.9}$$

with the Hamiltonian constraint

$$\begin{aligned} H_0 = &\frac{1}{j}\frac{F'p_s^2}{F''^2a^3} - \frac{1}{j}\frac{p_a p_s}{a^2F''} - 2ja^3F's^2 - jkaF' + 3ja^3sF + 3j\sqrt{k}a^2(sF'-F) + 3jFa^3\lambda\bar{\lambda} \\ &- 2\frac{1}{j}p_s\frac{\lambda\bar{\pi} - \bar{\lambda}\pi}{F''a^3} - j\sqrt{k}a^2F'\lambda\bar{\lambda} - \frac{1}{j}\frac{\pi\bar{\pi}}{F''a^3} + 2is(\lambda\bar{\pi} + \bar{\lambda}\pi) - iF'\frac{\lambda\bar{\pi}+\bar{\lambda}\pi}{F''} \\ &- i\sqrt{k}\frac{\lambda\bar{\pi}+\bar{\lambda}\pi}{a} - ja^3F'^2\frac{\lambda\bar{\lambda}}{F''}, \end{aligned} \tag{6.10}$$

and the complex conjugate supersymmetric constraints

$$\bar{S} = \bar{\lambda}\Big(iap_a - i\frac{F'p_s}{F''} + j\sqrt{k}a^2F' - 2ja^3sF' + 3ja^3F\Big) + \bar{\pi}\Big(\frac{1}{j}\frac{p_s}{F''a^3} - i\frac{\sqrt{k}}{a} + is\Big) - 2i\lambda\bar{\lambda}\bar{\pi}, \tag{6.11a}$$

$$-S = \lambda\Big(iap_a - i\frac{F'p_s}{F''} - j\sqrt{k}a^2F' + 2ja^3sF' - 3ja^3F\Big) + \pi\Big(-\frac{1}{j}\frac{p_s}{F''a^3} - i\frac{\sqrt{k}}{a} + is\Big) - 2i\lambda\bar{\lambda}\pi. \tag{6.11b}$$

The basic Poisson brackets are

$$\{a, p_a\} = 1, \qquad \{s, p_s\} = 1, \tag{6.12a}$$

$$\{\lambda, \bar{\pi}\} = 1, \qquad \{\bar{\lambda}, \pi\} = -1. \tag{6.12b}$$

## 6.1 Quantization

The field variables are promoted to operators such that $\bar{\lambda} \equiv \lambda^\dagger, \bar{\pi} \equiv \pi^\dagger$, $\bar{S} \equiv S^\dagger$. The basic (anti)commutators are $[a, p_a] = i\hbar$, $[s, p_s] = i\hbar$, $[\lambda, \bar{\pi}]_+ = i\hbar$, and $[\bar{\lambda}, \pi]_+ = -i\hbar$.

In order to handle the fermionic sector we define

$$A_\pm = \lambda \pm i\pi/2\hbar, \qquad \bar{A}_\pm = \bar{\lambda} \mp i\bar{\pi}/2\hbar, \tag{6.13}$$

with all anti-commutators among them vanishing, except for

$$\{A_\pm, \bar{A}_\pm\} = \pm 1. \tag{6.14}$$

As in Section 4.2.2, the ground state $|0\rangle$ is defined such that $A_\pm|0\rangle = 0$. There are three more independent states, namely,

$$|1\rangle = \bar{A}_+|0\rangle, \tag{6.15a}$$

$$|2\rangle = \bar{A}_-|0\rangle, \tag{6.15b}$$

$$|3\rangle = \bar{A}_+\bar{A}_-|0\rangle. \tag{6.15c}$$

The set of four states is orthogonal. Assuming the ground state has a well defined norm, for the other states we have

$$\langle 0|0\rangle = \langle 1, 1\rangle = -\langle 2, 2\rangle = -\langle 3, 3\rangle. \tag{6.16}$$

Therefore, once we choose a physical state having positive norm, the negative states will have to be discarded by choosing zero coefficients. This is certainly possible, since the equations are homogeneous. A generic state is of the form

$$|\psi(a,\phi)\rangle = \psi_0(a,s)|0\rangle + \psi_1(a,s)|1\rangle + \psi_2(a,s)|2\rangle + \psi_3(a,s)|3\rangle \tag{6.17}$$

with coefficient functions defined in the configuration space of bosonic variables. We choose the usual representation $p_a = -i\hbar\partial_a$, $p_s = -i\hbar\partial_s$.

Boundary conditions on the coefficient wave-functions must be supplemented for the observable operators to be hermitian. Since the scale factor is positive semi-definite, we shall impose $\psi_i(0,\phi) = 0$. On the other hand, $s$ can take, in principle, any value. In the examples below, the wavefunction decreases sufficiently fast when $s \to \infty$, so that it is normalizable.

Solving for $\lambda, \pi$, etc., in terms of $A_\pm$ and $\bar{A}_\pm$, and using (6.15), we get the following relations

$$2\lambda|\psi\rangle = (\psi_1 - \psi_2)|0\rangle + \psi_3|1\rangle + \psi_3|2\rangle, \tag{6.18a}$$
$$2\bar{\lambda}|\psi\rangle = \psi_0|1\rangle + \psi_0|2\rangle - (\psi_1 - \psi_2)|3\rangle, \tag{6.18b}$$
$$\pi|\psi\rangle = -i\hbar\left((\psi_1 + \psi_2)|0\rangle - \psi_3|1\rangle + \psi_3|2\rangle\right), \tag{6.18c}$$
$$\bar{\pi}|\psi\rangle = i\hbar\left(\psi_0|1\rangle - \psi_0|2\rangle + (\psi_1 + \psi_2)|3\rangle\right), \tag{6.18d}$$
$$2\lambda\bar{\lambda}\pi|\psi\rangle = -i\hbar\psi_3(|1\rangle + |2\rangle), \tag{6.18e}$$
$$2\lambda\bar{\lambda}\bar{\pi}|\psi\rangle = -i\hbar\psi_0(|1\rangle + |2\rangle). \tag{6.18f}$$

As before, we shall use Weyl ordering for the quantum operators. Thus, they read

$$\begin{aligned} S = \hbar&\left(-a\partial_a - 1 + \frac{F'}{F''}\partial_s - \frac{F'''F'}{2F''^2} + \frac{ja^3}{\hbar}\left(\frac{\sqrt{k}}{a}F' - 2F's + 3F\right)\right)\lambda + 2i\lambda\bar{\lambda}\pi \\ &- i\left(s + \frac{\hbar}{ja^3F''}\partial_s - \frac{\hbar}{2ja^3}\frac{F'''}{F''^2} - \frac{\sqrt{k}}{a}\right)\pi = A\lambda + B\pi + 2i\lambda\bar{\lambda}\pi, \end{aligned} \tag{6.19a}$$

$$\begin{aligned} \bar{S} = \hbar&\left(a\partial_a + 1 - \frac{F'}{F''}\partial_s + \frac{F'''F'}{2F''^2} + \frac{ja^3}{\hbar}\left(\frac{\sqrt{k}}{a}F' - 2F's + 3F\right)\right)\bar{\lambda} - 2i\lambda\bar{\lambda}\bar{\pi} \\ &+ i\left(s - \frac{\hbar}{ja^3F''}\partial_s + \frac{\hbar}{2ja^3}\frac{F'''}{F''^2} - \frac{\sqrt{k}}{a}\right)\bar{\pi} = A'\bar{\lambda} + B'\bar{\pi} - 2i\lambda\bar{\lambda}\bar{\pi}. \end{aligned} \tag{6.19b}$$

From the quantum constraints $S|\Psi\rangle = 0 = \bar{S}|\Psi\rangle$ and the orthogonality of the states $|i\rangle$, we get, after some rearrangements, the following pair of equations for $\psi_0$,

$$0 = \left(a\partial_a - 1 + \frac{2j}{\hbar}\sqrt{k}a^2F' - \frac{3j}{\hbar}a^3(sF' - F)\right)\psi_0, \tag{6.20a}$$
$$0 = \left(\partial_s - \frac{F'''}{2F''} - \frac{j}{\hbar}a^3F''s + \frac{j}{\hbar}\sqrt{k}a^2F''\right)\psi_0. \tag{6.20b}$$

$\psi_3$ satisfies almost identical equations,

$$0 = \left(a\partial_a - 1 - \frac{2j}{\hbar}\sqrt{k}a^2F' + \frac{3j}{\hbar}a^3(sF' - F)\right)\psi_3, \tag{6.21a}$$

$$0 = \left(\partial_s - \frac{F'''}{2F''} + \frac{j}{\hbar}a^3F''s - \frac{j}{\hbar}\sqrt{k}a^2F''\right)\psi_3, \tag{6.21b}$$

whereas, for the intermediate amplitudes $\psi_1$ and $\psi_2$, we get the system of coupled equations

$$0 = \left(a\partial_a + 1 - \frac{F'}{F''}\partial_s + \frac{F'''F'}{2F''^2}\right)(\psi_1 - \psi_2) + 2\left(s - \frac{\sqrt{k}}{a}\right)(\psi_1 + \psi_2), \tag{6.22}$$

$$0 = \left(\partial_s - \frac{F'''}{2F''}\right)(\psi_1 + \psi_2) - \frac{j^2a^6F''}{2\hbar^2}\left(\frac{\sqrt{k}}{a}F' - 2F's + 3F\right)(\psi_1 - \psi_2). \tag{6.23}$$

### Examples

The coefficient wave functions of the empty $\psi_0$ and filled $\psi_3$ states are determined uniquely,

$$\psi_0(a,s) = c_0a\sqrt{|F''|}\exp\left[-\frac{j}{\hbar}a^3\left(\frac{\sqrt{k}}{a}F' + F - sF'\right)\right] \tag{6.24a}$$

$$\psi_3(a,s) = c_3a\sqrt{|F''|}\exp\left[\frac{j}{\hbar}a^3\left(\frac{\sqrt{k}}{a}F' + F - sF'\right)\right] \tag{6.24b}$$

where, $c_0, c_3$ are normalization constants.

Considering that $\langle 0|0\rangle = -\langle 1|1\rangle$, only one of the states, whose corresponding bosonic amplitude is square integrable, is chosen to be a physical state and assigned positive norm.

Integrating over the scale-factor domain,

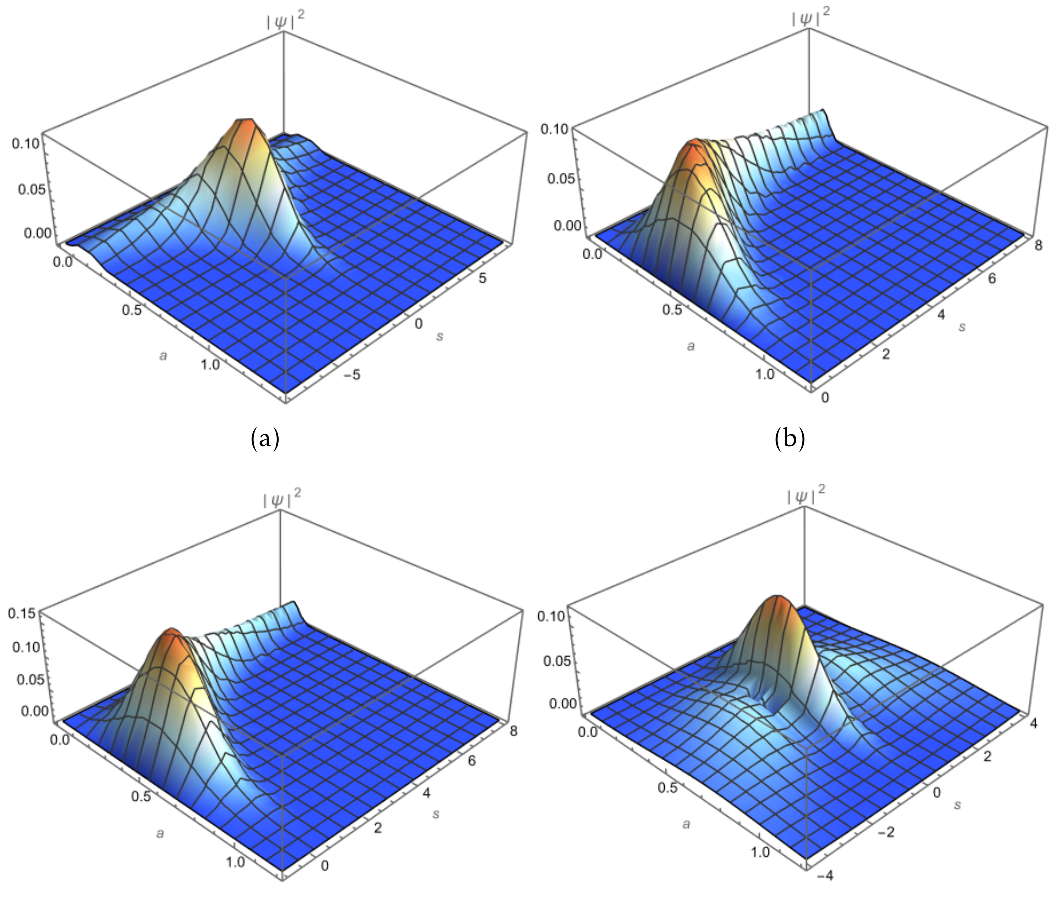
$$\int_0^\infty da|\psi_0(a,s)|^2 = \frac{\hbar}{6j}\frac{|F''|}{F - sF'} \qquad F - sF' > 0, \tag{6.25}$$

$$\int_0^\infty da|\psi_3(a,s)|^2 = -\frac{\hbar}{6j}\frac{|F''|}{F - sF'} \qquad F - sF' < 0, \tag{6.26}$$

The square integrability of the wave function depends on whether or not $F''/(F - sF')$ goes to zero sufficiently fast when $s$ approaches infinity. Depending on the function $F$, the domain of the wavefunction can be restricted to half the real axis only. Some examples of normalizable wavefunctions are listed in Table 6.1.

| | $F(s)$ | $\psi(a,s)$ | Domain |
|---|---|---|---|
| (a) | $-1+s+\frac{1}{2}s^2$ | $\psi_3 = ca\exp[-3a^3(1+\frac{1}{2}s^2)]$ | $s\in(-\infty,\infty)$ |
| (b) | $-1+s+\frac{1}{6}s^3$ | $\psi_3 = ca\sqrt{s}\exp[-3a^3(1+\frac{1}{3}s^3)]$ | $s\in[0,\infty)$ |
| (c) | $-1+s+\frac{1}{2}s^2+\frac{1}{6}s^3$ | $\psi_3 = ca\sqrt{1+s}\exp[-3a^3(1+\frac{1}{2}s^2+\frac{1}{3}s^3)]$ | $s\in[-1,\infty)$ |
| (d) | $(1+\frac{1}{2}s^2)^{-1}$ | $\psi_0 = ca\sqrt{\lvert 2-3s^2\rvert}(1+\frac{1}{2}s^2)^{-3/2}\exp[-3a^3(1+\frac{3}{2}s^2)/(1+\frac{1}{2}s^2)^2]$ | $s\in(-\infty,\infty)$ |

Table 6.1: Examples of square integrable wave-functions. The corresponding plots of $|\psi|^2$ are shown in Figure (6.1). The constant these examples acts like a $\Lambda$, without it the wavefunction is not square integrable.




(a) (b) (c) (d)

Figure 6.1: Plots of the squared magnitude of the wave functions of Table 6.1.

### Problem of time

As an application of the $F(\mathcal{R})$ action, we use an exact solution of the quantum supersymmetric equation, as an effective time wave-function following the procedure developed in [53], that consists of using the scalar field as a time parameter.

Given a solution to the WDW equation, $\psi(a,\phi)$, the time wave function is defined by $\Psi(a,t) = \psi(a,\phi)/(\int da|\psi(a,\phi)|^2)^{1/2}|_{\phi=t}$. The resulting wavefunction is normalized and satisfies a continuity equation.

For our $F(\mathcal{R})$ action, such a time wave function would be of the form

$$\Psi(a,t) = \sqrt{\frac{6j}{\hbar}}a\sqrt{|F(t)-tF'(t)|}\exp\left[-\frac{j}{\hbar}a^3|F(t)-tF'(t)|\right] \tag{6.27}$$

The mean value of the scale factor is, therefore,

$$\langle a\rangle(t) = \left(\frac{\hbar}{2j}\right)^{\frac{1}{3}}\Gamma(4/3)|tF'(t)-F(t)|^{-\frac{1}{3}} \tag{6.28}$$

For definiteness, we set $\hbar = 1$, $j = 3$. As an example we consider the following function

$$F(t) = \left(l+\frac{1}{2}t^2\right)^{-1} + \exp\left[-\left(1+\frac{1}{2}t^2\right)^{-1}\right] \tag{6.29}$$

Since the quantity $F - tF'$ is positive definite and finite over the real axis, we must use $\psi_0$. On the other hand, $F''$ remains finite but crosses the real axis (twice, since it is an even function), thus, it changes sign and the wavefunction must be adjusted according to the region. Anyway, the effective wavefunction does not depend on $F''$.

The squared magnitude of the time wavefunction is shown in Figure 6.2. The mean value of the scale factor, given by (6.28), as well as the associated co-moving Hubble radius, are shown in Figure 6.3. The latter exhibits the typical behavior during an inflationary period. By choosing $l = 10^{-70}$ we can obtain, from $t = 0$ to $t = 10$, a total number of e-folds $n \approx 53$.

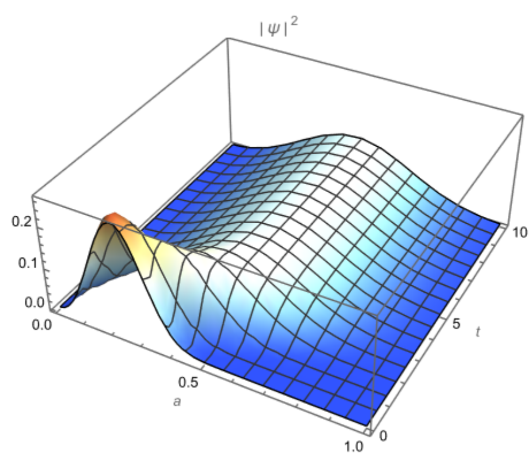
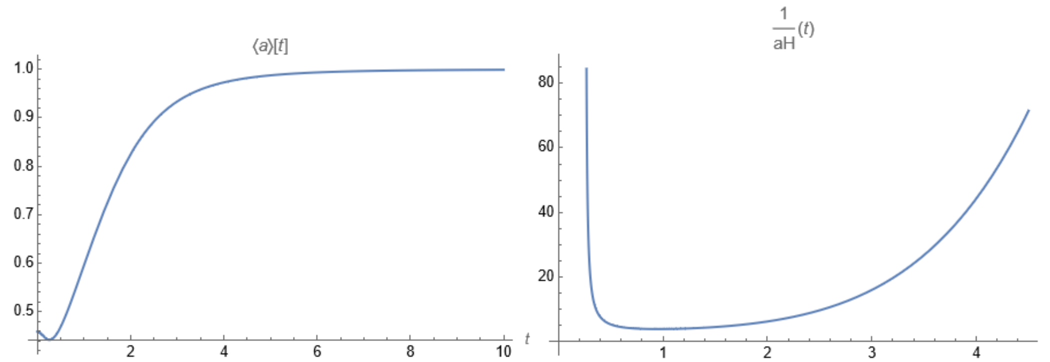



Figure 6.2: Plot of $\psi^2(a,t)$, for the effective wavefunction with time-dependence. We take $\hbar = 1$, $j = 3$ and $l = 0.1$. The plot is symmetric with respect to the axis $t = 0$.

That end this section, which is only an early study of the quantized $F(\mathcal{R})$ theory, to show its potential applications in quantum cosmology despite not corresponding to a direct generalization of the purely bosonic $f(R)$ theory, as discussed at the beginning of this chapter. In the remaining, we discuss some other aspects of the action at the classical level.




(a) (b)

Figure 6.3: (a) Mean value of the scale-factor for the model derived from (6.29), (b) shows the corresponding Hubble radius.

## 6.2 Scalar-tensor formulation 1

Consider the superfield analogue of (5.12),

$$L_F^{\Phi} = j\mathcal{E}\mathcal{A}^3\Big((\mathcal{R}-\Phi)F'(\Phi)+F(\Phi)\Big) \tag{6.30}$$

Variation with respect to $\Phi$ yields the superfield equation of motion, $F''(\Phi)(\mathcal{R}-\Phi)$. Thus, assuming $F$ is not a constant, $\Phi=\mathcal{R}$, and we recover the $F(\mathcal{R})$ action (6.4).

