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On the (n, m)-fold hyperspace suspension of a continuum

TESIS

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BUAP

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Por lo que se le autoriza a proceder con los trámites y realizar el examen de grado en la fecha que se le asigne.

A T E N T A M E N T E.
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Dedicatoria

A mis padres:
Gerardo Hernández Martínez,
Gloria Adriana Valdez Contreras,
por su apoyo incondicional.

A Nayeli Berenice Quiñones Baldazo,
por su amor y compañía.

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Introduction

Throughout the years, the study of hyperspaces has acquired a notorious importance within the theory of continua. Recall that a continuum X is a nonempty connected, compact and metric space, and a hyperspace of a continuum is family of closed subsets of X sharing certain properties.

In 1979 Sam B. Nadler, Jr. introduced the *hyperspace suspension of a continuum* X as the quotient space $C_1(X)/F_1(X)$, denoted by $HS(X)$, see [61]. Later, in 2004 Sergio Macías introduced the *n-fold hyperspace suspension of a continuum* X, for each $n \in \mathbb{N}$ and $n \geq 2$ as the quotient space $C_n(X)/F_n(X)$, denoted by $HS_n(X)$, see [50], where there are described some general properties about this hyperspace. As usual, once there is a new hyperspace around, the natural question about the uniqueness arises. We say that a continuum X has unique hyperspace $\mathcal{K}(X)$ if for any continuum Y such that $\mathcal{K}(X)$ is homeomorphic to $\mathcal{K}(Y)$, then X is homeomorphic to Y. More information about the uniqueness of hyperspaces $HS_n(X)$ for some families of continua can be found in [27], [29] and [31]. Afterwards, in 2018 J.G. Anaya, D. Maya and F. Vázquez-Juárez introduced, for $n, m \in \mathbb{N}$ with $m \leq n$, the (n, m)-fold hyperspace suspension of a continuum X, denoted by $HS^n_m(X)$, and defined as the quotient space $C_n(X)/F_m(X)$ obtained from $C_n(X)$ by shrinking $F_m(X)$ to a one-point set with the quotient topology, see [6]. Moreover, in that paper it is also proved the uniqueness of the hyperspace $HS^n_m(X)$ for finite graphs. Particularly, the aim for this project comes from the idea of finding more families of continua having unique hyperspace $HS^n_m(X)$ as well as extending key properties of the hyperspace that are satisfied in the case $n = m$. We will focus our attention in families of almost meshed and meshed locally connected continua. Furthermore, since we study general properties about this hyperspace, the comparison between the case $n = m$ and $n > m$ is questioned. This problem gives place to the question: is there a topological property that is satisfied by the hyperspace $HS_n^n(X)$ but fails at $HS_m^n(X)$? An answer for this question is approached in this work.

In chapter 1, general theory of continua is presented, as well as the definitions of the hyperspaces we shall be working with, such as $2^X, C_n(X)$ and $F_n(X)$, for $n \in \mathbb{N}$.

In chapter 2, we present the construction of the (n, m)-fold hyperspace suspension of a continuum and several general properties about this hyperspace are studied, such as being contractible, unicoherent, colocal connected among others. Results about general properties are condensed in a paper which was submitted and then accepted at Revista Integración, see [24]. Furthermore, a positive answer for the already asked question is given: Given a continuum X and $n, m \in \mathbb{N}$ with $2 \leq m < n$, is there a topological property P that holds on $HS_n^n(X)$ but not on $HS_m^n(X)$?

Finally, in chapter 3 we present a family of continua having unique hyperspace $HS_m^n(X)$. All the work done in order to find the homeomorphism between continua X and Y can be found in [23]. Moreover, we give some examples of continua without unique hyperspace which can also be found in [23]. At the end of this work, some open questions are posed which may extend the research done here.

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Contents

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Chapter 1

Preliminaries

In this chapter, we will go through several notions of continuum theory and its hyperspaces. In the first section, there are some verified properties of hyperspaces of a compact topological space. Consequently, all results described in this section will work for continua and its hyperspaces. In the second section, we present some examples of continua, as well as general properties that will help us to characterize some hyperspaces.

1.1 Hyperspaces of a topological space

As it was stated before, the study of hyperspaces of a topological space has gained importance through the last quarter of century. We begin this section reviewing several topological concepts of hyperspaces.

Definition 1.1. *Let (X, τ) be a nonempty topological space. A hyperspace of X is a collection of closed subsets of X along with the Vietoris topology, which would be defined shortly.*

Let (X, τ) be a nonempty topological space. Out attention will be focused into hyperspaces that satisfy the property of being T_1. For each $n \in \mathbb{N}$, consider the following hyperspaces:

$$\begin{aligned} CL(X) &= \{A \subset X : A \neq \emptyset \text{ and } A \text{ is a closed subset of } X\}, \\ C_n(X) &= \{A \in CL(X) : A \text{ has at most } n \text{ components}\}, \\ F_n(X) &= \{A \in CL(X) : A \text{ has at most } n \text{ points}\}. \end{aligned}$$

Observation 1.2. *If* $n = 1$*, we write* $C(X)$ *to denote* $C_1(X)$ *for convenience. Moreover, if* X *is compact, then* $2^X = CL(X)$*.*

Allow us to define the Vietoris topology for the hyperspace $CL(X)$.

Definition 1.3. *Let* (X, τ) *be a topological space. The* **Vietoris topology** *for* $CL(X)$ *is the smallest topology,* τ_V *such that* $CL(X)$ *satisfies the following condition:*

$\{A \in CL(X) : A \subset U\} \in \tau_V$ *for each* $U \in \tau$*, and* $\{A \in CL(X) : A \subset B\}$ *is a closed subset under* τ_V *for each* B *closed subset of* X *under* τ*.*

Indeed, τ_V is a well-defined topology (see [60, Theorem 0.11]). As τ_V is a topology, next step is to find a base for it. To do so, consider the following definition:

Definition 1.4. *Let* X *be a topological space,* $n \in \mathbb{N}$ *and* $U_1, U_2, ..., U_n \subset X$*. Consider the set:*

$$\langle U_1, U_2, ..., U_n \rangle = \{A \in CL(X) : A \subset \bigcup_{i=1}^{n} U_i,\ A \cap S_i \neq \emptyset \text{ for each } i \in \{1, ..., n\}\}.$$

These sets represent a basic element in the Vietoris topology:

Theorem 1.5. *[42, Theorem 1.2] Let* X *be a topological space. Consider the collection*

$$\mathcal{B}_V = \{\langle U_1, ..., U_n \rangle : U_i \in \tau, \text{ for each } i \in \{1, ..., n\},\ n \in \mathbb{N}\}.$$

Then, $\mathcal{B}_V$ *is a base for the Vietoris topolgy* τ_V*.*

Next step is to induce a metric to the hyperspace $CL(X)$. If X is a topological compact space with a metric, we may refer to X as a *compactum*. Consider the following definitions:

Definition 1.6. *Let* X *be a compactum with metric* d*. For each* $x \in X$ *and* $A \in CL(X)$ *the distance from* x *to* A *is given by*

$$d(x, A) = \inf\{d(x, a) \colon a \in A\}. \tag{1.1}$$

Let $r > 0$*, the* **generalized ball** *centered at* A *with radius* r *respect to the metric* d *as*

$$N_d(r, A) = \{x \in X \colon d(x, A) < r\}. \tag{1.2}$$

Now, we are ready to establish a metric for hyperspaces.

Definition 1.7. *Let (X, d) be a compactum. The function $H_d : CL(X) \times CL(X) \to \mathbb{R}^+$ is defined for each pair $A, B \in CL(X)$ as*

$$H_d(A, B) = \inf\{r > 0\colon A \subset N_d(r, B), \quad B \subset N_d(r, A)\}. \tag{1.3}$$

This function is indeed a metric, as the next result is given.

Theorem 1.8. *[42, Theorem 2.2] For a compactum (X, d), the function H_d previously defined is a metric for $CL(X)$, and it is known as* **Hausdorff metric**.

Given a compactum (X, d), once the Hausdorff metric is defined for $CL(X)$, raises the natural problem of comparing the topology induced by the metric H_d, which we will denoted by τ_{H_d} and the Vietoris topology. Indeed, the metric topology and the Vietoris topology are equal, as the next result shows it.

Theorem 1.9. *[42, Theorem 3.1] If (X, d) is a compactum, then $CL(X)$ is metrizable. Moreover, $\tau_V = \tau_{H_d}$.*

1.2 Continuum Theory

In this section, fundamental aspects of continuum theory are studied.