Clearly, the component Lagrangian will not depend on time-derivatives of the matter fields, $\phi,\eta,\bar\eta,G$. Further, $G$ appears as a Lagrange multiplier enforcing the constraint $\phi=s$. Thus, eliminating $G$, we get

$$\begin{aligned} L_F^{\phi} = jNa^3\Big[&F'(\phi)\Big(-\frac{\dot N\dot a}{N^3a}+\frac{\ddot a}{N^2a}+\frac{\dot a^2}{N^2a^2}+\frac{k}{a^2}+2\phi^2-3\frac{\sqrt{k}}{a}\phi+i\frac{\lambda\dot{\bar\lambda}+\bar\lambda\dot\lambda}{N}-i\frac{\dot\psi\bar\lambda+\dot{\bar\psi}\lambda}{N}\\ &-\dot a\frac{\psi\bar\lambda+\bar\psi\lambda}{Na}i+2\psi\bar\psi\lambda\bar\lambda-2\sqrt{k}\frac{\psi\bar\lambda-\bar\psi\lambda}{a}+4\lambda\bar\lambda\phi-\sqrt{k}\frac{\lambda\bar\lambda}{a}-\frac{\psi\dot{\bar\lambda}+\bar\psi\dot\lambda}{N}i+3(\psi\bar\lambda-\bar\psi\lambda)\phi\Big)\\ &+F''(\phi)\Big(-\frac{\dot\lambda\bar\eta+\dot{\bar\lambda}\eta}{N}i-2i\dot a\frac{\lambda\bar\eta+\bar\lambda\eta}{Na}+i\dot a\frac{\psi\bar\eta+\bar\psi\eta}{Na}+\psi\bar\psi(\lambda\bar\eta-\bar\lambda\eta)-\sqrt{k}\frac{\psi\bar\eta-\bar\psi\eta}{a}+\eta\bar\eta\\ &+2(\lambda\bar\eta-\bar\lambda\eta)\phi-\sqrt{k}\frac{\lambda\bar\eta-\bar\lambda\eta}{a}+(\psi\bar\eta-\bar\psi\eta)\phi\Big)-3F(\phi)\Big(\phi-\frac{\sqrt{k}}{a}+\lambda\bar\lambda+\psi\bar\lambda-\bar\psi\lambda\Big)\Big] \end{aligned} \tag{6.31}$$

We could easily integrate out $\eta$, $\bar\eta$, to recover the original theory (6.5), but we already know what happens in that formulation. Instead, we integrate by parts the first term on the third line of (6.31) to introduce a time derivative for $\eta$ (this is only to get similar expressions for $\lambda$ and $\eta$). Also, we integrate by parts to get rid of $\ddot{a}$, $\dot{N}$ and $\dot{\psi}$. The result is,

$$L_F^{\phi} = jNa^3\Bigg[F'(\phi)\Bigg(-\frac{\dot{a}^2}{N^2a^2}+\frac{k}{a^2}-3\sqrt{k}\frac{\phi}{a}+2\phi^2+i\frac{\lambda\dot{\bar{\lambda}}+\bar{\lambda}\dot{\lambda}}{N}+2\dot{a}\frac{\psi\bar{\lambda}+\bar{\psi}\lambda}{Na}i+2\psi\bar{\psi}\lambda\bar{\lambda}$$
$$-2\sqrt{k}\frac{\psi\bar{\lambda}-\bar{\psi}\lambda}{a}+4\lambda\bar{\lambda}\phi-\sqrt{k}\frac{\lambda\bar{\lambda}}{a}+3\phi(\psi\bar{\lambda}-\bar{\psi}\lambda)\Bigg)+F''(\phi)\Bigg(-\frac{\dot{a}\dot{\phi}}{N^2a}+i\frac{\lambda\dot{\bar{\eta}}+\bar{\lambda}\dot{\eta}}{N}$$
$$+i\dot{a}\frac{\lambda\bar{\eta}+\bar{\lambda}\eta}{Na}+i\dot{a}\frac{\psi\bar{\eta}+\bar{\psi}\eta}{Na}+\psi\bar{\psi}(\lambda\bar{\eta}-\bar{\lambda}\eta)+\eta\bar{\eta}-\sqrt{k}\frac{\psi\bar{\eta}-\bar{\psi}\eta}{a}+2\phi(\lambda\bar{\eta}-\bar{\lambda}\eta)$$
$$-\sqrt{k}\frac{\lambda\bar{\eta}-\bar{\lambda}\eta}{a}+i\dot{\phi}\frac{\psi\bar{\lambda}+\bar{\psi}\lambda}{N}+\phi(\psi\bar{\eta}-\bar{\psi}\eta)\Bigg)-3F(\phi)\Bigg(\phi-\frac{\sqrt{k}}{a}+\lambda\bar{\lambda}+\psi\bar{\lambda}-\bar{\psi}\lambda\Bigg)$$
$$+iF'''(\phi)\dot{\phi}\frac{\lambda\bar{\eta}+\bar{\lambda}\eta}{N}\Bigg]. \qquad (6.32)$$

To simplify even further the fermionic kinetic terms, we make the following re-scaling

$$\lambda \to a^{-3/2}F'^{-1/2}\lambda, \qquad \eta \to \sqrt{F'}F''^{-1}a^{-3/2}\eta, \qquad (6.33)$$

and we finally get

$$L_F^{\phi} = j\Bigg[-F'\frac{a\dot{a}^2}{N}-F''a^2\frac{\dot{a}\dot{\phi}}{N}+F'kNa-3Na^3F\phi-3\sqrt{k}Na^2F'\phi+3\sqrt{k}NFa^2+2Na^3F'\phi^2$$
$$+i(\lambda\dot{\bar{\lambda}}+\bar{\lambda}\dot{\lambda})+i(\lambda\dot{\bar{\eta}}+\bar{\lambda}\dot{\eta})-i\dot{a}\frac{\lambda\bar{\eta}+\bar{\lambda}\eta}{2a}+i\dot{\phi}F''\frac{\lambda\bar{\eta}+\bar{\lambda}\eta}{2F'}+i\dot{a}\sqrt{F'a}(\psi\bar{\eta}+\bar{\psi}\eta)+4N\phi\lambda\bar{\lambda}$$
$$+N\psi\bar{\psi}(\lambda\bar{\eta}-\bar{\lambda}\eta)-\sqrt{k}N\sqrt{F'a}(\psi\bar{\eta}-\bar{\psi}\eta)+2N\phi(\lambda\bar{\eta}-\bar{\lambda}\eta)-N\sqrt{k}\frac{\lambda\bar{\eta}-\bar{\lambda}\eta}{a}-3N\frac{F}{F'}\lambda\bar{\lambda}$$
$$+iF''\dot{\phi}a^{\frac{3}{2}}\frac{\psi\bar{\lambda}+\bar{\psi}\lambda}{\sqrt{F'}}+2i\sqrt{F'a}\dot{a}(\psi\bar{\lambda}+\bar{\psi}\lambda)+2N\psi\bar{\psi}\lambda\bar{\lambda}-2\sqrt{k}N\sqrt{F'a}(\psi\bar{\lambda}-\bar{\psi}\lambda)+NF'\frac{\eta\bar{\eta}}{F''}$$
$$-N\frac{\sqrt{k}}{a}\lambda\bar{\lambda}+Na^{\frac{3}{2}}\phi\sqrt{F'}(\psi\bar{\eta}-\bar{\psi}\eta)+3Na^{\frac{3}{2}}\phi\sqrt{F'}(\psi\bar{\lambda}-\bar{\psi}\lambda)-3Na^{\frac{3}{2}}F\frac{\psi\bar{\lambda}-\bar{\psi}\lambda}{\sqrt{F'}}\Bigg] \qquad (6.34)$$

Performing the Legendre transformation, we get

$$H_0 = -\frac{1}{j}\frac{p_a p_\phi}{F''a^2}+\frac{1}{j}\frac{F'p_\phi^2}{F''^2a^3}-jkF'a+ja^3\phi(3F-2\phi F')+ip_a\frac{\lambda\bar{\eta}+\bar{\lambda}\eta}{2F'a^2}-3ip_\phi\frac{\lambda\bar{\eta}+\bar{\lambda}\eta}{2F''a^3}$$
$$-j\frac{\lambda\bar{\lambda}\eta\bar{\eta}}{F'a^3}-2j\phi(\lambda\bar{\eta}-\bar{\lambda}\eta)+j\sqrt{k}\frac{\lambda\bar{\eta}-\bar{\lambda}\eta}{a}-4j\phi\lambda\bar{\lambda}+j\sqrt{k}\frac{\lambda\bar{\lambda}}{a}-jF'\frac{\eta\bar{\eta}}{F''}$$
$$+3j\sqrt{k}a^2\phi F'-3j\sqrt{k}a^2F+3jF\frac{\lambda\bar{\lambda}}{F'}, \qquad (6.35)$$

as well as,

$$\bar{S} = \frac{ip_a}{\sqrt{F'a}}\bar{\lambda} + \frac{ip_\phi\sqrt{F'}}{F''a^{3/2}}\bar{\eta} + \frac{j}{2}\frac{\bar{\lambda}\eta\bar{\eta}}{\sqrt{F'}a^{3/2}} + \frac{j}{2}\frac{\lambda\bar{\lambda}\bar{\eta}}{\sqrt{F'}a^{3/2}} + 2j\sqrt{k}\sqrt{F'a}\bar{\lambda} + j\sqrt{k}\sqrt{F'a}\bar{\eta}$$
$$- ja^{\frac{3}{2}}\phi\sqrt{F'}\bar{\eta} - 3ja^{\frac{3}{2}}\phi\sqrt{F'}\bar{\lambda} + 3j\frac{a^{\frac{3}{2}}F}{\sqrt{F'}}\bar{\lambda} \tag{6.36a}$$

$$-S = \frac{ip_a}{\sqrt{F'a}}\lambda + \frac{ip_\phi\sqrt{F'}}{F''a^{3/2}}\eta - \frac{j}{2}\frac{\lambda\eta\bar{\eta}}{\sqrt{F'}a^{3/2}} - \frac{j}{2}\frac{\lambda\bar{\lambda}\eta}{\sqrt{F'}a^{3/2}} - 2j\sqrt{k}\sqrt{F'a}\lambda - j\sqrt{k}\sqrt{F'a}\eta$$
$$+ ja^{\frac{3}{2}}\phi\sqrt{F'}\eta + 3ja^{\frac{3}{2}}\phi\sqrt{F'}\lambda - 3j\frac{a^{\frac{3}{2}}F}{\sqrt{F'}}\lambda. \tag{6.36b}$$

We have the following second-class constraints

$$C_\lambda = \pi_\lambda + ij\lambda = 0, \qquad C_{\bar{\lambda}} = \pi_{\bar{\lambda}} - ij\bar{\lambda} = 0, \tag{6.37}$$
$$C_\eta = \pi_\eta + ij\lambda = 0, \qquad C_{\bar{\eta}} = \pi_{\bar{\eta}} - ij\bar{\lambda} = 0. \tag{6.38}$$

Basic Poisson brackets are

$$\{a, p_a\} = 1, \qquad \{\varphi, p_\varphi\} = 1, \tag{6.39}$$
$$\{\lambda, \pi_{\bar{\lambda}}\} = 1 \qquad \{\bar{\lambda}, \pi_\lambda\} = -1, \tag{6.40}$$
$$\{\eta, \pi_{\bar{\eta}}\} = 1 \qquad \{\bar{\eta}, \pi_\eta\} = -1, \tag{6.41}$$

whereas the matrix of Poisson brackets between the second-class constraints is

$$C = \begin{Bmatrix} \{C_\lambda, C_\lambda\} & \{C_\lambda, C_{\bar{\lambda}}\} & \{C_\lambda, C_\eta\} & \{C_\lambda, C_{\bar{\eta}}\} \\ \{C_{\bar{\lambda}}, C_\lambda\} & \{C_{\bar{\lambda}}, C_{\bar{\lambda}}\} & \{C_{\bar{\lambda}}, C_\eta\} & \{C_{\bar{\lambda}}, C_{\bar{\eta}}\} \\ \{C_\eta, C_\lambda\} & \{C_\eta, C_{\bar{\lambda}}\} & \{C_\eta, C_\eta\} & \{C_\eta, C_{\bar{\eta}}\} \\ \{C_{\bar{\eta}}, C_\lambda\} & \{C_{\bar{\eta}}, C_{\bar{\lambda}}\} & \{C_{\bar{\eta}}, C_\eta\} & \{C_{\bar{\eta}}, C_{\bar{\eta}}\} \end{Bmatrix} = j\begin{Bmatrix} 0 & 2i & 0 & i \\ 2i & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{Bmatrix} \tag{6.42}$$

Basic non-vanishing Dirac brackets are

$$\{a, p_a\}_D = 1, \qquad \{\phi, p_\phi\}_D = 1, \tag{6.43a}$$
$$\{\lambda, \bar{\eta}\}_D = -ij^{-1}, \qquad \{\bar{\lambda}, \eta\}_D = -ij^{-1}, \tag{6.43b}$$
$$\{\eta, \bar{\eta}\}_D = 2ij^{-1} \tag{6.43c}$$

Note that $\{\lambda, \bar{\lambda}\}_D = 0$.

With these results, it is not evident that the quantization based on this Hamiltonian formulation yields the same results as those of Section 6.1, since both the algebra of Dirac brackets and the $S$'s look so different. To see that we get equivalent wave functions, without actually solving the quantum equations, it suffices to re-write the results of this section in terms of the following set of canonical variables,

$$a = a, \qquad P_a = p_a - 3ij\frac{\lambda'\bar{\eta}' + \bar{\lambda}'\eta'}{2a}, \tag{6.44a}$$

$$\phi = \phi, \qquad P_\phi = p_\phi - ijF''\frac{\lambda'\bar{\eta}' + \bar{\lambda}'\eta'}{2F'}, \tag{6.44b}$$

$$\lambda = a^{-3/2}F'^{-1/2}\lambda', \qquad \Pi = -ija^{3/2}\sqrt{F'}(\lambda' + \eta'), \tag{6.44c}$$

$$\eta = \sqrt{F'}F''^{-1}a^{-3/2}\eta', \qquad \bar{\Pi} = ija^3\left(F'\bar{\lambda} + F''\bar{\eta}\right). \tag{6.44d}$$

For these, we get the non-vanishing Dirac brackets

$$\{a, P_a\}_D = 1, \qquad \{\phi, p_\phi\}_D = 1, \tag{6.45a}$$

$$\{\lambda, \bar{\pi}\}_D = 1, \qquad \{\bar{\lambda}, \pi\}_D = -1, \tag{6.45b}$$

Thus, we get an algebra of Dirac brackets isomorphic to the algebra of Poisson brackets (6.12). Moreover, re-writing the Hamiltonian and supersymmetric constraints (6.35), (6.36), respectively, in terms of these new variables, we also recover the expressions (6.10) and (6.11) of the original higher-order theory (6.5).

## 6.3 Scalar-tensor formulation 2

In this last section we show how to write the $F(\mathcal{R})$ action as an ordinary supersymmetric Einstein gravity-matter action. For simplicity, we restrict to $k = 0$.

Integrating by parts (6.31), we get

$$\begin{aligned}
L = jNa^3\Bigg[&-\frac{\dot{a}^2F'}{N^2a^2} + 2F'\phi^2 - 3F\phi - \frac{\dot{a}\dot{\phi}F''}{N^2a} - 3F(\psi\bar{\lambda} - \bar{\psi}\lambda) - 3F\lambda\bar{\lambda} \\
&+F'\left(i\frac{\lambda\dot{\bar{\lambda}} + \bar{\lambda}\dot{\lambda}}{N} + \frac{2i\dot{a}}{Na}(\psi\bar{\lambda} + \bar{\psi}\lambda) + 2\psi\bar{\psi}\lambda\bar{\lambda} + 4\lambda\bar{\lambda}\phi + 3\phi(\psi\bar{\lambda} - \bar{\psi}\lambda)\right) \\
&+F''\Bigg(2\phi(\lambda\bar{\eta} - \bar{\lambda}\eta) + \eta\bar{\eta} - i\frac{\dot{\lambda}\bar{\eta} + \dot{\bar{\lambda}}\eta}{N} + \frac{i\dot{a}}{Na}(\psi\bar{\eta} + \bar{\psi}\eta) + \psi\bar{\psi}(\lambda\bar{\eta} - \bar{\lambda}\eta) \\
&\qquad -\frac{2i\dot{a}}{Na}(\lambda\bar{\eta} + \bar{\lambda}\eta) + \frac{i\dot{\phi}}{N}(\psi\bar{\lambda} + \bar{\psi}\lambda) + \phi(\psi\bar{\eta} - \bar{\psi}\eta)\Bigg)\Bigg].
\end{aligned} \tag{6.46}$$

The superfield generalization of the Weyl-Legendre transformation (5.13)-5.16 is

$$\mathcal{A} = e^{-\frac{c}{2}\Gamma}\mathcal{B}, \tag{6.47a}$$

$$\mathcal{E} = e^{-\frac{c}{2}\Gamma}\mathcal{D}, \tag{6.47b}$$

$$F'(\Phi) = e^{c\Gamma}. \tag{6.47c}$$

with three new superfields: $\mathcal{B} = \tilde{a}(1 + i\Theta\bar{\rho} + i\bar{\Theta}\rho + \ldots)$, $\mathcal{D} = \tilde{N}(1 + i\Theta\bar{\delta} + i\bar{\Theta}\delta) + \ldots$, and $\Gamma = \varphi + i\Theta\bar{\gamma} + i\bar{\Theta}\gamma + \ldots$.

The transformation of the components is

$$N = e^{-\frac{c}{2}\varphi}\tilde{N}, \qquad \psi = \delta - c\gamma/2, \tag{6.48a}$$

$$a = e^{-\frac{c}{2}\varphi}\tilde{a}, \qquad \lambda = \rho - c\gamma/2, \tag{6.48b}$$

$$F'(\phi) = e^{c\varphi}, \qquad \eta = ce^{c\varphi}[F''(\phi)]^{-1}\gamma. \tag{6.48c}$$

Deriving the l.h.s., of (6.48c) with respect to $\varphi$, yields

$$F''(\phi) = ce^{c\varphi}\left(\frac{d\phi}{d\varphi}\right)^{-1} \equiv ce^{c\varphi}l^{-1}(\varphi), \tag{6.49}$$

that is,

$$\eta = l(\varphi)\gamma. \tag{6.50}$$

Performing this transformation on the Lagrangian (6.46), with $c^2 = 2\kappa^2/3$, leads, after relabeling $\tilde{a} \to a$, $\tilde{N} \to N$, $\delta \to \psi$, $\rho \to \lambda$ and $\gamma \to \eta$, to

$$\begin{aligned}
L_F^{\varphi} = jNa^3\Big[&\frac{-\dot{a}^2}{N^2a^2} + \frac{\dot{\varphi}^2}{2jN^2} - e^{-c\varphi}\phi\Big(3e^{-c\varphi}F(\phi) - 2\phi\Big) + e^{-c\varphi/2}\Big(i\frac{\lambda\dot{\bar{\lambda}} + \bar{\lambda}\dot{\lambda}}{N} - i\frac{\eta\dot{\bar{\eta}} + \bar{\eta}\dot{\eta}}{2jN}\\
&-2i\dot{a}\frac{\lambda\bar{\psi} + \bar{\lambda}\psi}{Na} + 2i\dot{\varphi}\frac{\lambda\bar{\eta} + \bar{\lambda}\eta}{jN} - ic\dot{a}\frac{\lambda\bar{\eta} + \bar{\lambda}\eta}{Na} - ic\frac{\dot{\lambda}\bar{\eta} + \dot{\bar{\lambda}}\eta}{2N} - ic\frac{\lambda\dot{\bar{\eta}} + \bar{\lambda}\dot{\eta}}{2N} - i\dot{\varphi}\frac{\psi\bar{\eta} + \bar{\psi}\eta}{jN}\Big)\\
&+e^{-c\varphi}\Big(2\lambda\bar{\lambda}\psi\bar{\psi} + 4\phi\lambda\bar{\lambda} - c\lambda\bar{\lambda}(\psi\bar{\eta} - \bar{\psi}\eta) + 3\phi(\lambda\bar{\psi} - \bar{\lambda}\psi) - \frac{c}{2}\phi(\psi\bar{\eta} - \bar{\psi}\eta) + cl(\varphi)\eta\bar{\eta}\\
&-3c\frac{\phi}{2}(\lambda\bar{\eta} - \bar{\lambda}\eta) + \frac{\lambda\bar{\lambda}}{j}\eta\bar{\eta} - \frac{\psi\bar{\psi}}{j}\eta\bar{\eta} - \frac{\phi}{j}\eta\bar{\eta}\Big) - 3e^{-2c\varphi}F(\phi)\Big(\lambda\bar{\lambda} - c(\lambda\bar{\eta} - \bar{\lambda}\eta)\\
&+\frac{3}{2j}\eta\bar{\eta} + \lambda\bar{\psi} - \bar{\lambda}\psi - \frac{c}{2}(\psi\bar{\eta} - \bar{\psi}\eta)\Big)\Big].
\end{aligned} \tag{6.51}$$

A further re-scaling $\lambda \to e^{\frac{c}{4}\varphi}\lambda, \eta \to e^{\frac{c}{4}\varphi}\eta$, yields standard fermionic kinetic terms

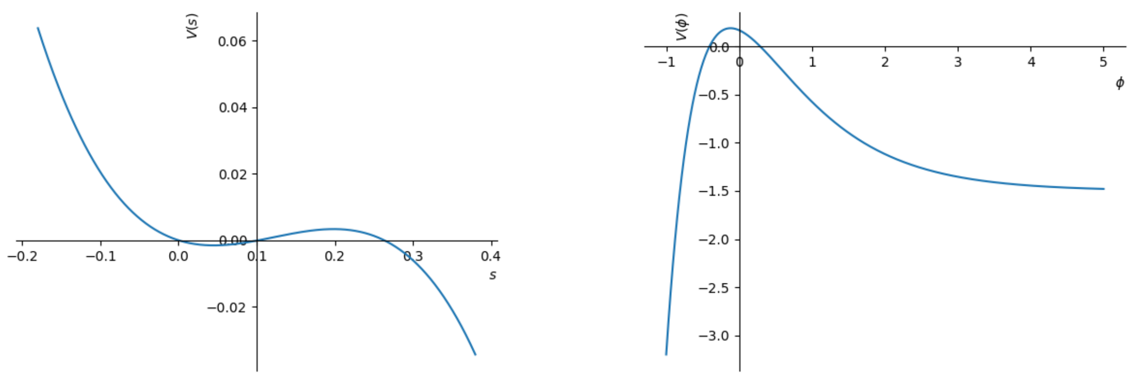
$$\begin{aligned}
L_F^{\varphi} = jNa^3\Big[&\frac{-\dot{a}^2}{N^2a^2} + \frac{\dot{\varphi}^2}{2jN^2} - e^{-c\varphi}\phi\Big(3e^{-c\varphi}F(\phi) - 2\phi\Big) + i\frac{\lambda\dot{\bar{\lambda}} + \bar{\lambda}\dot{\lambda}}{N} - i\frac{\eta\dot{\bar{\eta}} + \bar{\eta}\dot{\eta}}{2jN} + \frac{\lambda\bar{\lambda}\eta\bar{\eta}}{j}\\
&-ic\frac{\lambda\dot{\bar{\eta}} + \bar{\lambda}\dot{\eta}}{2N} - ic\frac{\dot{\lambda}\bar{\eta} + \dot{\bar{\lambda}}\eta}{2N} + 3i\dot{\varphi}\frac{\lambda\bar{\eta} + \bar{\lambda}\eta}{2jN} - ic\dot{a}\frac{\lambda\bar{\eta} + \bar{\lambda}\eta}{Na} - 3e^{-7c\varphi/4}F(\phi)\Big(\lambda\bar{\psi} - \bar{\lambda}\psi\\
&-\frac{c}{2}(\psi\bar{\eta} - \bar{\psi}\eta)\Big) - 3e^{-3c\varphi/2}F(\phi)\Big(3\frac{\eta\bar{\eta}}{2j} - c(\lambda\bar{\eta} - \bar{\lambda}\eta) + \lambda\bar{\lambda}\Big) + e^{-3c\varphi/4}\Big(3\phi(\lambda\bar{\psi} - \bar{\lambda}\psi)\\
&-\frac{c}{2}\phi(\psi\bar{\eta} - \bar{\psi}\eta)\Big) + e^{-c\varphi/2}\Big(4\phi\lambda\bar{\lambda} + cl(\varphi)\eta\bar{\eta} - \frac{\psi\bar{\psi}\eta\bar{\eta}}{j} - \frac{3}{2}c\phi(\lambda\bar{\eta} - \bar{\lambda}\eta) + 2\lambda\bar{\lambda}\psi\bar{\psi}\\
&-\frac{\phi}{j}\eta\bar{\eta}\Big) - e^{-c\varphi/4}\Big(2i\dot{a}\frac{\lambda\bar{\psi} + \bar{\lambda}\psi}{Na} + c\lambda\bar{\lambda}(\psi\bar{\eta} - \bar{\psi}\eta) + i\dot{\varphi}\frac{\psi\bar{\eta} + \bar{\psi}\eta}{jN}\Big)\Big].
\end{aligned} \tag{6.52}$$

From this, we identify the scalar potential

$$V(\varphi) = j\left(3F\phi e^{-2c\varphi} - 2\phi^2 e^{-c\varphi}\right)\Big|_{\phi(\varphi)} \tag{6.53}$$

where $\phi = \phi(\varphi)$.

For the examples discussed below (6.6), we get $V_1(\varphi) = \frac{3M^2}{8\kappa^2}\left(-e^{\beta\varphi} + 5 - 7e^{-\beta\varphi} + 3e^{-2\beta\varphi}\right)$, $V_2(\varphi) = \frac{M^2}{2\kappa^2}\left(-\frac{3}{2} + 3e^{-\beta\varphi} - \frac{4}{3}e^{-2\beta\varphi}\right)$. The profile of these potentials, shown in Fig. 5, are in accord with the conclusion there that their corresponding $f(R)$-actions have little to do with inflation.




(a) (b)

Figure 6.4: (a) Scalar potential in the Einstein frame associated to $F^1(\mathcal{R}) = \mathcal{R} + \alpha^{\frac{1}{2}}\mathcal{R}^2$. (b) Scalar potential for $F^2(\mathcal{R}) = \frac{2}{3}(\mathcal{R} + 2\alpha\mathcal{R}^3)$. From its corresponding $f(R)$ it is no surprise that its profile corresponds to a negative sign Starobinsky potential plus positive cosmological constant.

# Chapter 7

# Higher-derivative theories: The model of Starobinsky

In this last chapter, we put forward two supersymmetric extensions of the model of Starobinsky with N=1 and N=2 supersymmetry [56]. We perform the Hamiltonian formulation of their equivalent second-order formulations and discuss their quantization.