Definition 1.10. *We say that X is a* **continuum** *if it is a nondegenerate compactum and connected space. A* **subcontinuum** *is a subset of X which is a continuum itself.*

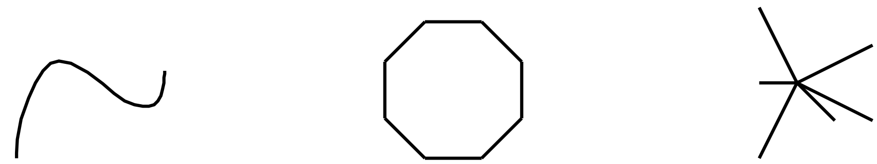
Observation 1.11. *Form now on, if X is a compactum or a continuum, we will write 2^X instead of $CL(X)$.*

Now we shall see several examples of continua.

Definition 1.12. *An* **arc** *is any space homeomorphic to $[0, 1]$.*

Definition 1.13. *A* **simple closed curve** *is any space homeomorphic to $\mathcal{S}^1$, where $\mathcal{S}^1$ is the unit circle.*

Definition 1.14. *A* **simple n-od** *is a finite graph that is the union of n arcs emanating from a single point v and otherwise disjoint to one another.*



(a) Arc (b) Simple Closed Curve (c) 6-od

Figure 1.1: Examples of continua

Theorem 1.15. *[63, Exercise 8.40 (b)] Let X be a locally connected continuum. If X does not contain a simple triod, then X is an arc or a simple closed curve.*

Definition 1.16. *Let α be an arc in the space M and let h be a homeomorphism between $[0,1]$ and α. The* **end points** *of α are the elements $h(0)$ and $h(1)$.*

Now we can use this examples of continua to construct a more complex space.

Definition 1.17. *A* **finite graph** *is a continuum which can be written as a union of finitely many arcs, whose pairwise intersection is at most one or both of its end points.*

Observe that the arc, simple closed curve and the simple n-od are all finite graphs. The following definition will be used in the case of finite graphs.

Definition 1.18. *Let G be a finite graph. An arc α is said to be a* **free arc** *in G if α without its end points is an open subset of G.*

An even more complex class of continua is given.

Definition 1.19. *A locally connected continuum that contains no simple closed curves is said to be a* **dendrite**.

Notice that a finite graph without simple closed curves is a particular example of a dendrite. For further information about dendrites and their construction, see [13]. Figure 1.2 shows the **Universal dendrite**, denoted by D_ω.

Here are some properties about dendrites that will be used later.

Theorem 1.20. *[13, Theorem 1.3] Every subcontinuum of a dendrite is a dendrite.*

Theorem 1.21. *[13, Theorem 1.4]*

(i) X is a dendrite if and only if each subcontinuum of X is a strong deformation retract of X.

(ii) X is a dendrite if and only if X is locally connected, 1-dimensional and contractible.

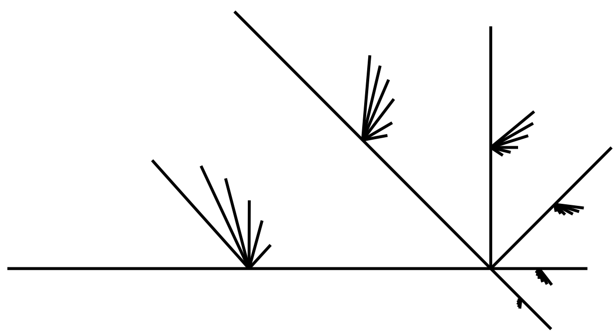


Figure 1.2: Universal dendrite

In Section 3 we discuss more families of continua known as almost meshed and meshed.

Definition 1.22. *Let (X, τ) be a metric space, $A \subset X$ and β a cardinal number. The subset A is said to be of* **order** *less or equal than β on X, denoted by $\mathrm{ord}_X(A) \leq \beta$ if for each $U \in \tau$ such that $A \subset U$, there exists $V \in \tau$ such that $A \subset V \subset U$ and $| \mathrm{fr}(V) | \leq \beta$. Furthermore, A has order equal to β if $\mathrm{ord}_X(A) \leq \beta$ and $\mathrm{ord}_X(A) \not\leq \alpha$ for each $\alpha < \beta$. If $A = \{p\}$, we write $\mathrm{ord}_X(p)$.*

From the definition of order, several sets can be identified within a finite graph.

Definition 1.23. *The set of end points of a continuum X, denoted by $E(X)$, is defined as those points in X of order equal to 1, and the set of those points in X with order greater or equal than 3 is denoted by $R(X)$ and it is known as the set of ramification points.*

Do all continua contain a nondegenerate proper subcontinuum? The question is answered in the next result, which will be useful later.