## 7.1 N=1 action

The basic features of the N=1 superspace are summarized in Section 3.5. We recall here that it has local coordinates $(t,\Theta)$, where $\Theta$ is a real odd parity Grassmann number.

We define the real scale factor superfield as follows

$$\mathcal{A}(t,\Theta) = a(t)\Big[1 + i\Theta\lambda(t)\Big]. \tag{7.1}$$

where we use an expansion analogous to (6.1). From (3.58), the transformation of the components under local supersymmetry is

$$\delta_\zeta a = -i\zeta a\lambda, \qquad \delta_\zeta \lambda = \zeta\left(\frac{\dot{a}}{aN} - i\psi\lambda\right). \tag{7.2}$$

As with the N=2 case, we define the $k=0$ curvature superfield[7]

$$\mathcal{R}[\mathcal{A}] = i\mathcal{A}^{-1}\nabla_\tau\nabla_\Theta\mathcal{A} + i\mathcal{A}^{-2}\nabla_\tau\mathcal{A}\nabla_\Theta\mathcal{A}, \tag{7.3}$$

where covariant derivatives are given in (3.56). Thus, we have

$$\mathcal{R}(t,\Theta) = -2\frac{\dot{a}\lambda}{Na} - \frac{\dot{\lambda}}{N} - \frac{\dot{a}\psi}{Na} + \Theta\left(\frac{\ddot{a}}{N^2a} - \frac{\dot{N}\dot{a}}{N^3a} + \frac{\dot{a}^2}{N^2a^2} + i\frac{\lambda\dot{\psi}}{N} - 6i\frac{\dot{a}\psi\lambda}{Na} - 2i\frac{\psi\dot{\lambda}}{N} + 2i\frac{\lambda\dot{\lambda}}{N}\right). \tag{7.4}$$

[7] Non-vanishing spatial curvature can be introduced via an interaction with a Goldstino field $\beta(t)$. Specifically, we add to the lagrangian density in (7.5) a term $-k\mathcal{E}\mathcal{A}\mathcal{B}$, where $\mathcal{B} = \beta + \Theta\left(-1 + i\beta\psi + i\beta N^{-1}\dot{\beta}\right)$ is a Goldstino superfield [25, 48].

The ordinary supersymmetric FRW action follows from the Lagrangian density $\mathcal{L}_1 = \mathcal{E}\mathcal{A}^3\mathcal{R}$, where $\mathcal{E}$ is the N=1 density superfield given in (3.59). From this we obtain the following component Lagrangian

$$L_1 = \frac{3}{\kappa^2} N a^3 \left[ \frac{\ddot{a}}{N^2 a} - \frac{\dot{N}\dot{a}}{N^3 a} + \frac{\dot{a}^2}{N^2 a^2} + i\frac{\lambda\dot{\psi}}{N} - i\frac{\dot{a}}{Na}\psi\lambda - i\frac{\psi\dot{\lambda}}{N} - i\frac{\lambda\dot{\lambda}}{N} \right]. \tag{7.5}$$

Since $\mathcal{R}$ is an anti-commuting superfield ($\mathcal{R}^2 = 0$), we cannot construct higher-order polynomials as we did for the N=2 curvature superfield in Chapter 6. Nonetheless, it is possible to construct an exact supersymmetric model of Starobinsky by considering the product of $\mathcal{R}$ with its fermionic covariant derivative.

We write the supersymmetric Starobinsky Lagrangian in the form $L_S = L_1 + \alpha L_2$, where $L_2$ is contribution from

$$\mathcal{L}_2 = \frac{3}{\kappa^2} \mathcal{E}\mathcal{A}^3 \mathcal{R} \nabla_\Theta \mathcal{R}. \tag{7.6}$$

Integrating over $\Theta$, we get the total Lagrangian

$$\begin{aligned} L_S = \frac{3}{\kappa^2} N a^3 \Bigg[ & \frac{\ddot{a}}{N^2 a} - \frac{\dot{N}\dot{a}}{N^3 a} + \frac{\dot{a}^2}{N^2 a^2} + \alpha \left( \frac{\ddot{a}}{N^2 a} - \frac{\dot{N}\dot{a}}{N^3 a} + \frac{\dot{a}^2}{N^2 a^2} \right)^2 - i\frac{\lambda\dot{\lambda}}{N} + i\frac{\lambda\dot{\psi}}{N} - i\frac{\psi\dot{\lambda}}{N} \\ & - i\frac{\dot{a}\psi\lambda}{Na} + \alpha \Bigg( i\frac{\dot{\lambda}\ddot{\lambda}}{N^3} - i\frac{\ddot{a}\lambda\dot{\lambda}}{N^3 a} - 8i\frac{\dot{a}\ddot{a}\psi\lambda}{N^3 a^2} - 9i\frac{\dot{a}^3\psi\lambda}{N^3 a^3} + 7i\frac{\dot{a}^2\lambda\dot{\lambda}}{N^3 a^2} + 2i\frac{\dot{a}\lambda\ddot{\lambda}}{N^3 a} - i\frac{\dot{N}\dot{a}\lambda\dot{\lambda}}{N^4 a} \\ & - i\frac{\ddot{a}\dot{a}\psi\lambda}{N^3 a^2} + 4i\frac{\dot{N}\dot{a}}{N^4 a}\psi\dot{\lambda} - 4i\frac{\ddot{a}\psi\dot{\lambda}}{N^3 a} - i\frac{\dot{a}^2\psi\dot{\lambda}}{N^3 a^2} + 4\psi\lambda\frac{\dot{\psi}\dot{\lambda}}{N^2} - i\frac{\dot{a}\dot{\psi}\dot{\lambda}}{N^3 a} - i\frac{\ddot{a}\psi\dot{\lambda}}{N^3 a} + 4i\frac{\dot{a}^2\lambda\dot{\psi}}{N^3 a^2} \\ & + 9i\frac{\dot{N}\dot{a}^2\psi\lambda}{N^4 a^2} + i\frac{\dot{a}\psi\ddot{\lambda}}{N^3 a} + i\frac{\dot{a}^2\psi\dot{\psi}}{N^3 a^2} - 2i\frac{\dot{N}\dot{a}}{N^4 a}\lambda\dot{\psi} + 2i\frac{\ddot{a}\lambda\dot{\psi}}{N^3 a} \Bigg) \Bigg] \end{aligned} \tag{7.7}$$

Because of the term quadratic in $\ddot{a}$, the scale factor satisfies the expected fourth-order equation of motion, now with fermionic contributions. On the other hand, the term $\dot{\lambda}\ddot{\lambda}$ yields a third-order equation of motion arising from the generalized Lagrange equation of motion $(d^2/dt^2)(\partial L/\partial\ddot{\lambda}) + \ldots$. Therefore, there is a tripling of fermionic degrees of freedom compared to the ordinary case, as we require not only the initial value of $\lambda$, but also of its first two time-derivatives.

### 7.1.1 Scalar-tensor formulation

As for the bosonic action, a Hamiltonian formulation can be obtained in different ways, and the simplest is the one of type BF, eq. 5.25. In our case, we can rewrite the lagrangian density (7.6) in terms of $\mathcal{R}$ and another odd parity superfield $\Phi = \eta + \Theta\phi$ (with $\eta\eta = 0$), as follows

$$\mathcal{L}_2^{\Phi} = \mathcal{E}\mathcal{A}^3(2\mathcal{R} - \Phi)\nabla_\Theta\Phi + \mathcal{E}\Phi\mathcal{R}\nabla_\Theta\mathcal{A}^3. \tag{7.8}$$

The equivalence can be verified at the superfield level. Using (3.60), we get the equation of motion $2\mathcal{A}^3\nabla_\Theta(\mathcal{R} - \Phi) + (\nabla_\Theta\mathcal{A}^3)(\mathcal{R} - \Phi) = 0$. Therefore, $\Phi = \mathcal{R}$ and, substituting this $\Phi$

back into (7.8), we recover (7.6). Note that, since $\mathcal{R}$ is nilpotent, the last term on the r.h.s of (7.8) does not contribute when we replace $\Phi$ by $\mathcal{R}$.

Therefore, the total equivalent scalar-tensor action has Lagrangian density

$$\mathcal{L}_S^{\Phi} = \frac{3}{\kappa^2}\mathcal{E}\mathcal{A}^3\left[\mathcal{R} + \alpha\left((2\mathcal{R} - \Phi)\nabla_\theta\Phi + 3\mathcal{A}^{-1}\Phi\mathcal{R}\nabla_\theta\mathcal{A}\right)\right], \tag{7.9}$$

which yields, after integrating by parts, the following component Lagrangian,

$$\begin{aligned} L_S^{\phi} = \frac{3}{\kappa^2}Na^3\Bigg[&-\frac{\dot{a}^2}{a^2N^2}(1+2\alpha\phi) - 2\frac{\alpha\dot{a}\dot{\phi}}{aN^2} - \alpha\phi^2 + 2i\frac{\dot{a}\psi\lambda}{aN} - i\frac{\lambda\dot{\lambda}}{N} + \alpha\Bigg(2i\frac{\dot{\eta}\dot{\lambda}}{N^2} + i\frac{\dot{\eta}\eta}{N} \\ &-3i\frac{\dot{a}\phi}{aN}\psi\lambda + 7i\frac{\dot{a}\dot{\eta}\lambda}{aN^2} - 2i\frac{\phi\psi\dot{\lambda}}{N} - 2i\frac{\dot{a}\psi\dot{\eta}}{aN^2} + 2i\frac{\dot{\phi}\psi\lambda}{N} + i\frac{\phi\lambda\dot{\lambda}}{N} + 3i\phi\eta\lambda + 9i\frac{\dot{a}^2\eta\lambda}{N^2a^2} \\ &-6\frac{\psi\eta\lambda\dot{\lambda}}{N} + 6i\frac{\dot{a}\eta\dot{\lambda}}{N^2} - 3i\frac{\dot{a}^2\psi\eta}{a^2N^2}\Bigg)\Bigg]. \end{aligned} \tag{7.10}$$

This Lagrangian contains two bosons $a, \phi$, and two real fermions $\lambda, \eta$. One can verify that the equations of motion for $\phi, \eta$ are solved by $\eta = -2H\lambda - \dot{\lambda}$, $\phi = \left(\frac{1}{6}R + 2i\lambda\dot{\lambda}\right)$, as expected from the superfield solution.

### 7.1.2 Hamiltonian formulation

As with the $F(\mathcal{R})$ action, (7.10) still contains a term quadratic in the fermionic velocities, $\dot{\eta}\dot{\lambda}$. It is, however, sufficiently good to perform the usual Hamiltonian formulation.

With this N=1 supersymmetry, we use the usual definition for both bosonic and fermionic momenta. As usual, we get two primary (first-class) constraints $p_N = 0$, $\pi_\psi = 0$.

The Hamiltonian is given by the Legendre transformation $H_S^{\phi} = \dot{N}p_N + \dot{\psi}\pi_\psi + \dot{a}p_a + \dot{\phi}p_\phi + \dot{\lambda}\pi_\lambda + \dot{\eta}\pi_\eta - L$, which yields

$$H_S^{\phi} = NH_0 + \frac{1}{2}\Psi S \tag{7.11}$$

where the Hamiltonian and supersymmetric constraints are, respectively,

$$\begin{aligned} 0 \approx H_0 = &\frac{1}{4j\alpha^2}(1+2\alpha\phi)\frac{p_\phi^2}{a^3} - \frac{1}{2j\alpha}\frac{p_a p_\phi}{a^2} + j\alpha a^3\phi^2 + i\left(\frac{3}{j\alpha}\frac{p_\phi^2}{a^3} + \frac{ja^3}{2}(1-7\alpha\phi)\right)\eta\lambda \\ &+ \frac{7}{4j\alpha}\frac{p_\phi}{a^3}\lambda\pi_\lambda + \frac{1-\alpha\phi}{2\alpha}\lambda\pi_\eta + \frac{3}{2j\alpha}\frac{p_\phi}{a^3}\eta\pi_\eta + \frac{i}{2j\alpha}\frac{\pi_\lambda\pi_\eta}{a^3} - \frac{\eta\pi_\lambda}{2}, \end{aligned} \tag{7.12}$$

$$0 \approx S = i\left(ap_a - \frac{1-\alpha\phi}{2\alpha}p_\phi\right)\lambda - i\left(\frac{3}{4j\alpha}\frac{p_\phi^2}{a^3} + j\alpha a^3\phi\right)\eta + \frac{1}{2j\alpha}\frac{p_\phi}{a^3}\pi_\lambda + \phi\pi_\eta. \tag{7.13}$$

Note that $S$ is imaginary and $H$ real. The algebra of constraints, all of which are first-class, closes under Poisson brackets. In particular, the following holds, $\{S, S\} = -2H_0$.

### 7.1.3 Quantization

Basic (anti)commutators are

$$[a, p_a] = i\hbar, \qquad [\phi, p_\phi] = i\hbar, \tag{7.14}$$

$$[\lambda, \pi_\lambda]_+ = -i\hbar, \qquad [\eta, \pi_\eta]_+ = -i\hbar. \tag{7.15}$$

In accord with the assumption of reality of the classical fermionic variables, the quantum operators $\lambda$, $\eta$ are assumed to be hermitian. Then, for the anti-commutation relations (7.15) to be consistent, the fermionic momenta must be anti-hermitian. This has the consequence that we cannot define a set creation and annihilation fermionic operators as we did in (6.13). If we define, $A \equiv \lambda + c\pi_\lambda$ ($A^\dagger = \lambda - c^*\pi_\lambda$), then $[A, A^\dagger]_+ = -i\hbar(c - c^*)$ (c must be complex so that $A \neq A^\dagger$). However, we also have $[A, A]_+ = -i\hbar c \neq 0$.

Instead, we obtain a consistent quantization by representing the fermionic momenta as differential operators, namely,

$$\pi_\lambda = -i\hbar\partial_\lambda, \qquad \pi_\eta = -i\hbar\partial_\eta. \tag{7.16}$$

The wave-function will be of the form

$$\psi = \psi_0(a, \phi) + \lambda\psi_1(a, \phi) + \eta\psi_2(a, \phi) + \lambda\eta\psi_3(a, \phi) \tag{7.17}$$

As before, we choose Weyl ordering for the supersymmetry constraint operator. For simplicity, we replace the canonical pair $\phi$, $p_\phi$ with $q = 2\alpha\phi$, $p_q = (2\alpha)^{-1}p_\phi$. Thus,

$$S = i\left(\frac{ap_a + p_a a}{2} - p_q + \frac{qp_q + p_q q}{4}\right)\lambda - i\left(\frac{3\alpha}{j}\frac{p_q^2}{a^3} + \frac{j}{2}a^3 q\right)\eta + \frac{1}{j}\frac{p_q}{a^3}\pi_\lambda + \frac{q}{2\alpha}\pi_\eta, \tag{7.18}$$

$$\equiv A\lambda + B\eta + C\pi_\lambda + D\pi_\eta \tag{7.19}$$

Note that $S$ is anti-hermitian. For this ordering choice there corresponds a hermitian Hamiltonian operator proportional to the anti-commutator $[S, S]_+$. Using (7.19) and (C.5), we get

$$\begin{aligned}[S, S]_+ = {} & [A, C]_+[\lambda, \pi_\lambda]_+ + [B, D]_+[\eta, \pi_\eta]_+ + [A, B]_-[\lambda, \eta]_- + [A, C]_-[\lambda, \pi_\lambda]_- \\ & + [A, D]_-[\lambda, \pi_\eta]_- + [B, C]_-[\eta, \pi_\lambda] + [B, D]_-[\eta, \pi_\eta]_- + [C, D]_-[\pi_\lambda, \pi_\eta]_-. \end{aligned} \tag{7.20}$$

Computing the right-hand side of (7.20), and rearranging, we get

$$\begin{aligned} H_0 = {} & -\frac{1}{j}a^{-2}p_a p_q + \frac{1}{j}a^{-3}(1+q)p_q^2 + \frac{j}{4\alpha}a^3 q^2 - \frac{5i\hbar}{4j}a^{-3}p_q + i\left(\frac{12\alpha}{j}a^{-3}p_q^2 + \frac{j}{2}a^3\left(1 - \frac{7}{2}q\right)\right)\eta\lambda \\ & + \frac{7}{2j}a^{-3}p_q\lambda\pi_\lambda + \frac{1 - q/2}{2\alpha}\lambda\pi_\eta - \frac{1}{2}\eta\pi_\lambda + \frac{3}{j}\frac{p_q}{a^3}\eta\pi_\eta + \frac{i}{2j\alpha}\frac{\pi_\lambda\pi_\eta}{a^3} \end{aligned} \tag{7.21}$$

which differs from the classical expression by a term proportional to $\hbar$.

Applying $S$ (7.19) to the wavefunction yields

$$S\psi = -i\hbar\Big(C\psi_1 + D\psi_2\Big) + \lambda\Big(A\psi_0 + i\hbar D\psi_3\Big) + \eta\Big(B\psi_0 - i\hbar C\psi_3\Big) + \lambda\eta\Big(A\psi_2 - B\psi_1\Big), \tag{7.22}$$

Equating to zero each component in the fermionic expansion, leads to a system of four coupled PDE's. They are, respectively,

$$0 = \partial_q \psi_1 + \frac{ij}{2\hbar\alpha} a^3 q \psi_2, \tag{7.23a}$$

$$0 = \left( a\partial_a + \frac{3}{4} - (1 - q/2)\partial_q \right)\psi_0 + \frac{i}{2\alpha} q\psi_3, \tag{7.23b}$$

$$0 = \left( \partial_q^2 - \frac{j^2}{6\hbar^2\alpha} a^6 q \right)\psi_0 + \frac{i}{3\alpha}\partial_q \psi_3, \tag{7.23c}$$

$$0 = \left( \partial_q^2 - \frac{j^2}{6\alpha\hbar^2} a^6 q \right)\psi_1 + \frac{ij}{3\hbar\alpha} a^3 \left( a\partial_a + \frac{3}{4} - (1 - q/2)\partial_q \right)\psi_2. \tag{7.23d}$$

We can solve for $\psi_2$, $\psi_3$ from (7.23a), (7.23b), respectively (assuming $a^3 q \neq 0$, and $q \neq 0$), and substitute into the remaining equations to get

$$\left[ -aq\partial_a\partial_q + a\partial_a + q(1+q)\partial_q^2 - (1 + \frac{3}{4}q)\partial_q - \left( \frac{j^2}{4\hbar^2\alpha} a^6 q^3 - \frac{3}{4} \right) \right] \psi_0 = 0, \tag{7.24}$$

$$\left[ -aq\partial_a\partial_q + q(1+q)\partial_q^2 - (1 - \frac{11}{4}q)\partial_q - \frac{j^2}{4\hbar^2\alpha} a^6 q^3 \right] \psi_1 = 0. \tag{7.25}$$

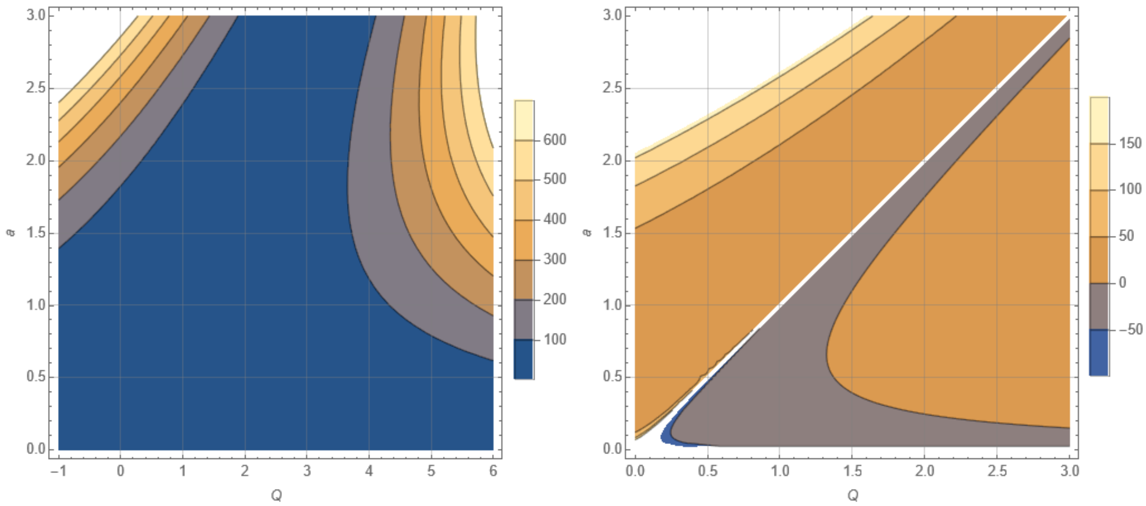
In terms of the variable $Q = a(1+q)$, $a' = a$, used in Chapter 5, these equations read,

$$\left[ \partial_a\partial_Q + \frac{3}{4a}\partial_Q - \frac{1}{(Q-a)}\partial_a + \left( \frac{j^2}{4\hbar^2\alpha} a^2 (Q-a)^2 - \frac{3}{4}\frac{1}{a(Q-a)} \right) \right] \psi_0(a,Q) = 0, \tag{7.26}$$

$$\left[ \partial_a\partial_Q - \left( \frac{7}{4a} - \frac{1}{Q-a} \right)\partial_Q + \frac{j^2}{4\hbar^2\alpha} a^2 (Q-a)^2 \right] \psi_1(a,Q) = 0 \tag{7.27}$$

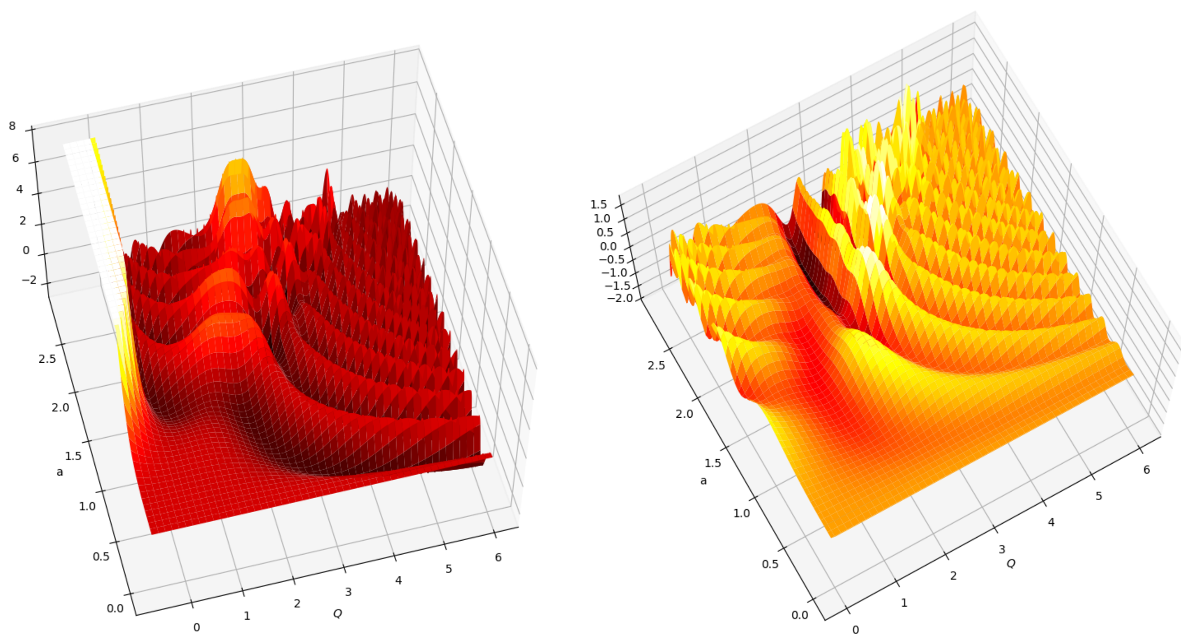
Note that they contain the pure bosonic Wheeler-DeWitt equation plus some extra terms. For $\psi_1$, the difference involves only a term linear in the momentum $p_Q$. For $\psi_0$, we have terms linear in both $p_a$ and $p_Q$, as well as a contribution to the scalar potential. Further, these terms carry coefficients that diverge when $a \to 0$ and $Q \to a$. At first sight, $\psi_0$ might seem more delicate due to the extra potential term $a^{-1}(Q-a)^{-1}$, that not only diverges, but is discontinuous at $Q = a$, see Figure 7.1. Despite this, we obtained numerical solutions that are continuous along this line. Figures 7.2 and 7.3 show such numerical solutions, for which we used the boundary conditions implied by the saddle-point approximation, $\psi(0,Q) = 1$. We also show $\psi_2$ and $\psi_3$ obtained using equations (7.23a) and (7.23b) (expressed in terms of the new coordinates), respectively.