Theorem 1.24. *[63, Corollary 5.5] Let X be a nondegenerate continuum. Then, X contains a nondegenerate proper subcontinuum. Furthermore, if A is a proper subcontinuum of X and U is an open subset of X such that $A \subset B \neq A$ and $B \subset U$.*

The following definitions are properties of continua, which in the next chapter are set to be extended for hyperspaces.

Definition 1.25. *A continuum X is* **decomposable** *if X can be written as the union of two proper subcontinua. A continuum which is not decomposable is said to be* **indecomposable**.

Example 1.26. *The simple closed curve S^1 is a decomposable continuum, while the Knaster continuum [44, 3, p. 205] is an example of an indecomposable continuum.*

Definition 1.27. *A continuum X is said to be* **hereditarily indecomposable** *if $A \cap B = \emptyset$, or $A \subset B$, or $B \subset A$, for each $A, B \in C(X)$.*

Definition 1.28. *A topological space Z_1 is* **contractible** *whenever the identity function of Z_1 onto Z_1 is homotopic to some constant function of Z_1.*

Example 1.29. *The closed disk D^1 is a contractible space.*

A consequence of being a contractible metric space is given.

Theorem 1.30. *[52, Theorem 1.3.11] If X is a contractible metric space, then X is arcwise connected.*

Definition 1.31. *A* **retraction** *is a continuous function $r : Y \to Y$ such that $r(r(y)) = r(y)$ for each $y \in Y$. A subset Z of Y is said to be a* **retract** *of Y if there exists a retraction of Y onto Z. Finally, a compactum K is an* **absolut retract** *whenever K is embbeded in a metric space Y, the embbeded copy of K is a retract of Y.*

Definition 1.32. *A continuum X has the* **property (b)** *provided that each map $f : X \to \mathcal{S}^1$ is homotopic to a constant.*

Definition 1.33. *A continuum is* **uniformly pathwise connected** *if it is the continuous image of the cone over the Cantor set. [43, Theorem 3.5].*

Definition 1.34. *A continuum is* **colocally connected** *whenever each one of its points has a local basis of open sets whose complements are connected.*

Definition 1.35. *The continuum X is* **aposyndetic** *if for each pair of points x and y of X, there exists a subcontinuum W of X such that $x \in \mathrm{int}_X(W) \subset W \subset X - \{y\}$. A continuum is* **finitely aposyndetic** *provided that for each finite subset F of X and point x of X not in F, there exists a subcontinuum W of X such that $x \in \mathrm{int}_X(W) \subset W \subset X - F$.*

Definition 1.36. *A continuum X has the* **property of Kelley** *provided that for each $\varepsilon > 0$, there exists $\delta > 0$ such that if p and q are two points of X satisfying $d(p, q) < \delta$, and if A is a subcontinuum of X containing p, then there exists a subcontinuum B of X such that $q \in B$ and $H(A, B) < \varepsilon$.*

For more information about the following examples, see [42, p. 167].

Example 1.37. *(i) Every locally connected continuum has the property of Kelley.*

(ii) The $\sin(1/x)$-continuum (also known as the topologist's sine curve) has the property of Kelley.

A result involving hereditarily indecomposable continua and the property Kelley is written.

Theorem 1.38. *[42, Theorem 20.6] Any hereditarily indecomposable continuum has the property of Kelley.*

Kelley's property will be a key concept to find a property satisfied by the n-fold hyperspace suspension, but fails to be satisfied by the (n, m)-fold hyperspace suspension.

Definition 1.39. *A continuum X is* **unicoherent** *provided that for each pair A and B of subcontinua of X such that $X = A \cup B$, $A \cap B$ is connected. We say that X is* **hereditarily unicoherent** *if each subcontinuum of X is unicoherent.*

A result that combines the notion of unicoherence and aposyndetic is given.

Lemma 1.40. *[7, Corollary 1] If X is an unicoherent and aposyndetic continuum, then X is finitely aposyndetic.*

An interesting result that connects the notion of property (b) and unicoherent is now given.

Lemma 1.41. *[48, Theorem 4.7] Let X be a connected metric space. If X has the property (b), then X is unicoherent.*

Allow us to present the notion of dimension.

Definition 1.42. *Let X be a separable metric space and $p \in X$. The* **dimension** *of the space X, denoted by $\dim[X]$, is defined as follows:*

1. $\dim[X] = -1$ *if and only if* $X = \emptyset$.