Since we are considering $k = 0$, the potential for $\psi_1$ is positive semi-definite. In comparison with the bosonic case for $k = 1$ of Section 5.3, the oscillations pass through the central channel of very small $U$ without decreasing considerably in amplitude. All these wave functions are oscillatory in the region of positive potential. For $\psi_0$, the effective potential does have a region of negative value, and the wavefunction grows exponentially, but predominantly near the axis $a = 0$. Along $Q = a$, the obtained wave function displays a good behavior (it is continuous and does not diverge) despite the discontinuous potential.




(a) (b)

Figure 7.1: (a) Scalar potential $(1/4\alpha)j^2a^2(Q-a)^2$ for $\psi_2$ and (b) $(1/4\alpha)j^2\alpha^{-1}a^2(Q-a)^2 - (3/4)a^{-1}(Q-a)^{-1}$ for $\psi_0$.

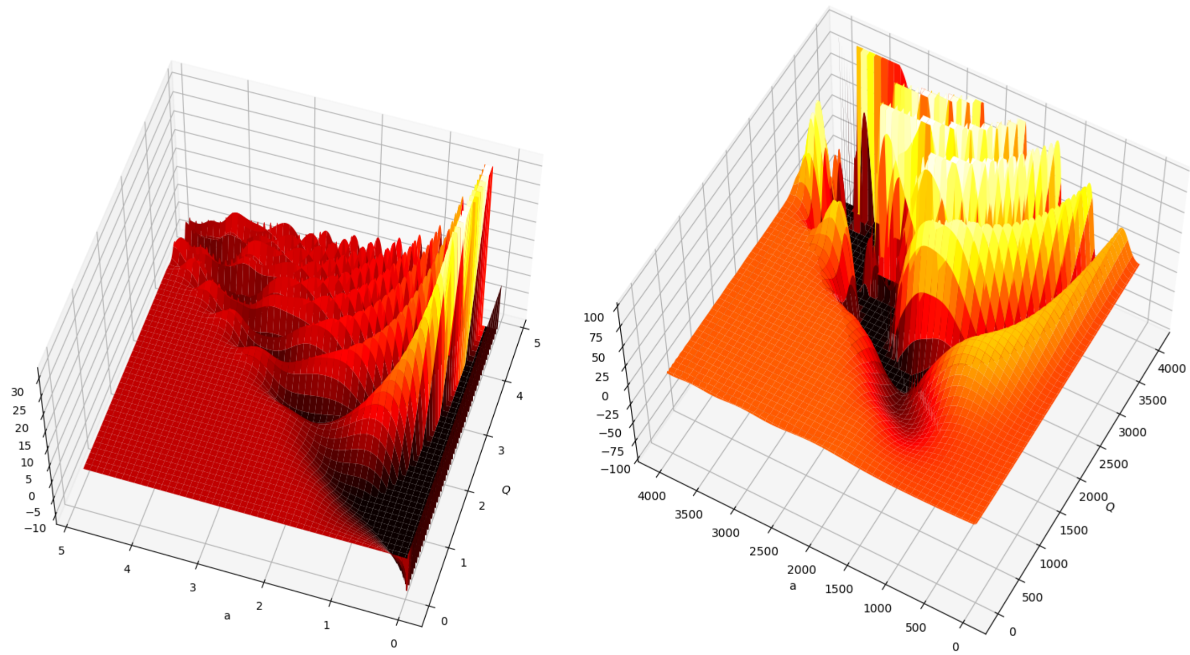



(a) (b)

Figure 7.2: (a) $\psi_1(a,Q)$ obtained by solving numerically equation (7.27) with boundary condition $\psi_1(0,Q)=1$. We set $\hbar=1$, $j=3$. (In the plot, we set values of $\psi_1$ above 1 equal to 1); (b) $\psi_2(a,Q)$ obtained from $\psi_1$ by using equation (7.23a).

### Alternative

We can also obtain a quantization scheme, and construct a space of fermionic states like in N=2 case if we abandon hermiticity, at least formally. This comes about from the realization




(a) (b)

Figure 7.3: $\psi_0(a,Q)$ obtained by solving numerically (7.26) with boundary condition $\psi_1(0,Q) = 1$. We set $\hbar = 1$, $j = 3$. (We only plot values of in the range $-12 \leq \psi \leq 32$); (b) $\psi_3(a,Q)$ computed using equation (7.23b).

that the basic anti-commutators (7.15) are also consistent if $\lambda \neq \lambda^\dagger$, but instead $\lambda^\dagger \equiv i\pi_\lambda/\hbar$ ($\pi_\lambda^\dagger = i\hbar\lambda$). Indeed, $[\lambda, \pi_\lambda]_+^\dagger = \pi_\lambda^\dagger \lambda^\dagger + \lambda^\dagger \pi_\lambda^\dagger = -[\lambda, \pi_\lambda]_+ = i\hbar$. Analogous considerations hold for $\eta$, $\pi_\eta$.

Thus, we define $\bar{\lambda}$, $\bar{\eta}$ such that $\pi_\lambda \equiv -i\hbar\bar{\lambda}$, $\pi_\eta \equiv -i\hbar\bar{\eta}$. Then, the basic anti-commutators can be written as

$$\{\lambda, \bar{\lambda}\} = 1, \qquad \{\eta, \bar{\eta}\} = 1 \tag{7.28}$$

Now, we define a ground state to be annihilated by $\lambda$ and $\eta$: $\lambda|0\rangle 00$, $\eta|0\rangle = 0$. Three more states can then be constructed, namely

$$|1\rangle = \bar{\lambda}|0\rangle, \tag{7.29a}$$
$$|2\rangle = \bar{\eta}|0\rangle, \tag{7.29b}$$
$$|3\rangle = \bar{\lambda}\bar{\eta}|0\rangle. \tag{7.29c}$$

They are orthogonal and, unlike the previous chapter, we can choose them all to have positive norm. An arbitrary state is of the form

$$|\psi(a,q)\rangle = \psi_0(a,q)|0\rangle + \psi_1(a,q)|1\rangle + \psi_2(a,q)|2\rangle + \psi_3(a,q)|3\rangle \tag{7.30}$$

Re-writing the supersymmetric constraint operator,

$$S = i\left(\frac{1}{2}(ap_a + p_a a) - p_q + \frac{qp_q + p_q q}{4}\right)\lambda - i\left(\frac{3\alpha}{j}\frac{p_q^2}{a^3} + \frac{j}{2}a^3 q\right)\eta - i\hbar\frac{1}{j}\frac{p_q}{a^3}\bar{\lambda} - \frac{i\hbar}{2\alpha}q\bar{\eta} \tag{7.31}$$

In this representation $S$ is not hermitian nor anti-hermitian. This might not be as bad as it sounds since we still postulate a zero-eigenvalue equation $S|\psi\langle= 0$ and there is no clear notion of unitary evolution.

However, our best argument in favor of this quantization (for the moment) is that it gives basically the same wave functions as with the differential representation for the fermionic momenta. From $S|\psi\rangle = 0$ and the orthogonality of the states $|i\rangle$, we get the following system of PDE's

$$0 = \left(\partial_q^2 - \frac{j^3}{6\hbar^2\alpha}a^6 q\right)\phi_2 - i\frac{ja^3}{3\hbar\alpha}\left(a\partial_a - (1-q/2)\partial_q + \frac{3}{4}\right)\phi_1, \tag{7.32a}$$

$$0 = i\frac{\hbar}{j}\frac{\partial_q}{a^3}\phi_2 + \frac{q}{2\alpha}\phi_1, \tag{7.32b}$$

$$0 = \left(\partial_q^2 - \frac{j^2}{6\hbar^2\alpha}a^6 q\right)\phi_3 - \frac{i}{3\alpha}\partial_q\phi_0, \tag{7.32c}$$

$$0 = i\left(a\partial_a - (1-q/2)\partial_q + \frac{3}{4}\right)\phi_3 + \frac{q}{2\alpha}\phi_0. \tag{7.32d}$$

which reproduce (7.23) by sending $\phi_0 \to -\psi_3$, $\phi_1 \to -\psi_2$, $\phi_2 \to \psi_1$, and $\phi_3 \to \psi_0$.

## 7.2 N=2 action

The experience with the N=1 model, suggests defining an action depending on the covariant derivatives of the curvature superfield. Considering that $\nabla_\theta\mathcal{R}$ is an odd parity complex superfield, and the lagrangian density must be real and of even parity, the natural candidate is $\nabla_\theta\mathcal{R}\nabla_{\bar{\theta}}\mathcal{R}$, which has the form of a superfield kinetic term[46]. As in the case with real fermions, we write

$$\mathcal{L} = \mathcal{L}_1 + \alpha\mathcal{L}_2, \tag{7.33}$$

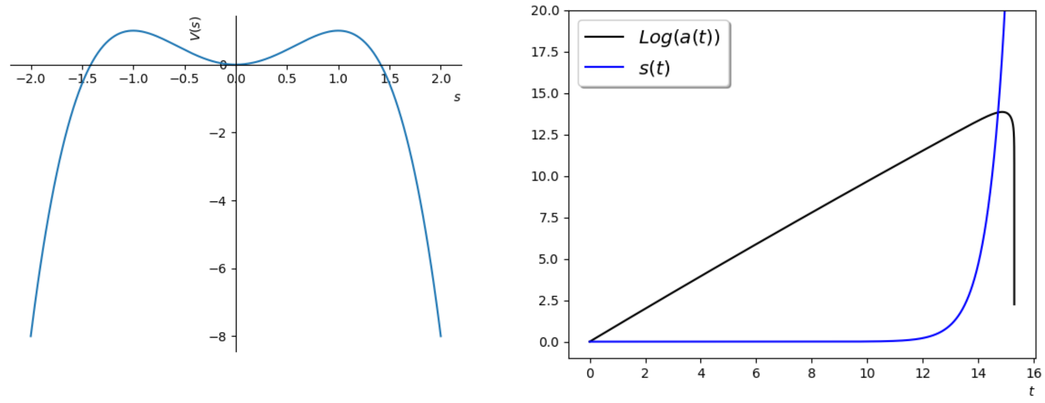
with $L_1$ the ordinary supersymmetric FRW Lagrangian of Chapter 4 and

$$\mathcal{L}_2 = \frac{3}{\kappa^2}\mathcal{E}\mathcal{A}^3\nabla_{\bar{\Theta}}\mathcal{R}\nabla_{\Theta}\mathcal{R}. \tag{7.34}$$

The bosonic part of the total component Lagrangian is

$$L^{\mathrm{bos}} = \frac{1}{2\kappa^2}Na^3\left[R_0 + \frac{\alpha}{6}R_0^2 + 6\alpha\dot{s}^2 - 6s^2 + 4\alpha R_0 s^2 + 24\alpha s^4\right]. \tag{7.35}$$

Thus, with the choice (7.34) the total Lagrangian (7.33) contains the FRW model of Starobinsky, plus extra terms for the scalar field $s$. The kinetic term $\dot{s}^2$ tells us that the former




(a) (b)

Figure 7.4: (a) Scalar potential $V(s)$. (b) Numerical solution with initial conditions $a = 1$, $H = 5M$, $\dot{H} = -\frac{1}{6}M^2$, $\ddot{H} = 0$, $s = 10^{-9}$, $\dot{s} = 0$ and $M = 0.2$.

auxiliary field $s$ is promoted to a dynamical field (not only a higher-derivative d.o.f. as with the $F(\mathcal{R})$ action). It is directly coupled to the curvature $Rs^2$ and has potential $V(s) = s^2 - 4\alpha s^4$. The equations of motion will be fourth order for $a$ and second order for $s$. In the fermionic sector we find $\dot{\lambda}\dddot{\bar{\lambda}} + \dot{\bar{\lambda}}\ddot{\lambda}$ yielding third-order equations of motion.

The scalar potential is unbounded from below, but it has a local minimum around $s = 0$. However, since the effective quadratic mass is $M^2 - \frac{2}{3}R$ (for the canonically normalized field $\tilde{s} = \sqrt{6}s/\kappa M$), we require not only $s$, but also $R$ to be sufficiently small in order to obtain a stable dynamics. If we set initial conditions $s = 0 = \dot{s}$, we obtain inflation, however, this is not a stable solution in the sense that small initial values of $s$ or $\dot{s}$, eventually cause the field amplitude of $s$ to blow up while $a$ goes to zero (see Figure 7.4 (b)).

This is certainly not an unlikely situation. Inflationary models derived from supergravity and more fundamental theories, e.g., string theory, contain several scalar fields. It is required that all of them but one, sit in a stable vacuum state during inflation. This is, of course, not the generic situation, it may happen that some of the additional fields develop instabilities pushing the overall dynamics away from the inflationary solution. This is the case with theories of extra-dimensions; after compactification one needs a mechanism to stabilize module fields [35].

To avoid a fine tuning of the initial conditions, we considered adding to the Lagrangian a sort of superpotential term in the form of a $F(\mathcal{R})$. It turns out that with an $\mathcal{R}^3$, we can cancel the coupling $Rs^2$ and negative sign fourth-power potential in (7.35).

Therefore, we propose the following superfield Lagrangian

$$\mathcal{L}_S = \frac{3}{\kappa^2}\mathcal{E}\mathcal{A}^3\left[\mathcal{R} + \alpha\left(\nabla_{\bar{\Theta}}\mathcal{R}\nabla_{\Theta}\mathcal{R} - \frac{4}{3}\mathcal{R}^3\right)\right], \tag{7.36}$$

leading to

$$
\begin{aligned}
L_S = \frac{3}{\kappa^2} N a^3 \Bigg[ & \frac{\ddot a}{N^2 a} - \frac{\dot N \dot a}{N^3 a} + \frac{\dot a^2}{N^2 a^2} + \alpha \left( \frac{\ddot a}{N^2 a} - \frac{\dot N \dot a}{N^3 a} + \frac{\dot a^2}{N^2 a^2} \right)^2 + \frac{\alpha \dot s^2}{N^2} - s^2 + s\lambda\bar\lambda \\
& - i\frac{\dot\psi\bar\lambda + \dot{\bar\psi}\lambda}{N} + 2\psi\bar\psi\lambda\bar\lambda - i\dot a\frac{\psi\bar\lambda + \bar\psi\lambda}{Na} - i\frac{\psi\dot{\bar\lambda} + \bar\psi\dot\lambda}{N} + i\frac{\lambda\dot{\bar\lambda} + \bar\lambda\dot\lambda}{N} + \alpha \Bigg( i\frac{\ddot{\bar\lambda}\dot\lambda + \ddot\lambda\dot{\bar\lambda}}{N^3} \\
& - 3s^3\psi\bar\psi + 4\frac{\dot s\dot a}{N^2}\frac{\lambda\bar\lambda}{a} - 11 s\dot a^2\frac{\psi\bar\lambda - \bar\psi\lambda}{N^2 a^2} - s\dot a\frac{\psi\dot{\bar\lambda} - \bar\psi\dot\lambda}{N^2 a} - 6\frac{s\ddot a\psi\bar\psi}{N^2 a} - 9i\dot a^3\frac{\psi\bar\lambda + \bar\psi\lambda}{N^3 a^3} \\
& - 9i\dot a\ddot a\frac{\psi\bar\lambda + \bar\psi\lambda}{N^3 a^2} + i\dot N\dot a\frac{\lambda\dot{\bar\lambda} + \bar\lambda\dot\lambda}{N^4 a} + 2i\dot N\dot a\frac{\dot\psi\bar\lambda + \dot{\bar\psi}\lambda}{N^4 a} - is^2\frac{\psi\dot{\bar\lambda} + \bar\psi\dot\lambda}{N} + 6s\dot N\frac{\dot a}{a}\frac{\psi\bar\psi}{N^3} \\
& - 2\dot\lambda\dot{\bar\lambda}\frac{\psi\bar\lambda - \bar\psi\lambda}{N^2} - s\frac{\dot\psi\dot{\bar\lambda} - \dot{\bar\psi}\dot\lambda}{N^2} + s\frac{\dot\lambda\dot{\bar\lambda}}{N^2} + 6\frac{\dot s\dot a}{a}\frac{\psi\bar\psi}{N^2} + \frac{\lambda\ddot{\bar\lambda} - \bar\lambda\ddot\lambda}{N^2}\psi\bar\psi + i\dot a\frac{\psi\ddot{\bar\lambda} + \bar\psi\ddot\lambda}{N^3 a} \\
& + 9is\frac{\psi\dot{\bar\lambda} + \bar\psi\dot\lambda}{N}\lambda\bar\lambda - 5\frac{\psi\dot{\bar\psi}\lambda\dot{\bar\lambda}}{N^2} - 5\frac{\bar\psi\dot\psi\bar\lambda\dot\lambda}{N^2} + \frac{\psi\dot{\bar\psi}\bar\lambda\dot\lambda}{N^2} + \frac{\bar\psi\dot\psi\lambda\dot{\bar\lambda}}{N^2} - 22\dot a\psi\bar\psi\frac{\lambda\dot{\bar\lambda} - \bar\lambda\dot\lambda}{N^2 a} \\
& - 2\frac{\dot\psi\dot{\bar\psi}}{N^2}\lambda\bar\lambda - 3is\psi\bar\psi\frac{\lambda\dot{\bar\lambda} + \bar\lambda\dot\lambda}{N} + s\frac{\psi\ddot{\bar\lambda} - \bar\psi\ddot\lambda}{N^2} - 4\frac{\psi\dot\psi\bar\lambda\dot{\bar\lambda}}{N^2} - 4\frac{\bar\psi\dot{\bar\psi}\lambda\dot\lambda}{N^2} - 3\dot s\frac{\psi\dot{\bar\lambda} - \bar\psi\dot\lambda}{N^2} \\
& + \frac{\dot\lambda\dot{\bar\lambda}}{N^2}\lambda\bar\lambda - 4\dot a s\frac{\dot\psi\bar\lambda - \dot{\bar\psi}\lambda}{N^2 a} - 2\frac{\dot N\dot a}{N^3}\psi\bar\psi\frac{\lambda\bar\lambda}{a} - 13\dot a\frac{\psi\dot{\bar\psi} - \bar\psi\dot\psi}{N^2 a}\lambda\bar\lambda + 4i\dot N\dot a\frac{\psi\dot{\bar\lambda} + \bar\psi\dot\lambda}{N^4 a} \\
& + 2s\frac{\lambda\ddot{\bar\lambda} - \bar\lambda\ddot\lambda}{N^2} + i\ddot a\frac{\lambda\dot{\bar\lambda} + \bar\lambda\dot\lambda}{N^3 a} - 2i\dot a\frac{\lambda\ddot{\bar\lambda} + \bar\lambda\ddot\lambda}{N^3 a} - 7i\dot a^2\frac{\lambda\dot{\bar\lambda} + \bar\lambda\dot\lambda}{N^3 a^2} + 10s\dot a\frac{\lambda\dot{\bar\lambda} - \bar\lambda\dot\lambda}{N^2 a} \\
& + 9s^2\psi\bar\psi\lambda\bar\lambda - 5i\ddot a\frac{\psi\dot{\bar\lambda} + \bar\psi\dot\lambda}{N^3 a} + 3\frac{s\dot a^2\psi\bar\psi}{N^2 a^2} - 4\frac{s\dot N\dot a}{N^3 a}\lambda\bar\lambda + 4\frac{s\ddot a\lambda\bar\lambda}{N^2 a} + is\frac{\psi\dot{\bar\psi} + \bar\psi\dot\psi}{N}\lambda\bar\lambda \\
& + 8\frac{s\dot a^2\lambda\bar\lambda}{N^2 a^2} + s\dot N\dot a\frac{\psi\bar\lambda - \bar\psi\lambda}{N^3 a} + 6is\psi\bar\psi\frac{\dot\psi\bar\lambda + \dot{\bar\psi}\lambda}{N} + 2is^2\frac{\dot\psi\bar\lambda + \dot{\bar\psi}\lambda}{N} + 6i\dot a s^2\frac{\psi\bar\lambda + \bar\psi\lambda}{Na} \\
& + \dot s\frac{\lambda\dot{\bar\lambda} - \bar\lambda\dot\lambda}{N^2} - \dot N\frac{\lambda\dot{\bar\lambda} - \bar\lambda\dot\lambda}{N^3}\psi\bar\psi + 2\frac{\ddot a\psi\bar\psi\lambda\bar\lambda}{N^2 a} - 2\dot a\frac{\dot\psi\bar\lambda - \dot{\bar\psi}\lambda}{N^2 a}\psi\bar\psi + 9i\dot N\dot a^2\frac{\psi\bar\lambda + \bar\psi\lambda}{N^4 a^2} \\
& + 3\dot a\frac{\psi\dot{\bar\lambda} - \bar\psi\dot\lambda}{N^2 a}\lambda\bar\lambda + \frac{\dot\psi\dot{\bar\lambda} - \dot{\bar\psi}\dot\lambda}{N^2}\lambda\bar\lambda - s\dot N\frac{\psi\dot{\bar\lambda} + \bar\psi\dot\lambda}{N^3} - i\dot a\frac{\dot\psi\dot{\bar\lambda} + \dot{\bar\psi}\dot\lambda}{N^3 a} - 7\dot a\dot s\frac{\psi\bar\lambda - \bar\psi\lambda}{N^2 a} \\
& - 53\dot a^2\frac{\psi\bar\psi\lambda\bar\lambda}{N^2 a^2} + 7is\dot s\frac{\psi\bar\lambda + \bar\psi\lambda}{N} - s\ddot a\frac{\psi\bar\lambda - \bar\psi\lambda}{N^2 a} - 2i\ddot a\frac{\dot\psi\bar\lambda + \dot{\bar\psi}\lambda}{N^3 a} - 4i\dot a^2\frac{\dot\psi\bar\lambda + \dot{\bar\psi}\lambda}{N^3 a^2} \\
& - 12\psi\bar\psi\frac{\dot\lambda\dot{\bar\lambda}}{N^2} - i\dot a^2\frac{\psi\dot{\bar\lambda} + \bar\psi\dot\lambda}{N^3 a^2} - is^2\frac{\psi\dot{\bar\psi} + \bar\psi\dot\psi}{N} - i\dot a^2\frac{\psi\dot{\bar\psi} + \bar\psi\dot\psi}{N^3 a^2} - 2\dot N a\frac{\lambda\dot{\bar\lambda} - \bar\lambda\dot\lambda}{N^3} \Bigg) \Bigg].
\end{aligned}
\tag{7.37}
$$

The linear and quadratic curvature terms remain without change, but $s$ is now a rather harmless massive scalar field[8].

[8] With $M$ constrained to be about $10^{-6} M_P$, $s$ is a pretty heavy field. Considered in a field-theory context, heavy fields are more benevolent with the picture provided by the Standard Model of Cosmology than lighter fields [35].