2. *Suppose that we have defined inductively that* $\dim[Y] \leq n - 1$ *for some* $n \in \mathbb{N}$ *and a space Y. Then, for a space X and a point $p \in X$, we define*

$$\dim_p[X] \leq n$$

if and only if p has a local base of open neighborhoods in X whose boundaries have dimension less or equal than $n - 1$.

3. $\dim[X] \leq n$ *if and only if* $\dim_p[X] \leq n$ *for each* $p \in X$.

4. $\dim[X] = n$ *if and only if* $\dim[X] \leq n$ *and* $\dim[X] \nleq n - 1$.

5. $\dim_p[X] = n$ *if and only if* $\dim_p[X] \leq n$ *and* $\dim_p[X] \nleq n - 1$.

6. $\dim[X] = \infty$ *if and only if* $\dim[X] \nleq n$ *for each* $n \in \mathbb{N}$.

7. $\dim_p[X] = \infty$ *if and only if* $\dim_p[X] \nleq n$ *for each* $n \in \mathbb{N}$.

The following result states that dimension is a topological property.

Theorem 1.43. *[64, Theorem 1.2] The notion of dimension at a point is topologically invariant.*

Now, allow us to review a result that will be used often.

Theorem 1.44. *[64, Theorem 3.2] If $B \subset A$ and $\dim[A] \leq n$, then $\dim[B] \leq n$.*

Recall that a set F_σ is the numerable union of closed sets.

Theorem 1.45. *[64, Theorem 7.1] A space which is the numerable union of F_σ-sets, each one of them with dimension less than n, is also of dimension less than n.*

Theorem 1.46. *[64, Theorem 7.3] Let $X = Y \cup Z$, where $\dim(Y) \leq n$ and $\dim(Z) \leq n$. If at least one of Y and Z is closed in X, then $\dim(X) \leq n$.*

Lemma 1.47. *[64, Exercise 7.4] The dimension of a nonempty metric space is not changed by adding a point to the space.*

Lemma 1.48. *[17, Theorem 6.1] Let (Z, d) be a compact space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence contained in Z. If every subsequence of $\{x_n\}_{n\in\mathbb{N}}$ converges to some point $y \in Z$, then $\{x_n\}_{n\in\mathbb{N}}$ converges to y.*

Consider the following results about extending continuous functions.

Theorem 1.49. *[17, Theorem 9.4, p. 83] Let X be a space and $\{A_\alpha\}_{\alpha\in J}$ is a covering of X such that either:*

(i) the sets A_α are all open or,

(ii) the sets A_α are all closed and form a finite neighborhood family.

For each $\alpha \in J$, let $f_\alpha : A_\alpha \to Y$ be continuous and $f_\alpha|_{A_\alpha \cap A_\beta} = f_\beta|_{A_\alpha \cap A_\beta}$ for each $\alpha, \beta \in J$. Then, there exists a continuous function $f : X \to Y$, which is an extension of each f_α.

Before we move on with more properties of continuous functions, recall that a **Hilbert cube** is a countable product of the interval $[0, 1]$ under the product topology, which is often denoted by $\mathcal{Q}$.

Theorem 1.50. *[42, Theorem 11.9.1] Any homeomorphism between two Z-sets of a Hilbert cube, $\mathcal{Q}$, can be extended to a homeomorphism of $\mathcal{Q}$ onto $\mathcal{Q}$.*

Theorem 1.51. *[17, Theorem 4.3, p. 126] Let X, Y be spaces with equivalence relations R, S, respectively, and let $f : X \to Y$ be a relation-preserving, continuous function. Then, passing to the quotient, the function $f_* : X/R \to Y/S$ is also continuous.*

Theorem 1.52. *[31, Corollary 2.7] A contractible compact metric space K which is a Z-set of $\mathcal{Q}$ satisfies that $\mathcal{Q}/K$ is homeomorphic to $\mathcal{Q}$.*

Theorem 1.53. *[31, Lemma 2.8] Let X be a non-degenerate continuum and Z be a nowhere dense subcontinuum of X. Let $q : X \to X/Z$ be the natural function of the quotient space X/Z. Then $\mathrm{bd}_{X/Z}(q(A)) = q(\mathrm{bd}_X(A))$, where A is any closed subset of X.*

Theorem 1.54. *[31, Lemma 2.9] Let X be a compact metric space, A and Z be closed subsets of X such that $A \cap Z \neq \emptyset$, and $q : X \to X/Z$ and $r : A \to A/(A \cap Z)$ be the natural function of the respective quotient spaces. Also, if Y is another closed subset of X such that $A \cap Y = A \cap Z$ and $p : X \to X/Y$ is the natural function of the quotient space, then*

(i) the function $g : q(A) \to A/A(\cap Z)$ given by $g(q(x)) = r(x)$ for each $x \in A$ is a homeomorphism,

(ii) the function $h : q(A) \to p(a)$ given by $h(q(x)) = p(x)$ for each $x \in A$ is a homeomorphism.

Theorem 1.55. *[58, Theorem 18.3] Let $X = A \cup B$ where A, B are closed subsets of X. Let $f : A \to Y$ and $g : B \to Y$ be continuous functions. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$, defined by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$.*

Theorem 1.56. *[44, Theorem 2, p. 434] If X is contractible with respecto to Y, then so is every set which can be obtained from X*

(i) by a retraction,

(ii) by a continuous monotone transformation (if X is compact),

(iii) by an open transformation (if X is compact).

Lemma 1.57. *[25, Proposition 1(a), p. 798] Let X be a continuum and $n \in \mathbb{N}$. If $V \subset X$ is an $n-$cell and U is an open set in X such that $U \cap V \neq \emptyset$, then there is an $n-$cell $\mathcal{T} \subset U \cap V$.*

We close this section with the definition of composants.

Definition 1.58. *The set C of all points of the space X, which can be joined with the point p by a closed, connected proper subset of X is known as the* **composant** *of the point p.*

A result regarding an indecomposable continuum X and its composants is showed.

Theorem 1.59. *[44, Theorem 7, p. 212] Every complete indecomposable space contains a countable infinity composants.*

1.3 Hyperspaces 2^X, $C_n(X)$, $F_n(X)$

In this section we review several results about hyperspaces $2^X, C_n(X), F_n(X)$ defined in last section for continua. In order to check arcwise connectedness for hyperspaces, allow us to recall the following definitions.

Definition 1.60. *A collection $\mathcal{N}$ of sets is known as* **nest** *whenever $N_1, N_2 \in \mathcal{N}$, then $N_1 \subset N_2$ or $N_2 \subset N_1$. A nest from N_1 to N_2 is a nest $\mathcal{N}$ such that for each $N \in \mathcal{N}$, $N_1 \subset N \subset N_2$.*

Definition 1.61. *Let X be a compactum and $\mathcal{H} \subset 2^X$. An* **order arc** *in $\mathcal{H}$ is an arc α such that α is a nest. If A_1, A_2 are arbitrary sets and α is a nest from A_1 to A_2, then we say that α is an order arc from A_1 to A_2.*

Theorem 1.62. *[60, Theorem 1.8] Let X be a compactum and $A_0, A_1 \in 2^X$ such that $A_0 \neq A_1$. The following conditions are equivalent:*

(i) there exists an order arc in 2^X from A_0 to A_1.

(ii) $A_0 \subset A_1$ and each component of A_1 intersects A_0.

Using this result, it follows that 2^X and $C(X)$ are arcwise connected continua for each $n \in \mathbb{N}$, as the following result states:

Theorem 1.63. *[42, Theorem 14.10] If X is a continuum, then 2^X and $C(X)$ are arcwise connected continua, for each $n \in \mathbb{N}$.*

An immediate result is given.

Theorem 1.64. *[60, Theorem 1.11] If α is an order arc in 2^X beginning with $A \in C(X)$, then $\alpha \subset C(X)$.*

More results about the $n-$fold hyperspace of a continuum are presented. They will be key to prove most of the general properties on the (n,m)-fold hyperspace.

Theorem 1.65. *[48, Theorem 3.4] Let X be a continuum and let $n \in \mathbb{N}$. Then $C_n(X)$ contains an n-cell.*

Theorem 1.66. *[48, Theorem 4.8] If X is a continuum, then for every $n \in \mathbb{N}$, each map from the circle, $\mathcal{S}^1$, is homotopic to a constant map. In particular, we have that $C_n(X)$ is unicoherent.*

Lemma 1.67. *[46, Theorem 2.1] If X is a continuum such that $\dim[X] = 2$, then $\dim[C(X)]$ is infinite.*

Theorem 1.68. *[54, Theorem 4.6] The hyperspaces $C_n([0,1])$ and $C_n(\mathcal{S}^1)$ are $2n-$dimensional Cantor manifolds, for each $n \in \mathbb{N}$.*

Theorem 1.69. *[56, Theorem 2.4] Let X a finite graph and A an element of $C_n(X)$. Then,*

(a) $\dim_A[C_n(X)] = 2n + \sum_{p \in R(X)\cap A} (\mathrm{ord}_X(p) - 2)$,

(b) $\dim[C_n(X)] = \dim_X[C_n(X)]$ *and,*

(c) $\dim[C_n(X)] = 2n + \dim[C(X)]$.