The equations of motion of the bosonic sector read

$$0 = \frac{3}{2}H^2M^2 + 9H^2\dot{H} + \dot{H}M^2 + \frac{9}{2}\dot{H}^2 + 6H\ddot{H} + \dddot{H} + \frac{\kappa^2M^2}{4}(\dot{\tilde{s}}^2 - M^2\tilde{s}^2), \tag{7.38a}$$

$$0 = \ddot{\tilde{s}} + 3H\dot{\tilde{s}} + M^2\tilde{s}. \tag{7.38b}$$

where $\tilde{s}$ is the canonically normalized field, $\tilde{s} = \sqrt{6\alpha/\kappa^2}s$. Numerical solutions to equations (7.38) are shown in Figure 7.5 for different initial values of $s$ and $\dot{s}$. As inflation takes place, the field is driven to the minimum of the potential. Kinetic energy $M^2\dot{\tilde{s}}^2$ is quickly dissipated by the friction term $H\dot{\tilde{s}}$ on the left hand side of (7.38b). In contrast, this same friction term sustains a high value of the field for longer (acting as a cosmological constant), so that we get more inflation (measured in e-folds) when most of the initial energy of $s$ is of potential type.

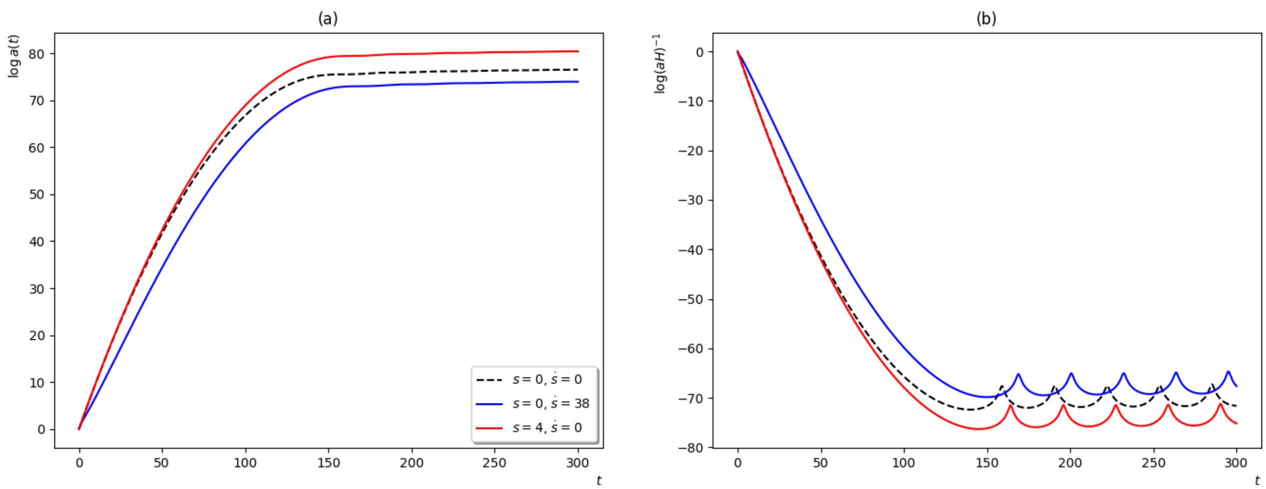



Figure 7.5: Numerical solutions to equations (7.38). The initial conditions for the scale factor are the same as in Figure 1. Here we have an additional scalar field. (a) shows the scale factor whereas (b) shows the comoving Hubble length for pure kinetic initial energy (blue color) and pure potential initial energy (red color) of the field $s$. The dotted line is the same as in the pure Starobinky dynamics of Figure (5.1).

### 7.2.1 Scalar-tensor formulation

A superfield Lagrangian equivalent to (7.36) is the following

$$\mathcal{L}_S^{\Phi} = \frac{3}{\kappa^2}\mathcal{E}\mathcal{A}^3\left[\mathcal{R} + \alpha\left(\nabla_{\bar{\Theta}}\Phi\nabla_{\Theta}\mathcal{R} - \nabla_{\Theta}\Phi\nabla_{\bar{\Theta}}\mathcal{R} - \nabla_{\bar{\Theta}}\Phi\nabla_{\Theta}\Phi - \frac{4}{3}\mathcal{R}^3\right)\right]. \tag{7.39}$$

where $\Phi = \phi + i\Theta\bar{\eta} + i\bar{\Theta}\eta + \Theta\bar{\Theta}G$. The equivalence can be verified using the superfield equation of motion for $\Phi$. From (3.52), we get $\nabla_{\bar{\Theta}}\mathcal{A}^3\nabla_{\Theta}(\mathcal{R}-\Phi) - \nabla_{\Theta}\mathcal{A}^3\nabla_{\bar{\Theta}}(\mathcal{R}-\Phi) - \mathcal{A}^3[\nabla_{\Theta}, \nabla_{\bar{\Theta}}](\mathcal{R}-$

$\Phi) = 0$. Therefore, the solution is given by $\Phi = \mathcal{R} + c$, where $c$ is a constant. Replacing $\Phi$ by $\mathcal{R} + c$ into (7.39) returns (7.34). The component Lagrangian depending on $\Phi$ is

$$\begin{aligned}L_2^{\phi} \doteq \frac{3a^3}{\kappa^2}\Bigg[&2\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right)G - G^2 + 4Gs^2 + 2\dot{\phi}\dot{s} - \dot{\phi}^2 + i(\eta\dot{\bar{\eta}} + \bar{\eta}\dot{\eta}) + 2(\dot{\lambda}\dot{\bar{\eta}} - \dot{\bar{\lambda}}\dot{\eta}) + 3s\eta\bar{\eta} \\ &+3i\lambda\bar{\lambda}(\dot{\lambda}\bar{\eta} + \dot{\bar{\lambda}}\eta) - Gi(\lambda\dot{\bar{\lambda}} + \bar{\lambda}\dot{\lambda}) - 3is(\dot{\lambda}\bar{\eta} + \dot{\bar{\lambda}}\eta) + 4\frac{\dot{a}}{a}(\lambda\dot{\bar{\eta}} - \bar{\lambda}\dot{\eta}) + 3\frac{\dot{a}}{a}(\dot{\lambda}\bar{\eta} - \dot{\bar{\lambda}}\eta) \\ &+3\dot{\phi}(\lambda\dot{\bar{\lambda}} - \bar{\lambda}\dot{\lambda}) + 4is(\lambda\dot{\bar{\eta}} + \bar{\lambda}\dot{\eta}) + 3G(\lambda\bar{\eta} - \bar{\lambda}\eta) + 3\frac{\dot{a}^2}{a^2}(\lambda\bar{\eta} - \bar{\lambda}\eta) + 3i\dot{s}(\lambda\bar{\eta} + \bar{\lambda}\eta) \\ &+12\frac{\dot{a}}{a}\dot{\phi}\lambda\bar{\lambda} - 3\frac{\ddot{a}}{a}(\lambda\bar{\eta} - \bar{\lambda}\eta) - 3i\dot{\phi}(\lambda\bar{\eta} + \bar{\lambda}\eta) + 3\lambda\bar{\lambda}\eta\bar{\eta} - 4Gs\lambda\bar{\lambda}\Bigg]. \qquad (7.40)\end{aligned}$$

As expected from the superfield expression (7.39), $\phi$ is an ignorable coordinate. We eliminate $\dot{\phi}$ from the theory by choosing $p_\phi$ equal to zero. On the other hand, it can be seen that $G$ plays the role of the scalaron, we just need to eliminate the coupling $Gs^2$ by making a shift $G \to \varphi = G - 2s^2$. Doing this, and also integrating by parts, we get the final form of the equivalent lagrangian,

$$\begin{aligned}L_S^{\varphi} = \frac{3}{\kappa^2}Na^3\Bigg[&-\frac{\dot{a}^2}{N^2a^2}(1+2\alpha\varphi) - 2\frac{\alpha\dot{a}\dot{\varphi}}{N^2a} - \alpha\varphi^2 + \frac{\alpha\dot{s}^2}{N^2} - s^2 + i\frac{\lambda\dot{\bar{\lambda}} + \bar{\lambda}\dot{\lambda}}{N} + 2\psi\bar{\psi}\lambda\bar{\lambda} + s\lambda\bar{\lambda} \\ &+2i\dot{a}\frac{\psi\bar{\lambda} + \bar{\psi}\lambda}{Na} + \alpha\Bigg(2\frac{\dot{\lambda}\dot{\bar{\eta}} - \dot{\bar{\lambda}}\dot{\eta}}{N^2} + 3\dot{\lambda}\dot{\bar{\lambda}}\frac{\psi\bar{\lambda} - \bar{\psi}\lambda}{N^2} - 3is\frac{\dot{\lambda}\bar{\eta} + \dot{\bar{\lambda}}\eta}{N} + 9\dot{a}^2\frac{\lambda\bar{\eta} - \bar{\lambda}\eta}{N^2a^2} + 8s\frac{\dot{\lambda}\dot{\bar{\lambda}}}{N^2} \\ &+6is\dot{a}\frac{\psi\bar{\eta} + \bar{\psi}\eta}{Na} + 7is\dot{s}\frac{\psi\bar{\lambda} + \bar{\psi}\lambda}{N} - i\varphi\frac{\lambda\dot{\bar{\lambda}} + \bar{\lambda}\dot{\lambda}}{N} + 2is\frac{\psi\dot{\bar{\eta}} + \bar{\psi}\dot{\eta}}{N} + 12s^3\lambda\bar{\lambda} - 2\dot{s}\frac{\psi\dot{\bar{\lambda}} - \bar{\psi}\dot{\lambda}}{N^2} \\ &-4\varphi s\psi\bar{\psi} + 4is\frac{\lambda\dot{\bar{\eta}} + \bar{\lambda}\dot{\eta}}{N} - \frac{\dot{a}^2\psi\bar{\psi}}{2N^2a^2}\lambda\bar{\lambda} - 2\dot{a}\frac{\psi\dot{\bar{\eta}} - \bar{\psi}\dot{\eta}}{N^2a} - 8s\dot{a}\frac{\psi\dot{\bar{\lambda}} - \bar{\psi}\dot{\lambda}}{N^2a} + 18is^2\dot{a}\frac{\psi\bar{\lambda} + \bar{\psi}\lambda}{Na} \\ &-2\varphi\psi\bar{\psi}\lambda\bar{\lambda} - 9i\dot{a}\frac{\psi\bar{\eta} + \bar{\psi}\eta}{2Na}\lambda\bar{\lambda} + 3(\psi\bar{\lambda} - \bar{\psi}\lambda)\eta\bar{\eta} + 3\varphi(\lambda\bar{\eta} - \bar{\lambda}\eta) - \frac{9}{2}\lambda\bar{\lambda}\frac{\dot{\lambda}\dot{\bar{\lambda}}}{N^2} - 4\varphi s\lambda\bar{\lambda} \\ &-\varphi s(\psi\bar{\lambda} - \bar{\psi}\lambda) + 3i\frac{\psi\bar{\lambda}\dot{\bar{\lambda}}\eta}{N} - 3i\frac{\psi\lambda\dot{\bar{\lambda}}\bar{\eta}}{N} + 3i\frac{\bar{\psi}\bar{\lambda}\dot{\lambda}\eta}{N} - 3i\frac{\bar{\psi}\lambda\dot{\lambda}\bar{\eta}}{N} - 6i\frac{\psi\bar{\lambda}\dot{\lambda}\bar{\eta}}{N} + 6i\frac{\eta\bar{\psi}\lambda\dot{\bar{\lambda}}}{N} \\ &+3i\frac{\psi\dot{\bar{\eta}} + \bar{\psi}\dot{\eta}}{N}\lambda\bar{\lambda} + 32\frac{s\dot{a}^2\lambda\bar{\lambda}}{a^2N^2} - 3\dot{a}^2\frac{\psi\bar{\eta} - \bar{\psi}\eta}{N^2a^2} + 7\dot{a}\frac{\lambda\dot{\bar{\eta}} - \bar{\lambda}\dot{\eta}}{N^2a} - \frac{3}{2}\lambda\bar{\lambda}\eta\bar{\eta} - 2i\varphi\frac{\psi\dot{\bar{\lambda}} + \bar{\psi}\dot{\lambda}}{N} \\ &+16s\dot{a}\frac{\lambda\dot{\bar{\lambda}} - \bar{\lambda}\dot{\lambda}}{N^2a} - 3i\varphi\dot{a}\frac{\psi\bar{\lambda} + \bar{\psi}\lambda}{Na} - \frac{9}{2}s^2\psi\bar{\psi}\lambda\bar{\lambda} + 15i\frac{\dot{\lambda}\bar{\eta} + \dot{\bar{\lambda}}\eta}{2N}\lambda\bar{\lambda} + 6s\psi\bar{\psi}(\lambda\bar{\eta} - \bar{\lambda}\eta) \\ &+3s\eta\bar{\eta} + 3\dot{s}\frac{\lambda\dot{\bar{\lambda}} - \bar{\lambda}\dot{\lambda}}{N^2} - 3\dot{a}\frac{\psi\dot{\bar{\lambda}} - \bar{\psi}\dot{\lambda}}{2N^2a}\lambda\bar{\lambda} - \dot{a}\psi\bar{\psi}\frac{\lambda\dot{\bar{\lambda}} - \bar{\lambda}\dot{\lambda}}{N^2a} - 16s\dot{a}^2\frac{\psi\bar{\lambda} - \bar{\psi}\lambda}{N^2a^2} + 4\frac{\dot{a}\dot{s}\psi\bar{\psi}}{N^2a} \\ &-7\dot{a}\dot{s}\frac{\psi\bar{\lambda} - \bar{\psi}\lambda}{N^2a} + 3s^2(\psi\bar{\eta} - \bar{\psi}\eta) + 6s^2(\lambda\bar{\eta} - \bar{\lambda}\eta) - 27s\frac{\psi\bar{\eta} - \bar{\psi}\eta}{2}\lambda\bar{\lambda} + is\psi\bar{\psi}\frac{\lambda\dot{\bar{\lambda}} + \bar{\lambda}\dot{\lambda}}{N} \\ &+i\frac{\eta\dot{\bar{\eta}} + \bar{\eta}\dot{\eta}}{N} + 2i\psi\bar{\psi}\frac{\lambda\dot{\bar{\eta}} + \bar{\lambda}\dot{\eta}}{N} - 21is\frac{\psi\dot{\bar{\lambda}} + \bar{\psi}\dot{\lambda}}{2N}\lambda\bar{\lambda} + 4is^2\frac{\psi\dot{\bar{\lambda}} + \bar{\psi}\dot{\lambda}}{N} + 10is^2\frac{\lambda\dot{\bar{\lambda}} + \bar{\lambda}\dot{\lambda}}{N}\end{aligned}$$

$$+12\frac{\dot{s}\dot{a}\lambda\bar{\lambda}}{N^2a}-2\psi\bar{\psi}\frac{\dot{\lambda}\dot{\bar{\lambda}}}{N^2}+6s^3(\psi\bar{\lambda}-\bar{\psi}\lambda)+2i\dot{\varphi}\frac{\psi\bar{\lambda}+\bar{\psi}\lambda}{N}+6\dot{a}\frac{\dot{\lambda}\bar{\eta}-\dot{\bar{\lambda}}\eta}{N^2a}+8\frac{s\dot{a}^2\psi\bar{\psi}}{N^2a^2}\Bigg)\Bigg]. \tag{7.41}$$

### 7.2.2 Hamiltonian formulation

In this case, we return to the notation of Section 6.1 for the fermionic momenta, $\pi_\lambda = \partial L/\partial\dot{\bar{\lambda}}$, $\pi_{\bar{\lambda}} = -\partial L/\partial\dot{\lambda}$, and so on.

Primary constraints arise, as always, from the momenta of gauge fields: $p_N = 0$, $\pi_\psi = 0$, $\pi_{\bar{\psi}} = 0$. The remaining velocities can be solved for and, computing $H = \dot{N}p_N - \dot{\psi}p_{\bar{\psi}} + \dot{\bar{\psi}}\pi_\psi + \dot{a}p_a + \dot{\varphi}p_\varphi + \dot{s}p_s - \dot{\lambda}\pi_{\bar{\lambda}} + \dot{\bar{\lambda}}\pi_\lambda - \dot{\eta}\pi_{\bar{\eta}} + \dot{\bar{\eta}}\pi_\eta - L$, we get the familiar form $H = NH_0 + \frac{1}{2}(\Psi\bar{S} - \bar{\Psi}S)$, where

$$\begin{aligned}
H_0 = {}& \frac{\kappa^2}{12\alpha^2}(1+2\alpha\varphi)\frac{p_\varphi^2}{a^3} - \frac{\kappa^2}{6\alpha}\frac{p_a p_\varphi}{a^2} + \frac{\kappa^2}{12\alpha}\frac{p_s^2}{a^3} + \frac{3}{\kappa^2}a^3(\alpha\varphi^2+s^2) + \frac{\kappa^2}{6\alpha}\frac{\pi_\lambda\pi_{\bar{\eta}} - \pi_{\bar{\lambda}}\pi_\eta}{a^3} \\
& - \frac{3\kappa^2}{4\alpha}\frac{p_s p_\varphi + 2sp_\varphi^2}{a^3}\lambda\bar{\lambda} + \frac{\kappa^2}{4\alpha}\frac{p_s + 4sp_\varphi}{a^3}(\lambda\pi_{\bar{\eta}} - \bar{\lambda}\pi_\eta) - \frac{is}{2}(\eta\pi_{\bar{\eta}} + \bar{\eta}\pi_\eta) - \frac{6\alpha}{\kappa^2}a^3 s\eta\bar{\eta} \\
& - \frac{7\kappa^2}{12}\frac{p_\varphi}{a^3}(\lambda\pi_{\bar{\lambda}} - \bar{\lambda}\pi_\lambda) - \frac{18\alpha}{\kappa^2}a^3\lambda\bar{\lambda}\eta\bar{\eta} - \frac{i}{2}(\varphi + 6s^2)(\lambda\pi_{\bar{\eta}} + \bar{\lambda}\pi_\eta) - \frac{2\kappa^2}{3\alpha}\frac{s\pi_\eta\pi_{\bar{\eta}}}{a^3} \\
& - \frac{\kappa^2}{2\alpha}\frac{p_\varphi}{a^3}(\eta\pi_{\bar{\eta}} - \bar{\eta}\pi_\eta) + 2is(\lambda\pi_{\bar{\lambda}} + \bar{\lambda}\pi_\lambda) + \frac{\kappa^2}{\alpha}p_\varphi^2\frac{\lambda\bar{\eta} - \bar{\lambda}\eta}{a^3} + \frac{3}{\kappa^2}a^3 s(3-4\alpha s^2)\lambda\bar{\lambda} \\
& + \frac{3a^3}{2\kappa^2}(1 - 7\alpha\varphi - 6\alpha s^2)(\lambda\bar{\eta} - \bar{\lambda}\eta) - \frac{3}{4}i(p_s + 5sp_\varphi)(\lambda\bar{\eta} + \bar{\lambda}\eta) + \frac{i}{2}(\eta\pi_{\bar{\lambda}} + \bar{\eta}\pi_\lambda) \\
& - \frac{15i}{4}\lambda\bar{\lambda}(\eta\pi_{\bar{\eta}} + \bar{\eta}\pi_\eta) + \frac{i}{2\alpha}(\lambda\pi_{\bar{\eta}} + \bar{\lambda}\pi_\eta),
\end{aligned} \tag{7.42a}$$

$$\begin{aligned}
S = {}& \left(iap_a - \frac{3}{\kappa^2}a^3 s - \frac{\kappa^2}{4\alpha}\frac{p_\varphi p_s}{a^3} + \frac{i}{2}\varphi p_\varphi - \frac{ip_\varphi}{2\alpha}(1 - 6\alpha s^2) + \frac{12}{\kappa^2}\alpha a^3 s(\varphi + s^2)\right)\lambda - 2i\lambda\bar{\lambda}\pi_\lambda \\
& + \left(\frac{15}{4}ip_\varphi + \frac{18\alpha}{\kappa^2}a^3 s\right)\lambda\bar{\lambda}\eta + \left(\frac{i}{2}(p_s - 3p_\varphi s) + \frac{\kappa^2}{4\alpha}\frac{p_\varphi^2}{a^3} + \frac{3\alpha}{\kappa^2}a^3(\varphi + 2s^2)\right)\eta + \frac{9\alpha}{2\kappa^2}a^3\lambda\eta\bar{\eta} \\
& + \frac{\kappa^2}{6\alpha}\frac{p_s}{a^3}\pi_\eta + \left(is - \frac{\kappa^2}{6\alpha}\frac{p_\varphi}{a^3}\right)\pi_\lambda - \frac{3i}{2}(\lambda\bar{\eta} + \bar{\lambda}\eta)\pi_\eta - i(\varphi + 2s^2)\pi_\eta = \bar{S}^*,
\end{aligned} \tag{7.42b}$$

The bosonic sector of (7.42a) contains that of (7.12), as it should, plus the Hamiltonian of the scalar field $s$.

Usually, the supersymmetric constraint amounts to a partial differential equation of lower order than the Wheeler-DeWitt equation, which represents an enormous simplification when looking for analytic solutions [6]. That is not quite the case here since there are terms quadratic in the momenta of the scalars in (7.13) and (7.42b), although the equations are significantly simpler.

### 7.2.3 Quantization

Basic (anti)commutators are $\{a, p_a\} = i\hbar$, $\{s, p_s\} = i\hbar$, $\{G, p_G\} = i\hbar$, $\{\lambda, \pi_{\bar{\lambda}}\} = i\hbar$, $\{\bar{\lambda}, \pi_\lambda\} = -i\hbar$, $\{\eta, \pi_{\bar{\eta}}\} = i\hbar$, $\{\bar{\eta}, \pi_\eta\} = -i\hbar$.

As in Chapter 6, we define the following sets of creation and annihilation operators

$$A_\pm = \lambda \pm i\frac{\pi_\lambda}{2\hbar}, \qquad \bar{A}_\pm = \bar{\lambda} \mp i\frac{\pi_{\bar{\lambda}}}{2\hbar}, \tag{7.43a}$$

$$B_\pm = \eta \pm i\frac{\pi_\eta}{2\hbar}, \qquad \bar{B}_\pm = \bar{\eta} \mp i\frac{\pi_{\bar{\eta}}}{2\hbar}, \tag{7.43b}$$

with the only non-vanishing anti-commutators

$$\{A_\pm, \bar{A}_\pm\} = \pm 1, \qquad \{B_\pm, \bar{B}_\pm\} = \pm 1. \tag{7.44}$$

We have a total of sixteen independent states,

$$\begin{bmatrix} |00\rangle & |01\rangle & |02\rangle & |03\rangle \\ |10\rangle & |11\rangle & |12\rangle & |13\rangle \\ |20\rangle & |21\rangle & |22\rangle & |23\rangle \\ |30\rangle & |31\rangle & |32\rangle & |33\rangle \end{bmatrix} = \begin{bmatrix} 1 & \bar{B}_+ & \bar{B}_- & \bar{B}_+\bar{B}_- \\ \bar{A}_+ & \bar{A}_+\bar{B}_+ & \bar{A}_+\bar{B}_- & \bar{A}_+\bar{B}_+\bar{B}_- \\ \bar{A}_- & \bar{A}_-\bar{B}_+ & \bar{A}_-\bar{B}_- & \bar{A}_-\bar{B}_+\bar{B}_- \\ \bar{A}_+\bar{A}_- & \bar{A}_+\bar{A}_-\bar{B}_+ & \bar{A}_+\bar{A}_-\bar{B}_- & \bar{A}_+\bar{A}_-\bar{B}_+\bar{B}_- \end{bmatrix} |00\rangle \tag{7.45}$$

where the ground state $|00\rangle$ is such that $A_\pm|00\rangle = 0 = B_\pm|00\rangle$.

Denoting the states as $|\mu\nu\rangle$ where $\mu, \nu = 0, 1, 2, 3$ (no connection with relativity), their norms are related as $\langle 00|00\rangle = \langle 10|10\rangle = -\langle 20|20\rangle = -\langle 30|30\rangle$, $\langle 00|00\rangle = \langle 01|01\rangle = -\langle 02|02\rangle = -\langle 03|03\rangle$. The remaining states satisfy $\langle \mu\nu|\mu\nu\rangle = \langle \mu 0|\mu 0\rangle\langle 0\nu|0\nu\rangle$.

Solving equations (7.43) for the original set of fermionic variables

$$\lambda = \frac{1}{2}(A_+ + A_-), \qquad \bar{\lambda} = \frac{1}{2}(\bar{A}_+ + \bar{A}_-), \tag{7.46a}$$

$$\pi_\lambda = -i\hbar(A_+ - A_-), \qquad \pi_{\bar{\lambda}} = i\hbar(\bar{A}_+ - \bar{A}_-), \tag{7.46b}$$

$$\eta = \frac{1}{2}(B_+ + B_-), \qquad \bar{\eta} = \frac{1}{2}(\bar{B}_+ + \bar{B}_-), \tag{7.46c}$$

$$\pi_\eta = -i\hbar(B_+ - B_-), \qquad \pi_{\bar{\eta}} = i\hbar(\bar{B}_+ - \bar{B}_-). \tag{7.46d}$$

A generic state is of the form

$$|\Psi\rangle = \psi_{\mu\nu}(a, G, s)|\mu\nu\rangle \tag{7.47}$$

The supersymmetric constraint operators read, under Weyl ordering,

$$\begin{aligned} S = &\left(\frac{i}{2}\{a, p_a\} - \frac{3}{\kappa^2}a^3 s - \frac{\kappa^2}{4\alpha}\frac{p_G p_s}{a^3} + \frac{i}{4}\{G, p_G\} - \frac{ip_G}{2\alpha}(1 - 6\alpha s^2) + \frac{12\alpha}{\kappa^2}a^3 s(G + s^2)\right)\lambda \\ &+ \frac{\kappa^2}{6\alpha}\frac{p_s}{a^3}\pi_\eta - i\lambda[\bar{\lambda}, \pi_\lambda] + 3\left(\frac{5}{4}ip_G + \frac{6\alpha}{\kappa^2}a^3 s\right)\lambda\bar{\lambda}\eta + \left(\frac{i}{2}\left(p_s - 3p_G s\right) + \frac{\kappa^2}{4\alpha}\frac{p_G^2}{a^3} + \frac{3\alpha}{\kappa^2}a^3(G + 2s^2)\right)\eta \\ &+ \left(is - \frac{\kappa^2}{6\alpha}\frac{p_G}{a^3}\right)\pi_\lambda - \frac{3i}{4}\lambda[\bar{\eta}, \pi_\eta] - \frac{3i}{2}\bar{\lambda}\eta\pi_\eta - i(G + 2s^2)\pi_\eta + \frac{9\alpha}{2\kappa^2}a^3\lambda\eta\bar{\eta} \end{aligned} \tag{7.48}$$

$$\equiv A\lambda - iB\pi_\eta - 2i\lambda\bar{\lambda}\pi_\lambda + 3C\lambda\bar{\lambda}\eta + D\eta + iF\pi_\lambda - i\frac{3}{2}(\lambda\bar{\eta} + \bar{\lambda}\eta)\pi_\eta - iH\pi_\eta + K\lambda\eta\bar{\eta}, \tag{7.49}$$

and

$$
\begin{aligned}
-\bar{S} &= \left(\frac{i}{2}\{a,p_a\} + \frac{3}{\kappa^2}a^3 s + \frac{\kappa^2}{4\alpha}\frac{p_G p_s}{a^3} + \frac{i}{4}\{G,p_G\} - \frac{ip_G}{2\alpha}(1-6\alpha s^2) - \frac{12\alpha}{\kappa^2}a^3 s(G+s^2)\right)\bar{\lambda} \\
&\quad - \frac{\kappa^2}{6\alpha}\frac{p_s}{a^3}\pi_{\bar{\eta}} + i\bar{\lambda}[\lambda,\pi_{\bar{\lambda}}] + 3\left(\frac{5}{4}ip_G - \frac{6\alpha}{\kappa^2}a^3 s\right)\lambda\bar{\lambda}\bar{\eta} + \left(\frac{i}{2}(p_s - 3p_G s) - \frac{\kappa^2}{4\alpha}\frac{p_G^2}{a^3} - \frac{3\alpha}{\kappa^2}a^3(G+2s^2)\right)\bar{\eta} \\
&\quad + \left(is + \frac{\kappa^2}{6\alpha}\frac{p_G}{a^3}\right)\pi_{\bar{\lambda}} + \frac{3i}{2}\lambda\bar{\eta}\pi_{\bar{\eta}} + \frac{3i}{4}\bar{\lambda}[\eta,\pi_{\bar{\eta}}] - i(G+2s^2)\pi_{\bar{\eta}} - \frac{9\alpha}{2\kappa^2}a^3\bar{\lambda}\eta\bar{\eta} \qquad (7.50) \\
&\equiv A'\bar{\lambda} + iB'\pi_{\bar{\eta}} + 2i\bar{\lambda}\lambda\pi_{\bar{\lambda}} + 3C'\lambda\bar{\lambda}\bar{\eta} + D'\bar{\eta} + iF'\pi_{\bar{\lambda}} + i\frac{3}{2}(\lambda\bar{\eta}+\bar{\lambda}\eta)\pi_{\bar{\eta}} - iH'\pi_{\bar{\eta}} - K'\bar{\lambda}\eta\bar{\eta} \qquad (7.51)
\end{aligned}
$$

From $S|\Psi\rangle = 0$, we get a set of fifteen equations

$$
\begin{aligned}
0 &= \left(\frac{A}{2}+\hbar F\right)\psi_{10} + \left(-\frac{A}{2}+\hbar F\right)\psi_{20} + \left(-B\hbar + \frac{D}{2} - \hbar H\right)\psi_{01} + \left(-B\hbar - \frac{D}{2} - \hbar H\right)\psi_{02}, \\
0 &= \left(\frac{A}{2} - \hbar F - \hbar\right)\psi_{30} + \left(B\hbar + \frac{3}{8}C - \frac{D}{2} + \hbar H\right)\psi_{11} + \left(B\hbar - \frac{3}{8}C + \frac{D}{2} + \hbar H\right)\psi_{12} - \frac{3}{8}C\psi_{21} + \frac{3}{8}C\psi_{22} + \frac{3}{4}\hbar\psi_{03}, \\
0 &= \left(\frac{A}{2} + \hbar F - \hbar\right)\psi_{30} + \left(B\hbar - \frac{3}{8}C - \frac{D}{2} + \hbar H\right)\psi_{21} + \left(B\hbar + \frac{3}{8}C + \frac{D}{2} + \hbar H\right)\psi_{22} + \frac{3}{8}C\psi_{11} - \frac{3}{8}C\psi_{12} + \frac{3}{4}\hbar\psi_{03}, \\
0 &= \left(-B\hbar + \frac{D}{2} - \hbar H\right)\psi_{31} + \left(-B\hbar - \frac{D}{2} - \hbar H\right)\psi_{32} + \frac{3}{4}\hbar\psi_{23} - \frac{3}{4}\hbar\psi_{13}, \\
0 &= \left(\frac{A}{2} + \hbar F - \frac{3}{8}\hbar - \frac{K}{8}\right)\psi_{11} + \left(-\frac{A}{2} + \hbar F + \frac{3}{8}\hbar + \frac{K}{8}\right)\psi_{21} + \left(B\hbar + \frac{D}{2} + \hbar H\right)\psi_{03} + \left(\frac{K}{8} - \frac{3}{8}\hbar\right)\psi_{12} + \left(\frac{3}{8}\hbar - \frac{K}{8}\right)\psi_{22}, \\
0 &= \left(\frac{A}{2} - \hbar F - \frac{11}{8}\hbar - \frac{K}{8}\right)\psi_{31} + \left(-B\hbar + \frac{3}{8}C - \frac{D}{2} - \hbar H\right)\psi_{13} - (3/2)C\psi_{23} + \left(\frac{K}{8} - \frac{3}{8}\hbar\right)\psi_{32}, \\
0 &= \left(\frac{A}{2} + \hbar F - \frac{11}{8}\hbar - \frac{K}{8}\right)\psi_{31} + \left(-B\hbar - \frac{3}{8}C - \frac{D}{2} - \hbar H\right)\psi_{23} + \frac{3}{8}C\psi_{13} + \left(\frac{K}{8} - \frac{3}{8}\hbar\right)\psi_{32}, \\
0 &= \left(B\hbar + \frac{D}{2} + \hbar H\right)\psi_{33}, \\
0 &= \left(\frac{A}{2} + \hbar F - \frac{3}{8}\hbar + \frac{K}{8}\right)\psi_{12} + \left(-\frac{A}{2} + \hbar F + \frac{3}{8}\hbar - \frac{K}{8}\right)\psi_{22} + \left(-B\hbar + \frac{D}{2} - \hbar H\right)\psi_{03} + \left(-\frac{3}{8}\hbar - \frac{K}{8}\right)\psi_{11} + \left(\frac{3}{8}\hbar + \frac{K}{8}\right)\psi_{21}, \\
0 &= \left(\frac{A}{2} - \hbar F - \frac{11}{8}\hbar + \frac{K}{8}\right)\psi_{32} + \left(B\hbar + \frac{3}{8}C - \frac{D}{2} + \hbar H\right)\psi_{13} - \frac{3}{8}C\psi_{23} + \left(-\frac{3}{8}\hbar - \frac{K}{8}\right)\psi_{31}, \\
0 &= \left(\frac{A}{2} + \hbar F - \frac{11}{8}\hbar + \frac{K}{8}\right)\psi_{32} + \left(B\hbar - \frac{3}{8}C - \frac{D}{2} + \hbar H\right)\psi_{23} + \frac{3}{8}C\psi_{13} + \left(-\frac{3}{8}\hbar - \frac{K}{8}\right)\psi_{31}, \\
0 &= \left(-B\hbar + \frac{D}{2} - \hbar H\right)\psi_{33}, \\
0 &= \left(\frac{A}{2} + \hbar F - 3\hbar/4\right)\psi_{13} + \left(-\frac{A}{2} + \hbar F + 3\hbar/4\right)\psi_{23}, \\
0 &= \left(\frac{A}{2} - \hbar F - 7\hbar/4\right)\psi_{33}, \\
0 &= \left(\frac{A}{2} + \hbar F - 7\hbar/4\right)\psi_{33},
\end{aligned}
$$

while, from $-\bar{S}|\Psi\rangle = 0$, we get another set of fifteen equations,

$$0 = \left(\frac{A'}{2} - \hbar F' - \frac{7}{4}\hbar\right)\psi_{00},$$

$$0 = \left(\frac{A'}{2} + \hbar F' - \frac{7}{4}\hbar\right)\psi_{00},$$

$$0 = \left(-\frac{A'}{2} - \hbar F' + \frac{3}{4}\hbar\right)\psi_{10} + \left(\frac{A'}{2} - \hbar F' - \frac{3}{4}\hbar\right)\psi_{20},$$

$$0 = \left(-B'\hbar + \frac{D'}{2} + \hbar H'\right)\psi_{00},$$

$$0 = \left(\frac{A'}{2} - \hbar F' - \frac{11}{8}\hbar + \frac{K'}{8}\right)\psi_{01} + \left(B'\hbar + \frac{3}{8}C' - \frac{D'}{2} - \hbar H'\right)\psi_{10} - \frac{3}{8}C'\psi_{20} + \left(-\frac{3}{8}\hbar - \frac{K'}{8}\right)\psi_{02},$$

$$0 = \left(\frac{A'}{2} + \hbar F' - \frac{11}{8}\hbar + \frac{K'}{8}\right)\psi_{01} + \left(B'\hbar - \frac{3}{8}C' - \frac{D'}{2} - \hbar H'\right)\psi_{20} + \frac{3}{8}C'\psi_{10} + \left(-\frac{3}{8}\hbar - \frac{K'}{8}\right)\psi_{02},$$

$$0 = \left(-\frac{A'}{2} - \hbar F' + \frac{3}{8}\hbar - \frac{K'}{8}\right)\psi_{11} + \left(\frac{A'}{2} - \hbar F' - \frac{3}{8}\hbar + \frac{K'}{8}\right)\psi_{21} + \left(-B'\hbar + \frac{D'}{2} + \hbar H'\right)\psi_{30} + \left(\frac{3}{8}\hbar + \frac{K'}{8}\right)\psi_{12} + \left(-\frac{3}{8}\hbar - \frac{K'}{8}\right)\psi_{22},$$

$$0 = \left(B'\hbar + \frac{D'}{2} - \hbar H'\right)\psi_{00},$$

$$0 = \left(\frac{A'}{2} - \hbar F' - \frac{11}{8}\hbar - \frac{K'}{8}\right)\psi_{02} + \left(-B'\hbar + \frac{3}{8}C' - \frac{D'}{2} + \hbar H'\right)\psi_{10} - \frac{3}{8}C'\psi_{20} + \left(\frac{K'}{8} - \frac{3}{8}\hbar\right)\psi_{01},$$

$$0 = \left(\frac{A'}{2} + \hbar F' - \frac{11}{8}\hbar - \frac{K'}{8}\right)\psi_{02} + \left(-B'\hbar - \frac{3}{8}C' - \frac{D'}{2} + \hbar H'\right)\psi_{20} + \frac{3}{8}C'\psi_{10} + \left(\frac{K'}{8} - \frac{3}{8}\hbar\right)\psi_{01},$$

$$0 = \left(-\frac{A'}{2} - \hbar F' + \frac{3}{8}\hbar + \frac{K'}{8}\right)\psi_{12} + \left(\frac{A'}{2} - \hbar F' - \frac{3}{8}\hbar - \frac{K'}{8}\right)\psi_{22} + \left(B'\hbar + \frac{D'}{2} - \hbar H'\right)\psi_{30} + \left(\frac{3}{8}\hbar - \frac{K'}{8}\right)\psi_{11} + \left(\frac{K'}{8} - \frac{3}{8}\hbar\right)\psi_{21},$$

$$0 = \left(-B'\hbar - \frac{D'}{2} + \hbar H'\right)\psi_{01} + \left(-B'\hbar + \frac{D'}{2} + \hbar H'\right)\psi_{02} + \frac{3}{4}\hbar\psi_{10} - \frac{3}{4}\hbar\psi_{20},$$

$$0 = \left(\frac{A'}{2} - \hbar F' - \hbar\right)\psi_{03} + \left(B'\hbar - \frac{3}{8}C' + \frac{D'}{2} - \hbar H'\right)\psi_{11} + \left(B'\hbar + \frac{3}{8}C' - \frac{D'}{2} - \hbar H'\right)\psi_{12} + \frac{3}{8}C'\psi_{21} - \frac{3}{8}C'\psi_{22} + \frac{3}{4}\hbar\psi_{30},$$

$$0 = \left(\frac{A'}{2} + \hbar F' - \hbar\right)\psi_{03} + \left(B'\hbar + \frac{3}{8}C' + \frac{D'}{2} - \hbar H'\right)\psi_{21} + \left(B'\hbar - \frac{3}{8}C' - \frac{D'}{2} - \hbar H'\right)\psi_{22} - \frac{3}{8}C'\psi_{11} + \frac{3}{8}C'\psi_{12} + \frac{3}{4}\hbar\psi_{30},$$

$$0 = \left(-\frac{A'}{2} - \hbar F'\right)\psi_{13} + \left(\frac{A'}{2} - \hbar F'\right)\psi_{23} + \left(-B'\hbar - \frac{D'}{2} + \hbar H'\right)\psi_{31} + \left(-B'\hbar + \frac{D'}{2} + \hbar H'\right)\psi_{32}.$$

Remarkably, we get two decoupled components, those of the empty and filled fermionic states, $\psi_{00}$ and $\psi_{33}$ (as always with the N=2 supersymmetry), which satisfy four equations each, but only three of them are independent. On the other hand, the fourteen wave functions of the intermediate states are all coupled by remaining twenty-two equations.

Adding and subtracting the equations for $\psi_{00}$, we get

$$0 = \left(\hbar a\partial_a - \hbar + \frac{3}{\kappa^2}a^3 s - \hbar^2\frac{\kappa^2}{4\alpha}\frac{\partial_G\partial_s}{a^3} + \frac{\hbar}{2\alpha}\left(\alpha G - 1 + 6\alpha s^2\right)\partial_G - \frac{12\alpha}{\kappa^2}a^3 s(G+s^2)\right)\psi_0, \tag{7.52a}$$

$$0 = \left(s - \frac{\hbar\kappa^2}{6\alpha}\frac{\partial_G}{a^3}\right)\psi_0, \tag{7.52b}$$

$$0 = \left(\frac{\hbar}{2}(\partial_s - 3s\partial_G) + \frac{\hbar^2\kappa^2}{4\alpha}\frac{\partial_G^2}{a^3} - \frac{3\alpha}{\kappa^2}a^3(G+2s^2)\right)\psi_0, \tag{7.52c}$$

$$0 = \left(\frac{\hbar\kappa^2}{6\alpha}\frac{\partial_s}{a^3} - \left(G+2s^2\right)\right)\psi_0, \tag{7.52d}$$

whereas for $\psi_{33}$, we find

$$0 = \left(\hbar a\partial_a - \hbar - \frac{3}{\kappa^2}a^3 s + \hbar^2\frac{\kappa^2}{4\alpha}\frac{\partial_G\partial_s}{a^3} + \frac{\hbar}{2\alpha}\left(\alpha G - 1 + 6\alpha s^2\right)\partial_G + \frac{12\alpha}{\kappa^2}a^3 s(G+s^2)\right)\psi_{15}, \tag{7.53a}$$

$$0 = \left(s + \frac{\hbar\kappa^2}{6\alpha}\frac{\partial_G}{a^3}\right)\psi_{15}, \tag{7.53b}$$

$$0 = \left(\frac{\hbar}{2}(\partial_s - 3s\partial_G) - \frac{\hbar^2\kappa^2}{4\alpha}\frac{\partial_G^2}{a^3} + \frac{3\alpha}{\kappa^2}a^3(G+2s^2)\right)\psi_{15}, \tag{7.53c}$$

$$0 = \left(\frac{\hbar\kappa^2}{6\alpha}\frac{\partial_s}{a^3} + (G+2s^2)\right)\psi_{15}. \tag{7.53d}$$

We have unique solutions given by

$$\psi_{00} = a^{\frac{5}{2}}\exp\left(\frac{6\alpha}{\hbar\kappa^2}a^3 sG + \frac{4\alpha}{\hbar\kappa^2}a^3 s^3\right), \tag{7.54}$$

$$\psi_{33} = a^{\frac{5}{2}}\exp\left(-\frac{6\alpha}{\hbar\kappa^2}a^3 sG - \frac{4\alpha}{\hbar\kappa^2}a^3 s^3\right) \tag{7.55}$$

which, are relate to the no-boundary and wormhole states that corresponds to classically forbidden configurations. Note that these satisfy the boundary condition $\psi(0,G,s) = 0$. An interesting difference is that here both states can be defined to have positive norm, due to the occurrence of a double minus sign. As mentioned above, $\langle 33|33\rangle = \langle 30|30\rangle\langle 03|03\rangle = \langle 00|00\rangle$.

The problem of decoupling the remaining wavefunction components and solving the equations numerically will be addressed in future work.

# Chapter 8

# Conclusions

In this work, we studied classical and quantum aspects of higher derivative supersymmetric theories. We put forward two supersymmetric extensions of the FRW model of Starobinsky, with real and complex fermions, using a superfield formalism for 1D supergravity. In the case of N=1 supersymmetry, the supermultiplets only contain two components and no auxiliary fields. Despite the small number of degrees of freedom, it is possible to construct a Lagrangian whose bosonic sector contains exactly $R + \frac{\alpha}{6}R^2$. Then, we considered an N=2 complex multiplet containing two scalar bosons and one complex scalar fermion.

In comparison to N=1, the N=2 model required more maneuvering. A kinetic term for the curvature superfield leads to the bosonic term $R^2$, however, it also generates a kinetic term for the otherwise auxiliary of the scale factor multiplet. Moreover, this new field comes with a negative quartic potential preventing the inflationary solution, unless we set exactly vanishing initial conditions. We fixed this by including a superpotential term of the form $F(\mathcal{R})$. Choosing $F(\mathcal{R}) = -8\mathcal{R}^3$, the scalar potential of this extra field was significantly improved. The final Lagrangian contains, besides the FRW Starobinsky model, a minimally coupled massive scalar field.

By means of one additional superfield, we obtained equivalent actions leading to second-order theories. These are not yet in the form ordinary supersymmetric gravity-matter theories, since they have non-minimal coupling and there are quadratic terms if the fermionic velocities. Nonetheless, they are already suitable for the Hamiltonian formulation. In fact, the canonical analysis is somewhat simpler than that of ordinary fermionic theories, because we do not have to deal with second-class constraints. The full tensor-scalar duals would require not one but extra superfields to accommodate the new fermionic degrees of freedom of the higher derivative theory. The manifestly supersymmetric way to do this should involve the superfield generalization of the Legendre-Weyl transformation (as we did for the $F(\mathcal{R})$ action). However, for the two models constructed, it is the highest component of the extra superfield that plays the role of the scalaron. Hence, it might be the case that the superfield transformation involve not just $\Phi$, but $[\nabla_\Theta, \nabla_{\bar{\Theta}}]\Phi$. Determining the proper transformation rule will be a topic addressed in future work. Whatever the final form the action, the bosonic sectors will be of the form (5.17), and for the N=2 model (7.42a), the contribution of the extra field would have the form of a non-linear sigma model (cf. the two-field model in [26]).

Additionally, we studied the $F(\mathcal{R})$ action for N=2. This was our naive ansatz for a supersymmetric extension of the bosonic $f(R)$. It certainly generates function of $R$ and $s$, with $s$ a higher-derivative degree of freedom. However, by integrating out $s$, we generally get non-polynomial functions, not to mention severe restrictions on the range of the $R$. Nonetheless, it provides a very handy action to explore some classical aspects of higher-derivative theories. For example, we performed and showed the equivalence of the Hamiltonian formulations arising from the original higher-order form and the one derived from its dual form with an extra superfield. Also, for this action we obtain its full dual Einstein gravity-matter form by generalizing the Weyl-Legendre transformation.

The actions constructed are examples of so-called pseudo classical mechanics because the dynamical variables are elements of Grassmann algebras. These actions find application as quantum theories, which in this case takes us to quantum supersymmetric cosmology. The supersymmetric wave function has multiple bosonic components associated to fermionic states. Our study indicates that the wave functions of the empty and filled fermionic states decouple even with higher derivative theories (for N=2 supersymmetry): $F(\mathcal{R})$ (second-order) and Starobinsky (third-order). They generally satisfy simple PDE's that determine them uniquely. Also, these states can be seen as the analogues of the no-boundary and wormhole solutions, related to classically forbidden regions in configuration space. For the $F(R)$ action, one could obtain general exact solutions and, by give some examples of functions $F$ leading to square integrable wave-functions. For the N=2 model of Starobinsky, there is no freedom left in the exact solutions. Unfortunately, the obtained wave functions are not square integrable, which is not unusual in quantum cosmology. Perhaps obtaining more interesting exact solutions would require to enrich the model by adding, say non-vanishing spatial curvature and, especially, a positive cosmological constant.