The following theorem will be extended to our hyperspace, see Theorem 2.35.

Theorem 1.70. *[52, Theorem 6.9.3] A locally connected continuum X is a graph if and only if for each $n \in \mathbb{N}$, $C_n(X)$ is finite dimensional.*

Theorem 1.71. *[55, Lemma 2.3] For any continuum X and $n \in \mathbb{N}$, $C_n(X)$ is locally arcwise connected at X.*

Theorem 1.72. *[48, Theorem 6.3] A nondegenerate continuum X is indecomposable if and only if for each $n \in \mathbb{N}$ is not arcwise connected.*

Theorem 1.73. *[49, Theorem 3.7] Let $n \in \mathbb{N}$. If X is an indecomposable continuum having Kelley's property, then X is the only point at which $C_n(X)$ is locally connected.*

Theorem 1.74. *[52, Theorem 6.1.8] Let X be a continuum and let $n \in \mathbb{N}$. Then $C_n(X)$ is homeomorphic to the Hilbert cube if and only if X is locally connected and does not contain free arcs.*

Theorem 1.75. *[52, Theorem 6.5.3] A nondegenerate continuum X is indecomposable if and only if for each $n \in \mathbb{N}$, $C_n(X) - \{X\}$ is not arcwise connected.*

Theorem 1.76. *[52, Theorem 6.5.8] Let X be a continuum and $n \in \mathbb{N}$. Then, X is hereditarily indecomposable if and only if $C_n(X) - \{A\}$ is not arcwise connected, for any A nondegenerate subcontinuum of X.*

Theorem 1.77. *[52, Theorem 6.5.15] Let $n > 1$ and let X be an indecomposable continuum. Then $\mathcal{A}$ is an arc component of $C_n(X) - \{X\}$ if and only if there exists a finite number of composants $\mathcal{K}_1, \ldots, \mathcal{K}_n$ of X such that $\mathcal{A} = \langle \mathcal{K}_1, \ldots, \mathcal{K}_n \rangle_n$.*

Theorem 1.78. *[52, Theorem 6.5.16] Let X be an indecomposable continuum and $n \in \mathbb{N}$. If $\mathcal{A}$ is an arc component of $C_n(X) - \{X\}$, then $\mathcal{A} - F_n(X)$ is an arc component of $C_n(X) - (\{X\} \cup F_n(X))$.*

We recall some properties about $F_n(X)$.

Theorem 1.79. *[53, Theorem 2.3] If X is a contractible continuum, then $F_n(X)$ is also contractible for each $n \in \mathbb{N}$.*

Now we prove that $F_n(X)$ is arcwise connected whenever X is an arcwise connected continuum.

Lemma 1.80. *[44, Teo. 10, pp.146] If $\{X_i\}_{i=1}^n$ is a collection of arcwise connected spaces, then $\prod_{i=1}^{n} X_i$ is an arcwise connected space.*

Theorem 1.81. *If X is an arcwise connected continuum, then $F_n(X)$ is an arcwise connected continuum for each $n \in \mathbb{N}$*

Proof. Let $f_n : X^n \to F_n(X)$ be defined by $f_n((x_1, ..., x_n)) = \{x_1, ..., x_n\}$. Notice that f_n is a homeomorphism for each $n \in \mathbb{N}$ and since X^n is an arcwise connected space due to Lemma 1.80, we conclude that $f_n(X^n) = F_n(X)$ is an arcwise connected continuum. □

Is is important to notice that, in contrast with 2^X and $C(X)$, in order for $F_n(X)$ to be arcwise connected, it is needed that X is also an arcwise connected continuum.

To conclude this section, next results combines Theorem 1.63 and Theorem 1.81.