On the other hand, we still have other fourteen components of the wave function. Decoupling the set equations seems not that straightforward as with the other examples worked out, $F(\mathcal{R}$ and N=1 Starobinsky. A more detailed analysis of the quantum dynamics of the intermediate states will be the subject of future investigation. Our experience with the other examples, suggests that by decoupling the equations we obtain the bosonic WDW equation plus some extra terms. This is exactly what happened with the $N = 1$ model, for which we obtain numerical solutions behaving very similarly to the bosonic wavefunction. Thus, while it is the case that supersymmetry imposes severe restrictions on a sector of the theory, the dynamics of the intermediate states is, by contrast, as complex as that of the purely bosonic models. This conclusion is not totally unexpected since supersymmetric theories have mode degrees of freedom than their purely bosonic counterparts [44].

# Appendix A

# Saddle-point method

We compute the wavefunction in the momentum representation following [7]. The canonical pair that is useful is

$$\int d^3x\sqrt{h} = 2\pi^2\sigma^3 a^3, \qquad \frac{-K_E}{18\pi^2\sigma^2} = \frac{1}{6\pi^2\sigma^3}\frac{p_a'}{a^2}. \tag{A.1}$$

where $K_E = \frac{3}{\sigma}\frac{a'}{Na}$ and $p_a' = -\frac{aa'}{N}$ (the pair $a, p_a'$ is not helpful, since $p_a$ also has two classical extreme values).

Defining $k = \frac{\sigma}{9}K_E = \frac{a'}{3Na}$, the $k$-momentum wavefunction is given by the Laplace transform

$$\Phi_0[k_0] = \int_0^\infty \delta a\; e^{-k_0 a^3}\Psi_0[a] = \int_0^\infty \delta a\; e^{-I^{k_0}[a]}, \tag{A.2}$$

where $I^k[a]$ is the action suitable for keeping $k$ fixed on the boundary instead of $a$,

$$I^k \equiv ka^3 + I_E[a]. \tag{A.3}$$

The inverse transformation reads

$$\Psi_0[a_0] = \frac{i}{2\pi}\int_C dk\; e^{ka_0^3}\Phi_0[k], \tag{A.4}$$

with the contour of integration going from $-i\infty$ up to $i\infty$ to the right of any singularity of $\Phi_0[k]$. Since we are working in the momentum representation, the 3-surface boundary has a given value $k_0$, where

$$k = \frac{\lambda}{3}\cot\theta, \tag{A.5}$$

instead of the radius $a \propto \sin\theta$.

Evaluating the action on (A.3) yields,

$$I^{k_0} = \frac{1}{3\lambda^2}\Big(\kappa_0/(1+\kappa_0^2)^{\frac{1}{2}} - 1\Big). \tag{A.6}$$

where $\kappa_0 = \frac{3}{\lambda}k_0 = \cot\theta_0$.

Since there is only one extreme configuration $\theta_0$ giving the prescribed value of $k$ on the boundary, the approximation to the wavefunction (A.2) is

$$\Phi_0(k_0) = e^{-I^{k_0}}, \tag{A.7}$$

up to a a normalization constant. The corresponding zeroth order $\Psi_0[a_0]$ is given by (A.4)

$$\Psi_0[a_0] = \frac{i}{2\pi}\int_C dk\; e^{-I_E^k[a]}, \tag{A.8}$$

where $I_E^k[a]$ is the action suitable for fixing $a$ on the boundary,

$$I_E^k[a] = -\frac{\lambda}{3}\kappa a^3 + \frac{1}{3\lambda^2}\left(\frac{\kappa}{\sqrt{1+\kappa^2}} - 1\right). \tag{A.9}$$

The contour integral is computed using the steepest descend method.

**Case 1:** $\lambda a_0 < 1$

The action possesses saddle points at real values of opposite sign,

$$\kappa_0 = \pm\sqrt{(\lambda a_0)^{-2} - 1}. \tag{A.10}$$

They correspond to two different values $\theta_0 \leq \frac{\pi}{2}$ or $\theta = \pi - \theta_0$, at which a 3-sphere of radius $a_0$ fits in a 4-sphere of radius $\lambda$. Since $\kappa = \cot\theta$, the positive value corresponds to the configuration in which the 3-sphere of radius $a_0$ is boundary of less than a hemi-4-sphere, whereas the negative value corresponds to a four-geometry consisting of more than the hemi-4-sphere. Evaluating A.9) at the real values (A.10) yields

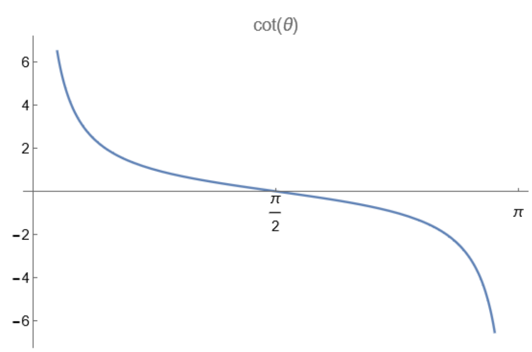
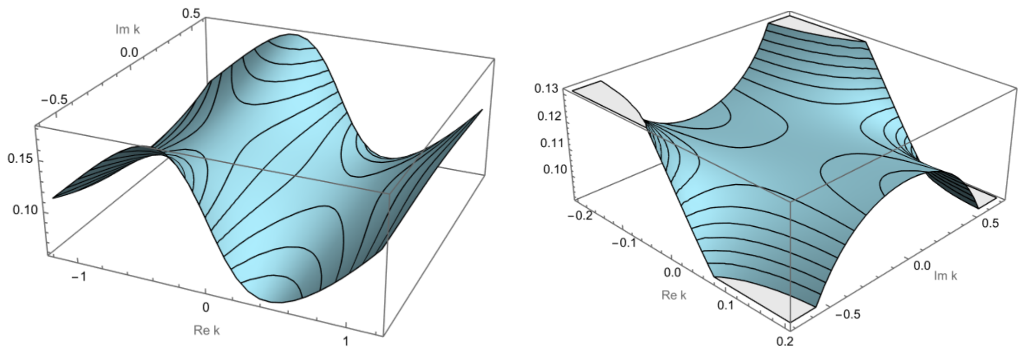



Figure A.1: A given value of the momentum $K$, there corresponds a single geometric configuration, in contrast to the scale factor which is proportional to $\sin\theta$.

$$I_\pm^k(a_0) = \frac{1}{3\lambda^2}\left(-1 \pm (1-\lambda^2 a_0^2)^{3/2}\right) \tag{A.11}$$

The steepest descent path going through the positive value lies along the direction $\arg\kappa = \frac{\pi}{2}$, while the one passing through the negative value lies along the $\arg\kappa = 0$. However, the integration contour in (A.8), parallel to the imaginary axis, can only be distorted into a *steepest descent* contour passing through the positive extreme value of (A.10). Therefore, the zeroth order approximation to the wave function is

$$\Psi_0[a_0] \propto \exp\left[\frac{1}{3\lambda^2}\left(1-(1-\lambda^2 a_0^2)^{3/2}\right)\right]. \tag{A.12}$$




(a) (b)

Figure A.2: Graphs of the analytic landscape $\mathfrak{Re}^2 F + \mathfrak{Im}^2 F$ for $F(\kappa)$ given by (A.9) for (a) $\lambda a < 1$ and (b) $\lambda a > 1$, with $\lambda = 1$, showing their pure real and pure imaginary saddle points, respectively.

**Case 2:** $\lambda a_0 > 1$

There is no real solution because a 3-sphere of radius $a_0 > 1/\lambda$ cannot fit anywhere in a 4-sphere of radius $1/\lambda$. However, we can consider complex geometries. Saddle points occur a purely imaginary values of $\kappa$,

$$\kappa_0 = \pm i\sqrt{1-(\lambda a_0)^{-2}} \tag{A.13}$$

yielding

$$I^k_{\pm}(a) = \frac{1}{3\lambda^2}\left(-1 \mp i(\lambda^2 a_0^2 - 1)^{3/2}\right) \tag{A.14}$$

The steepest descent path for the saddle point above the real axis lies along the direction $\arg\kappa = \frac{\pi}{4}$, whereas the other path lies on $\arg\kappa = \frac{3\pi}{4}$. The integration contour can be distorted into a steepest descent contour passing through both saddle points. The wavefunction is then

$$\begin{aligned}\Psi_0[a_0] &= \exp\left[\frac{1}{3\lambda^2} + i\frac{\pi}{2}\right]\left[\exp\left[i\left(\frac{(\lambda^2 a_0^2-1)^{3/2}}{3\lambda^2} - \frac{\pi}{4}\right)\right] + \exp\left[-i\left(\frac{(\lambda^2 a_0^2-1)^{3/2}}{3\lambda^2} - \frac{\pi}{4}\right)\right]\right] \\ &\propto e^{1/3\lambda^2}\cos\left[\frac{(\lambda^2 a_0^2-1)^{3/2}}{3\lambda^2} - \frac{\pi}{4}\right]\end{aligned} \tag{A.15}$$

# Appendix B

# Derivation of some new superspace expressions

## B.1 Transformation of the covariant multiplet

Using the vielbein, we have, $D_\tau = E_\tau{}^t \partial_t + E_\tau{}^{\underline{\theta}} \partial_{\underline{\theta}} = E_\tau{}^t \partial_t + E_\tau{}^{\underline{\theta}} (E_{\underline{\theta}}{}^\tau D_\tau + E_{\underline{\theta}}{}^{\underline{\Theta}} D_{\underline{\Theta}})$. Next, solving for $D_\tau$ yields, $D_\tau = (1 - E_\tau{}^{\underline{\theta}} E_{\underline{\theta}}{}^\tau)^{-1} (E_\tau{}^t \partial_t + E_\tau{}^{\underline{\theta}} E_{\underline{\theta}}{}^{\underline{\Theta}} D_{\underline{\Theta}})$. Now, using $0 = \delta_\tau{}^{\underline{\Theta}} = E_\tau{}^t E_t{}^{\underline{\Theta}} + E_\tau{}^{\underline{\theta}} E_{\underline{\theta}}{}^{\underline{\Theta}}$, we have

$$D_\tau = (1 - E_\tau{}^{\underline{\theta}} E_{\underline{\theta}}{}^\tau)^{-1} E_\tau{}^t (\partial_t - E_t{}^{\underline{\Theta}} D_{\underline{\Theta}}) \equiv K_\tau{}^t \partial_t + K_\tau{}^{\underline{\Theta}} D_{\underline{\Theta}}. \tag{B.1}$$

On the other hand, from (3.23) and the constraints imposed on the torsion components one gets,

$$[D_\tau, D_{\underline{\Theta}}] = 0, \tag{B.2a}$$

$$[D_\Theta, D_{\bar{\Theta}}]_+ = -T_{\Theta\bar{\Theta}}{}^\tau D_\tau, \tag{B.2b}$$

$$D_{\underline{\Theta}}^2 = 0. \tag{B.2c}$$

With (B.1) and (B.2) at hand, we are ready to compute the transformation of the covariant multiplet (3.35) under local supersymmetry (3.32). We will be omitting the superspace dependence, $F$ stands for $F(z) = F(t, \theta, \bar{\theta})$.

$$\begin{aligned}
\delta_\xi F &= -\xi^\tau D_\tau F - \xi^{\underline{\Theta}} D_{\underline{\Theta}} F \\
&= -\xi^\tau K_\tau{}^t \partial_t \phi - (\xi^\tau K_\tau{}^\Theta + \xi^\Theta) D_\Theta F + (\xi^\tau K_\tau{}^{\bar{\Theta}} + \xi^{\bar{\Theta}}) D_{\bar{\Theta}} F
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
\delta_\xi D_\Theta F &= -\xi^\tau D_\tau D_\Theta F - \xi^{\underline{\Theta}'} D_{\underline{\Theta}'} D_\Theta F = -\xi^\tau K_\tau{}^t \partial_t D_\Theta F - \left( \xi^\tau K_\tau{}^{\underline{\Theta}'} + \xi^{\underline{\Theta}'} \right) D_{\underline{\Theta}'} D_\Theta F \\
&= -\xi^\tau K_\tau{}^t \partial_t D_\Theta F + \left( \xi^\tau K_\tau{}^{\bar{\Theta}} + \xi^{\bar{\Theta}} \right) D_{\bar{\Theta}} D_\Theta F \\
&= -\xi^\tau K_\tau{}^t \partial_t D_\Theta F - (1/2) \left( \xi^\tau K_\tau{}^{\bar{\Theta}} + \xi^{\bar{\Theta}} \right) T_{\bar{\Theta}\Theta}{}^\tau D_\tau F + \left( \xi^\tau K_\tau{}^{\bar{\Theta}} + \xi^{\bar{\Theta}} \right) D_{[\bar{\Theta}} D_{\Theta]} F \\
&= -\xi^\tau K_\tau{}^t \partial_t D_\Theta F - (1/2) \left( \xi^\tau K_\tau{}^{\bar{\Theta}} + \xi^{\bar{\Theta}} \right) T_{\bar{\Theta}\Theta}{}^\tau \left( K_\tau{}^t \partial_t F + K_\tau{}^\Theta D_\Theta F - K_\tau{}^{\bar{\Theta}} D_{\bar{\Theta}} F \right) \\
&\quad + \left( \xi^\tau K_\tau{}^{\bar{\Theta}} + \xi^{\bar{\Theta}} \right) G,
\end{aligned} \tag{B.4}$$

$$
\begin{aligned}
\delta_\xi[D_{\bar\Theta},D_\Theta]F &= -\xi^\tau D_\tau[D_{\bar\Theta},D_\Theta]F - \xi^{\underline{\Theta}'}D_{\underline{\Theta}'}[D_{\bar\Theta},D_\Theta]F \\
&= -\xi^\tau K_\tau{}^t\partial_t[D_{\bar\Theta},D_\Theta]F - \left(\xi^\tau K_\tau{}^{\underline{\Theta}'} + \xi^{\underline{\Theta}'}\right)D_{\underline{\Theta}'}[D_{\bar\Theta},D_\Theta]F \\
&= -\xi^\tau K_\tau{}^t\partial_t[D_{\bar\Theta},D_\Theta]F - \left(\xi^\tau K_\tau{}^{\Theta} + \xi^{\Theta}\right)[D_\Theta,D_{\bar\Theta}]_+D_\Theta F - \left(\xi^\tau K_\tau{}^{\bar\Theta} + \xi^{\bar\Theta}\right)[D_{\bar\Theta},D_\Theta]_+D_{\bar\Theta}F \\
&= -\xi^\tau K_\tau{}^t\partial_t[D_{\bar\Theta},D_\Theta]F + \left(\xi^\tau K_\tau{}^{\Theta} + \xi^{\Theta}\right)T_{\Theta\bar\Theta}{}^\tau D_\tau D_\Theta F + \left(\xi^\tau K_\tau{}^{\bar\Theta} + \xi^{\bar\Theta}\right)T_{\bar\Theta\Theta}{}^\tau D_\tau D_{\bar\Theta}F \\
&= -\xi^\tau K_\tau{}^t\partial_t[D_{\bar\Theta},D_\Theta]F + \left(\xi^\tau K_\tau{}^{\Theta} + \xi^{\Theta}\right)T_{\Theta\bar\Theta}{}^\tau\left(K_\tau{}^t\partial_t D_\Theta F - K_\tau{}^{\bar\Theta}D_{\bar\Theta}D_\Theta F\right) \\
&+ \left(\xi^\tau K_\tau{}^{\bar\Theta} + \xi^{\bar\Theta}\right)T_{\bar\Theta\Theta}{}^\tau\left(K_\tau{}^t\partial_t D_{\bar\Theta}F + K_\tau{}^{\Theta}D_\Theta D_{\bar\Theta}F\right) \\
&= -\xi^\tau K_\tau{}^t\partial_t[D_{\bar\Theta},D_\Theta]F + \left(\xi^\tau K_\tau{}^{\Theta} + \xi^{\Theta}\right)T_{\Theta\bar\Theta}{}^\tau K_\tau{}^t\partial_t D_\Theta F + \left(\xi^\tau K_\tau{}^{\bar\Theta} + \xi^{\bar\Theta}\right)T_{\bar\Theta\Theta}{}^\tau K_\tau{}^t\partial_t D_{\bar\Theta} \\
&- \frac{1}{2}\left(\xi^\tau K_\tau{}^{\Theta} + \xi^{\Theta}\right)T_{\Theta\bar\Theta}{}^\tau K_\tau{}^{\bar\Theta}\left([D_{\bar\Theta},D_\Theta]F - T_{\bar\Theta\Theta}{}^\tau D_\tau F\right) \\
&- \frac{1}{2}\left(\xi^\tau K_\tau{}^{\bar\Theta} + \xi^{\bar\Theta}\right)T_{\bar\Theta\Theta}{}^\tau K_\tau{}^{\Theta}\left(T_{\Theta\bar\Theta}{}^\tau D_\tau F + [D_{\bar\Theta},D_\Theta]F\right) \\
&= -\xi^\tau K_\tau{}^t\partial_t[D_{\bar\Theta},D_\Theta]F + \left(\xi^\tau K_\tau{}^{\Theta} + \xi^{\Theta}\right)T_{\Theta\bar\Theta}{}^\tau K_\tau{}^t\partial_t D_\Theta F + \left(\xi^\tau K_\tau{}^{\bar\Theta} + \xi^{\bar\Theta}\right)T_{\bar\Theta\Theta}{}^\tau K_\tau{}^t\partial_t D_{\bar\Theta} \\
&- \frac{1}{2}\left(\xi^\tau K_\tau{}^{\Theta} + \xi^{\Theta}\right)T_{\Theta\bar\Theta}{}^\tau K_\tau{}^{\bar\Theta}\left([D_{\bar\Theta},D_\Theta]F - T_{\bar\Theta\Theta}{}^\tau\left(K_\tau^t\partial_t F + K_\tau{}^{\Theta}D_\Theta F - K_\tau{}^{\bar\Theta}D_{\bar\Theta}F\right)\right) \\
&- \frac{1}{2}\left(\xi^\tau K_\tau{}^{\bar\Theta} + \xi^{\bar\Theta}\right)T_{\bar\Theta\Theta}{}^\tau K_\tau{}^{\Theta}\left([D_{\bar\Theta},D_\Theta]F + T_{\Theta\bar\Theta}{}^\tau\left(K_\tau{}^t\partial_t F + K_\tau{}^{\Theta}D_\Theta F - K_\tau{}^{\bar\Theta}D_{\bar\Theta}F\right)\right).
\end{aligned}
\tag{B.5}
$$

Therefore, setting $\underline{\theta} = 0$, yields

$$\delta_\xi\phi = -\xi^\tau K_\tau{}^t|\dot\phi - (\xi^\tau K_\tau{}^{\Theta} + \xi^{\Theta})|\chi + (\xi^\tau K_\tau{}^{\bar\Theta} + \xi^{\bar\Theta})|\bar\chi, \tag{B.6}$$

$$\delta_\xi\chi = -\xi^\tau K_\tau{}^t|\dot\chi - \frac{1}{2}\left(\xi^\tau K_\tau{}^{\bar\Theta} + \xi^{\bar\Theta}\right)|T_{\bar\Theta\Theta}{}^\tau\left(K_\tau{}^t|\dot\phi + K_\tau{}^{\Theta}|\chi - K_\tau{}^{\bar\Theta}|\bar\chi\right) + \left(\xi^\tau K_\tau{}^{\bar\Theta} + \xi^{\bar\Theta}\right)|G, \tag{B.7}$$

$$
\begin{aligned}
\delta_\xi G &= -\xi^\tau K_\tau{}^t|\dot G + \left(\xi^\tau K_\tau{}^{\Theta} + \xi^{\Theta}\right)|T_{\Theta\bar\Theta}{}^\tau K_\tau{}^t|\dot\chi + \left(\xi^\tau K_\tau{}^{\bar\Theta} + \xi^{\bar\Theta}\right)|T_{\bar\Theta\Theta}{}^\tau K_\tau{}^t|\dot{\bar\chi} \\
&- \frac{1}{2}\left(\xi^\tau K_\tau{}^{\Theta} + \xi^{\Theta}\right)|T_{\Theta\bar\Theta}{}^\tau K_\tau{}^{\bar\Theta}|\left(G - T_{\bar\Theta\Theta}{}^\tau|\left(K_\tau^t|\dot\phi + K_\tau{}^{\Theta}|\chi - K_\tau{}^{\bar\Theta}|\bar\chi\right)\right) \\
&- \frac{1}{2}\left(\xi^\tau K_\tau{}^{\bar\Theta} + \xi^{\bar\Theta}\right)|T_{\bar\Theta\Theta}{}^\tau K_\tau{}^{\Theta}|\left(G + T_{\Theta\bar\Theta}{}^\tau\left(K_\tau{}^t|\dot\phi + K_\tau{}^{\Theta}|\chi - K_\tau{}^{\bar\Theta}|\bar\chi\right)\right)
\end{aligned}
\tag{B.8}
$$

## B.2 New vielbein

It is convenient to recall (3.38),

$$\Phi(z,\Theta) = e^{Z^{\underline{\Theta}}\mathcal{D}_{\underline{\Theta}}}F(z) = \left(1 + Z^{\Theta}D_\Theta - Z^{\bar\Theta}D_{\bar\Theta} + (1/2)Z^{\Theta}Z^{\bar\Theta}[D_\Theta,D_{\bar\Theta}]\right)F(z). \tag{B.9}$$

The fermionic covariant derivatives are the easiest to compute with some previous results. From (B.2), (B.9) and the nilpotency of the Θ-variables, we get

$$Z^{\bar\Theta}D_\Theta\Phi(z,\Theta) = Z^{\bar\Theta}D_\Theta F(z) = Z^{\bar\Theta}\partial_\Theta\Phi(z,\Theta), \tag{B.10a}$$

$$Z^{\Theta}D_{\bar\Theta}\Phi(z,\Theta) = Z^{\Theta}D_{\bar\Theta}F(z) = Z^{\Theta}\partial_{\bar\Theta}\Phi(z,\Theta). \tag{B.10b}$$

The other combinations are,

$$
\begin{aligned}
Z^{\Theta} D_{\Theta} \Phi(z,\Theta) &= Z^{\Theta}\Big(D_{\Theta} + Z^{\bar{\Theta}} D_{\Theta} D_{\bar{\Theta}}\Big)F(z) \\
&= Z^{\Theta}\Big(D_{\Theta} + (1/2)Z^{\bar{\Theta}}([D_{\Theta}, D_{\bar{\Theta}}] + [D_{\Theta}, D_{\bar{\Theta}}]_{+})\Big)F(z) \\
&= Z^{\Theta}\Big(D_{\Theta} - Z^{\bar{\Theta}} D_{[\bar{\Theta}} D_{\Theta]}\Big)F(z) - \frac{1}{2} Z^{\Theta} Z^{\bar{\Theta}} T_{\bar{\Theta}\Theta}{}^{\tau}\Big(K_{\tau}{}^{t}\partial_{t} + K_{\tau}{}^{\Theta} D_{\Theta} - K_{\tau}{}^{\bar{\Theta}} D_{\bar{\Theta}}\Big)F(z) \\
&= Z^{\Theta}\partial_{\Theta}\Phi(z,\Theta) - (1/2) Z^{\Theta} Z^{\bar{\Theta}} T_{\Theta\bar{\Theta}}{}^{\tau}\Big(K_{\tau}{}^{t}\partial_{t} + K_{\tau}{}^{\Theta}\partial_{\Theta} - K_{\tau}{}^{\bar{\Theta}}\partial_{\bar{\Theta}}\Big)\Phi(z,\Theta), \qquad \text{(B.11a)} \\
Z^{\bar{\Theta}} D_{\bar{\Theta}} \Phi(z,\Theta) &= Z^{\bar{\Theta}}\Big(D_{\bar{\Theta}} - Z^{\Theta} D_{\bar{\Theta}} D_{\Theta}\Big)F(z) \\
&= Z^{\bar{\Theta}}\Big(D_{\bar{\Theta}} - (1/2)Z^{\Theta}([D_{\bar{\Theta}}, D_{\Theta}] + [D_{\bar{\Theta}}, D_{\Theta}]_{+})\Big)F(z) \\
&= Z^{\bar{\Theta}}\Big(D_{\bar{\Theta}} - Z^{\Theta} D_{[\bar{\Theta}} D_{\Theta]}\Big)F(z) - (1/2) Z^{\Theta} Z^{\bar{\Theta}} T_{\bar{\Theta}\Theta}{}^{\tau}\Big(K_{\tau}{}^{t}\partial_{t} + K_{\tau}{}^{\Theta} D_{\Theta} - K_{\tau}{}^{\bar{\Theta}} D_{\bar{\Theta}}\Big)F(z) \\
&= Z^{\bar{\Theta}}\partial_{\bar{\Theta}}\Phi - (1/2) Z^{\Theta} Z^{\bar{\Theta}} T_{\bar{\Theta}\Theta}{}^{\tau}\Big(K_{\tau}{}^{t}\partial_{t} + K_{\tau}{}^{\Theta}\partial_{\Theta} - K_{\tau}{}^{\bar{\Theta}}\partial_{\bar{\Theta}}\Big)\Phi(z,\Theta). \qquad \text{(B.11b)}
\end{aligned}
$$

(B.10) and (B.11) will be used to write old covariant derivatives of new superfields in terms of simple partial derivatives with respect to the coordinates of the new superspace.

It is also worth recalling here the definition of the new covariant derivatives,

$$
\nabla_{A}\Phi(z,\Theta) = e^{Z^{\underline{\Theta}} D_{\underline{\Theta}}} D_{A} F(z) = (1 + Z^{\Theta} D_{\Theta} - Z^{\bar{\Theta}} D_{\bar{\Theta}} - Z^{\Theta} Z^{\bar{\Theta}} D_{[\bar{\Theta}} D_{\Theta]}) D_{A} F(z).
$$

We will denote $\Phi(z,\Theta)$ and $F(z)$, the superfields in the new and old superspace, respectively, just as $\Phi$ and $F$. Thus, we have,

$$
\begin{aligned}
\nabla_{\Theta}\Phi &= \Big(1 - Z^{\bar{\Theta}} D_{\bar{\Theta}} + (1/2) Z^{\Theta} Z^{\bar{\Theta}} D_{\Theta} D_{\bar{\Theta}}\Big) D_{\Theta} F \\
&= \Big(D_{\Theta} - (1/2) Z^{\bar{\Theta}}[D_{\bar{\Theta}}, D_{\Theta}] + (1/2) Z^{\bar{\Theta}} T_{\bar{\Theta}\Theta}{}^{\tau} D_{\tau} - (1/2) Z^{\Theta} Z^{\bar{\Theta}} T_{\Theta\bar{\Theta}}{}^{\tau} D_{\tau} D_{\Theta}\Big) F \\
&= \Big(\partial_{\Theta} + (1/2) Z^{\bar{\Theta}} T_{\bar{\Theta}\Theta}{}^{\tau} D_{\tau}\Big)\Phi = \Big(\partial_{\Theta} + (1/2) Z^{\bar{\Theta}} T_{\bar{\Theta}\Theta}{}^{\tau}\Big(K_{\tau}^{t}\partial_{t} + K_{\tau}^{\Theta} D_{\Theta} - K_{\tau}^{\bar{\Theta}} D_{\bar{\Theta}}\Big)\Big)\Phi \\
&= \Big[\partial_{\Theta} + (1/2) Z^{\bar{\Theta}} T_{\bar{\Theta}\Theta}{}^{\tau} K_{\tau}^{t}\partial_{t} - (1/2) T_{\bar{\Theta}\Theta}{}^{\tau} K_{\tau}^{\Theta} Z^{\bar{\Theta}}\partial_{\Theta} + (1/2) T_{\bar{\Theta}\Theta}{}^{\tau} K_{\tau}^{\bar{\Theta}} \times \\
&\quad \Big(\Big(Z^{\bar{\Theta}} + \frac{1}{2}\Theta Z^{\bar{\Theta}} T_{\bar{\Theta}\Theta}{}^{\tau} K_{\tau}{}^{\bar{\Theta}}\Big)\partial_{\bar{\Theta}} - \frac{1}{2} Z^{\Theta} Z^{\bar{\Theta}} T_{\bar{\Theta}\Theta}{}^{\tau} K_{\tau}{}^{t}\partial_{t} - \frac{1}{2}\Theta^{\Theta} Z^{\bar{\Theta}} T_{\bar{\Theta}\Theta}{}^{\tau} K_{\tau}{}^{\Theta}\partial_{\Theta}\Big)\Big]\Phi, \qquad \text{(B.12)} \\
\nabla_{\bar{\Theta}}\Phi &= \Big(1 + Z^{\Theta} D_{\Theta} - (1/2) Z^{\Theta} Z^{\bar{\Theta}} D_{\bar{\Theta}} D_{\Theta}\Big) D_{\bar{\Theta}} F \\
&= \Big(D_{\bar{\Theta}} + (1/2) Z^{\Theta}[D_{\Theta}, D_{\bar{\Theta}}] - (1/2) Z^{\Theta} T_{\Theta\bar{\Theta}}{}^{\tau} D_{\tau} + (1/2) Z^{\Theta} Z^{\bar{\Theta}} T_{\bar{\Theta}\Theta}{}^{\tau} D_{\tau} D_{\bar{\Theta}}\Big) F \\
&= \Big(\partial_{\bar{\Theta}} - (1/2) Z^{\Theta} T_{\Theta\bar{\Theta}}{}^{\tau} D_{\tau}\Big)\Phi = \Big(\partial_{\bar{\Theta}} - (1/2) Z^{\Theta} T_{\Theta\bar{\Theta}}{}^{\tau}\Big(K_{\tau}^{t}\partial_{t} + K_{\tau}^{\Theta} D_{\Theta} - K_{\tau}^{\bar{\Theta}} D_{\bar{\Theta}}\Big)\Big)\Phi \\
&= \Big[\partial_{\bar{\Theta}} - (1/2) Z^{\Theta} T_{\Theta\bar{\Theta}}{}^{\tau} K_{\tau}^{t}\partial_{t} - (1/2) T_{\Theta\bar{\Theta}}{}^{\tau} K_{\tau}^{\bar{\Theta}} Z^{\Theta}\partial_{\bar{\Theta}} + (1/2) T_{\Theta\bar{\Theta}}{}^{\tau} K_{\tau}^{\Theta} \times \\
&\quad \Big(\Big(Z^{\Theta} - \frac{1}{2} Z^{\Theta} Z^{\bar{\Theta}} T_{\Theta\bar{\Theta}}{}^{\tau} K_{\tau}{}^{\Theta}\Big)\partial_{\Theta} - \frac{1}{2} Z^{\Theta} Z^{\bar{\Theta}} T_{\Theta\bar{\Theta}}{}^{\tau} K_{\tau}{}^{t}\partial_{t} + \frac{1}{2} Z^{\Theta} Z^{\bar{\Theta}} T_{\Theta\bar{\Theta}}{}^{\tau} K_{\tau}{}^{\bar{\Theta}}\partial_{\bar{\Theta}}\Big)\Big]\Phi. \qquad \text{(B.13)}
\end{aligned}
$$

To obtain the bosonic covariant derivative, we first compute $D_{\underline{\Theta}}\Phi$ using the following,

$$\partial_t\Phi = \partial_t\Big(1 + Z^{\Theta}D_{\Theta} - Z^{\bar{\Theta}}D_{\bar{\Theta}} + (1/2)Z^{\Theta}Z^{\bar{\Theta}}[D_{\Theta}, D_{\bar{\Theta}}]\Big)F, \tag{B.14}$$

$$\partial_{\Theta}\Phi = \Big(D_{\Theta} + (1/2)Z^{\bar{\Theta}}[D_{\Theta}, D_{\bar{\Theta}}]\Big)F, \tag{B.15}$$

$$\partial_{\bar{\Theta}}\Phi = \Big(D_{\bar{\Theta}} + (1/2)Z^{\Theta}[D_{\Theta}, D_{\bar{\Theta}}]\Big)F, \tag{B.16}$$

$$Z^{\Theta}\partial_t\Phi = Z^{\Theta}\partial_t\Big(1 - Z^{\bar{\Theta}}D_{\bar{\Theta}}\Big)F, \tag{B.17}$$

$$Z^{\bar{\Theta}}\partial_t\Phi = Z^{\bar{\Theta}}\partial_t\Big(1 + Z^{\Theta}D_{\Theta}\Big)F, \tag{B.18}$$

$$Z^{\Theta}Z^{\bar{\Theta}}\partial_t\Phi = Z^{\Theta}Z^{\bar{\Theta}}\partial_t F, \tag{B.19}$$

$$Z^{\Theta}\partial_{\Theta}\Phi = Z^{\Theta}\Big(D_{\Theta} + (1/2)Z^{\bar{\Theta}}[D_{\Theta}, D_{\bar{\Theta}}]\Big)F, \tag{B.20}$$

$$Z^{\bar{\Theta}}\partial_{\Theta}\Phi = Z^{\bar{\Theta}}D_{\Theta}F, \tag{B.21}$$

$$Z^{\Theta}Z^{\bar{\Theta}}\partial_{\Theta}\Phi = Z^{\Theta}Z^{\bar{\Theta}}D_{\Theta}F, \tag{B.22}$$

$$Z^{\Theta}\partial_{\bar{\Theta}}\Phi = Z^{\Theta}D_{\bar{\Theta}}F, \tag{B.23}$$

$$Z^{\bar{\Theta}}\partial_{\bar{\Theta}}\Phi = Z^{\bar{\Theta}}\Big(D_{\bar{\Theta}} + (1/2)Z^{\Theta}[D_{\Theta}, D_{\bar{\Theta}}]\Big)F, \tag{B.24}$$

$$Z^{\Theta}Z^{\bar{\Theta}}\partial_{\bar{\Theta}}\Phi = Z^{\Theta}Z^{\bar{\Theta}}D_{\bar{\Theta}}F. \tag{B.25}$$

Using the above results, the fermionic covariant derivative can be computed,

$$\begin{aligned}
D_{\Theta}\Phi &= \Big(D_{\Theta} + Z^{\bar{\Theta}}D_{\Theta}D_{\bar{\Theta}} - (1/2)Z^{\Theta}Z^{\bar{\Theta}}D_{\Theta}D_{\bar{\Theta}}D_{\Theta}\Big)F \\
&= \Big(D_{\Theta} + (1/2)Z^{\bar{\Theta}}([D_{\Theta}, D_{\bar{\Theta}}] + [D_{\Theta}, D_{\bar{\Theta}}]_+) - (1/2)Z^{\Theta}Z^{\bar{\Theta}}[D_{\Theta}, D_{\bar{\Theta}}]_+D_{\Theta}\Big)F \\
&= \Big(D_{\Theta} + (1/2)Z^{\bar{\Theta}}[D_{\Theta}, D_{\bar{\Theta}}] - (1/2)Z^{\bar{\Theta}}T^{\tau}_{\Theta\bar{\Theta}}D_{\tau} + (1/2)Z^{\Theta}Z^{\bar{\Theta}}T^{\tau}_{\Theta\bar{\Theta}}D_{\tau}D_{\Theta}\Big)F \\
&= \Big(D_{\Theta} + (1/2)Z^{\bar{\Theta}}[D_{\Theta}, D_{\bar{\Theta}}]\Big)F - (1/2)Z^{\bar{\Theta}}T^{\tau}_{\Theta\bar{\Theta}}\Big(K_{\tau}{}^{t}\partial_t + K_{\tau}^{\Theta}D_{\Theta} - K_{\tau}^{\bar{\Theta}}D_{\bar{\Theta}}\Big)F \\
&+ (1/2)Z^{\Theta}Z^{\bar{\Theta}}T^{\tau}_{\Theta\bar{\Theta}}\Big(K_{\tau}{}^{t}\partial_t + K_{\tau}^{\Theta}\not{D}_{\Theta} - K_{\tau}^{\bar{\Theta}}D_{\bar{\Theta}}\Big)D_{\Theta}F \\
&= \Big(D_{\Theta} + (1/2)Z^{\bar{\Theta}}[D_{\Theta}, D_{\bar{\Theta}}]\Big)F - (1/2)Z^{\bar{\Theta}}T^{\tau}_{\Theta\bar{\Theta}}K_{\tau}{}^{t}\partial_t\Big(1 + Z^{\Theta}D_{\Theta}\Big)F \\
&- (1/2)Z^{\bar{\Theta}}T^{\tau}_{\Theta\bar{\Theta}}\Big(K_{\tau}^{\Theta}D_{\Theta} - K_{\tau}^{\bar{\Theta}}D_{\bar{\Theta}}\Big)F - (1/4)Z^{\Theta}Z^{\bar{\Theta}}T^{\tau}_{\Theta\bar{\Theta}}K_{\tau}^{\bar{\Theta}}\Big([D_{\bar{\Theta}}, D_{\Theta}] - T^{\tau}_{\Theta\bar{\Theta}}D_{\tau}\Big)F \\
&= \Big[D_{\Theta}F + (1/2)Z^{\bar{\Theta}}[D_{\Theta}, D_{\bar{\Theta}}]F\Big] - (1/2)T^{\tau}_{\Theta\bar{\Theta}}K_{\tau}{}^{t}\partial_t\Big[Z^{\bar{\Theta}}(1 + Z^{\Theta}D_{\Theta})F\Big] \\
&+ (1/2)T^{\tau}_{\Theta\bar{\Theta}}K_{\tau}^{\Theta}\Big[Z^{\bar{\Theta}}D_{\Theta}F\Big] - (1/2)T^{\tau}_{\Theta\bar{\Theta}}K_{\tau}^{\bar{\Theta}}Z^{\bar{\Theta}}\Big[D_{\bar{\Theta}}F - (1/2)Z^{\Theta}[D_{\bar{\Theta}}, D_{\Theta}]F\Big] \\
&+ (1/4)T^{\tau}_{\Theta\bar{\Theta}}K_{\tau}^{\bar{\Theta}}T^{\tau}_{\Theta\bar{\Theta}}\Big(K_{\tau}{}^{t}\Big[Z^{\Theta}Z^{\bar{\Theta}}\partial_t F\Big] + K_{\tau}{}^{\Theta}\Big[Z^{\Theta}Z^{\bar{\Theta}}D_{\Theta}F\Big] - K_{\tau}{}^{\bar{\Theta}}\Big[Z^{\Theta}Z^{\bar{\Theta}}D_{\bar{\Theta}}F\Big]\Big) \\
&= \partial_{\Theta}\Phi - (1/2)T^{\tau}_{\Theta\bar{\Theta}}K_{\tau}{}^{t}Z^{\bar{\Theta}}\partial_t\Phi + (1/2)T^{\tau}_{\Theta\bar{\Theta}}K_{\tau}^{\Theta}Z^{\bar{\Theta}}\partial_{\Theta}\Phi - (1/2)T^{\tau}_{\Theta\bar{\Theta}}K_{\tau}^{\bar{\Theta}}Z^{\bar{\Theta}}\partial_{\bar{\Theta}}\Phi \\
&+ (1/4)T^{\tau}_{\Theta\bar{\Theta}}K_{\tau}^{\bar{\Theta}}T^{\tau}_{\Theta\bar{\Theta}}\Big(K_{\tau}{}^{t}Z^{\Theta}Z^{\bar{\Theta}}\partial_t\Phi + K_{\tau}{}^{\Theta}Z^{\Theta}Z^{\bar{\Theta}}\partial_{\Theta}\Phi - K_{\tau}{}^{\bar{\Theta}}Z^{\Theta}Z^{\bar{\Theta}}\partial_{\bar{\Theta}}\Phi\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \Big(1 - (1/2)Z^{\bar\Theta} T_{\Theta\bar\Theta}{}^{\tau} K_{\tau}{}^{\Theta} + (1/4)Z^{\Theta} Z^{\bar\Theta} T_{\Theta\bar\Theta}{}^{\tau} K_{\tau}{}^{\bar\Theta} T_{\bar\Theta\Theta}{}^{\tau} K_{\tau}{}^{\Theta}\Big)\partial_{\Theta}\Phi \\
&+ \Big((1/2)Z^{\bar\Theta} T_{\Theta\bar\Theta}{}^{\tau} K_{\tau}{}^{\bar\Theta} - (1/4)Z^{\Theta} Z^{\bar\Theta} T_{\Theta\bar\Theta}{}^{\tau} K_{\tau}{}^{\bar\Theta} T_{\bar\Theta\Theta}{}^{\tau} K_{\tau}{}^{\bar\Theta}\Big)\partial_{\bar\Theta}\Phi \\
&+ \Big(-(1/2)Z^{\bar\Theta} T_{\Theta\bar\Theta}{}^{\tau} K_{\tau}^{t} + (1/4)Z^{\Theta} Z^{\bar\Theta} T_{\Theta\bar\Theta}{}^{\tau} K_{\tau}^{\bar\Theta} T_{\bar\Theta\Theta}{}^{\tau} K_{\tau}{}^{t}\Big)\partial_{t}\Phi,
\end{aligned}
\tag{B.26}
$$

The complex conjugate is obtained similarly. The result is,

$$
\begin{aligned}
D_{\bar\Theta}\Phi &= \Big(1 - (1/2)Z^{\Theta} T_{\bar\Theta\Theta}{}^{\tau} K_{\tau}{}^{\bar\Theta} - (1/4)Z^{\Theta} Z^{\bar\Theta} T_{\bar\Theta\Theta}{}^{\tau} K_{\tau}{}^{\Theta} T_{\Theta\bar\Theta}{}^{\tau} K_{\tau}^{\bar\Theta}\Big)\partial_{\bar\Theta}\Phi \\
&+ \Big((1/2)Z^{\Theta} T_{\bar\Theta\Theta}{}^{\tau} K_{\tau}{}^{\Theta} + (1/4)Z^{\Theta} Z^{\bar\Theta} T_{\bar\Theta\Theta}{}^{\tau} K_{\tau}{}^{\Theta} T_{\Theta\bar\Theta}{}^{\tau} K_{\tau}{}^{\Theta}\Big)\partial_{\Theta}\Phi \\
&+ \Big((1/2)Z^{\Theta} T_{\bar\Theta\Theta}{}^{\tau} K_{\tau}{}^{t} + (1/4)Z^{\Theta} Z^{\bar\Theta} T_{\bar\Theta\Theta}{}^{\tau} K_{\tau}{}^{\Theta} T_{\Theta\bar\Theta}{}^{\tau} K_{\tau}{}^{t}\Big)\partial_{t}\Phi.
\end{aligned}
\tag{B.27}
$$

Finally, for the bosonic covariant derivative,

$$
\begin{aligned}
\nabla_{\tau}\Phi &= \Big(1 + Z^{\Theta} D_{\Theta} - Z^{\bar\Theta} D_{\bar\Theta} - (1/2)Z^{\Theta} Z^{\bar\Theta}[D_{\bar\Theta}, D_{\Theta}]\Big)D_{\tau}F \\
&= D_{\tau}\Phi = \Big(K_{\tau}^{t}\partial_{t} + K_{\tau}^{\Theta} D_{\Theta} - K_{\tau}^{\bar\Theta} D_{\bar\Theta}\Big)\Phi,
\end{aligned}
\tag{B.28}
$$

Thus, we only substitute (B.26) and (B.27) on the l.h.s., of (B.28).

For the precise numerical values one reads off $K_{\tau}{}^{t}|$, $K_{\tau}{}^{\underline{\Theta}}|$ from the vielbein at the W-Z gauge and uses $T_{\Theta\bar\Theta}{}^{\tau} = 2i$.

# Appendix C

# Useful expressions

We write bosonic coordinates and their momenta as $q_a$, $p_a$, and fermionic ones by $\psi^\alpha$, $\pi_\alpha = \partial L/\partial\dot{\psi}^\alpha$. The generalized Poisson bracket is defined to be [49]

$$\{A,B\} = \frac{\partial A}{\partial q^a}\frac{\partial B}{\partial p_a} - \frac{\partial A}{\partial p_a}\frac{\partial B}{\partial q^a} + (-1)^{n_A}\left(\frac{\partial A}{\partial \psi^\alpha}\frac{\partial B}{\partial \pi_\alpha} + \frac{\partial A}{\partial \pi_\alpha}\frac{\partial B}{\partial \psi^\alpha}\right) \tag{C.1}$$

where $n_A$ is 0, or 1, depending on if $A$ is commuting or anti-commuting, respectively.

Note that in (C.1) the fermionic momenta are defined without introducing a minus sign. To compute the Poisson brackets in this work, we re-wrote (C.1) accordingly, to get

$$\begin{aligned}\{A,B\} = {} & \frac{\partial A}{\partial a}\frac{\partial B}{\partial p_a} - \frac{\partial A}{\partial p_a}\frac{\partial B}{\partial a} + \frac{\partial A}{\partial \varphi}\frac{\partial B}{\partial p_\varphi} - \frac{\partial A}{\partial p_\varphi}\frac{\partial B}{\partial \varphi} \\ & + (-1)^{n_A}\left(-\frac{\partial A}{\partial \lambda}\frac{\partial B}{\partial \pi_{\bar\lambda}} - \frac{\partial A}{\partial \pi_{\bar\lambda}}\frac{\partial B}{\partial \lambda} + \frac{\partial A}{\partial \bar\lambda}\frac{\partial B}{\partial \pi_\lambda} + \frac{\partial A}{\partial \pi_\lambda}\frac{\partial B}{\partial \bar\lambda}\right) \\ & + (-1)^{n_A}\left(-\frac{\partial A}{\partial \eta}\frac{\partial B}{\partial \pi_{\bar\eta}} - \frac{\partial A}{\partial \pi_{\bar\eta}}\frac{\partial B}{\partial \eta} + \frac{\partial A}{\partial \bar\eta}\frac{\partial B}{\partial \pi_\eta} + \frac{\partial A}{\partial \pi_\eta}\frac{\partial B}{\partial \bar\eta}\right).\end{aligned} \tag{C.2}$$

On the other hand, the Dirac bracket is given by

$$\{A,B\}_D = \{A,B\} - \{A,C_i\}C^{ij}\{C_j,B\}, \tag{C.3}$$

where $C_i$ stands for the second-class constraints and $C^{ij}$ is the inverse of the matrix of their Poisson brackets,

$$\left(C_{ij}\right) = \left(\{C_i,C_j\}\right). \tag{C.4}$$

Finally, if $a$, $b$ are commuting quantities, whereas $\alpha$, $\beta$ are anti-commuting, then

$$\begin{aligned}[a\alpha,b\beta]_+ &\equiv a\alpha b\beta + b\beta a\alpha = ab\alpha\beta + ba\beta\alpha \\ &= \frac{1}{2}\Big((ab-ba)\alpha\beta + ba\alpha\beta + ba\beta\alpha\Big) + \frac{1}{2}\Big(ab(\alpha\beta+\beta\alpha) - ab\beta\alpha + ba\beta\alpha\Big) \\ &= \frac{1}{2}\Big([a,b]\alpha\beta + ba[\alpha,\beta]_+\Big) + \frac{1}{2}\Big(ab[\alpha,\beta]_+ - [a,b]\beta\alpha\Big) \\ &= \frac{1}{2}\Big([a,b][\alpha,\beta] + [a,b]_+[\alpha,\beta]_+\Big)\end{aligned} \tag{C.5}$$

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