Lemma 1.82. *[52, Lemma 1.8.11] Let X be a compact metric space and let $\sigma : 2^{2^X} \to 2^X$ be defined by $\sigma(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} A$. Then, σ is well defined and continuous.*

Corollary 1.83. *$C_n(X)$ is an arcwise connected continuum for each $n \in \mathbb{N}$.*

Proof. Let $\sigma : 2^{2^X} \to 2^X$ be the function defined in Lema 1.82. Now, by Theorem 1.63 and Theorem 1.81, we have that $F_n(C(X))$ is an arcwise connected continuum. In the way σ was defined, it follows that $\sigma(F_n(C(X)) = C_n(X)$ which is an arcwise connected continuum by Lema 1.82. □

Chapter 2

On the (n, m)-fold hyperspace suspension of a continuum

In this chapter we will verify that if X is a continuum and $n, m \in \mathbb{N}$, then $HS^n_m(X)$ is a continuum for $n \geq m$.

2.1 Construction

In order to define the hyperspace, consider the next definition.

Definition 2.1. *Let (X, τ) be a topological space and $\mathcal{D}$ be a collection of nonempty pairwise disjoint subsets of X whose union is X. This collection of subsets $\mathcal{D}$ is known as* **partition** *of X. Let*

$$\tau(\mathcal{D}) = \{\mathcal{U} \subset \mathcal{D} \colon \bigcup_{U \in \mathcal{U}} U \in \tau\}. \tag{2.1}$$

The space $(\mathcal{D}, \tau(\mathcal{D}))$ is called quotient space of X, and $\tau(\mathcal{D})$ is known as the quotient topology.

If this partition is composed by closed sets, we will say that it is a closed partition.

Definition 2.2. *Let X be a continuum and $n, m \in \mathbb{N}$ with $n \geq m$. The (n, m)-fold hyperspace suspension of X, denoted by $HS^n_m(X)$, is the quotient space $C_n(X) \diagup F_m(X)$ which can be obtained by shrinking $F_m(X)$ to a point under the quotient topology.*

By the way the (n,m)-fold hyperspace suspension is constructed, it is crucial to show the partition from Definition 2.1:

$$\mathcal{D} = \{F_m(X)\} \cup \{\{A\}\colon A \in C_n(X) - F_m(X)\}. \tag{2.2}$$

Definition 2.3. *Let X and Y be topological spaces and $p : X \to Y$ a surjective function. The function p is called the* **quotient function** *if satisfies the following property: $U \subset Y$ is an open subset of Y if and only if $p^{-1}(U)$ is an open subset of X.*

For notation purposes, given a topological space X and a partition $\mathcal{D}$, we denote by $p_X : X \to \mathcal{D}$ the quotient function.

Theorem 2.4. *Let X be a topological space and $\mathcal{D}$ be a partition. The function $p_X : X \to \mathcal{D}$, where $p_X(x)$ is the unique element in the partition containing x, is a quotient function.*

Proof. Let $\mathcal{U} \subset \mathcal{D}$. Notice that $p_X^{-1}(\mathcal{U}) = \bigcup_{U\in\mathcal{U}} U$. From the quotient topology definition, and by Definition 2.3, we have that $\mathcal{U}$ is an open subset of $\mathcal{D}$ if and only if $p_X^{-1}(\mathcal{U})$ is an open subset of X. □

For the rest of this work, we will denote by $q_X^{(n,m)}$ the natural quotient function between $C_n(X)$ and $HS_m^n(X)$. Taking into account how the partition $\mathcal{D}$ from $C_n(X)$ was defined, the following observation is easy to see.

Remark 2.5. *The function $q_X|_{C_n(X)-F_m(X)}\colon C_n(X) - F_m(X) \to HS_m^n(X) - \{F_X^m\}$ is an homeomorphism.*

Next result will help us to show that $HS_m^n(X)$ is a continuum whenever X is a continuum.

Definition 2.6. *Let (X, τ) be a topological space. A partition $\mathcal{D}$ of X is an* **upper semicontinuous partition** *if whenever $D \in \mathcal{D}$, $U \in \tau$ and $D \subset U$, there exists $V \in \tau$ with $D \subset V$ satisfying that if $A \in \mathcal{D}$ and $A \cap V \neq \emptyset$, then $A \subset U$.*

An immediate consequence of this definition is given in the next result.

Theorem 2.7. *[59, Theorem 3.10] Every quotient space of an upper semicontinuous partition of a continuum is a continuum.*

We aim to prove that the (n,m)-fold hyperspace suspension of a continuum is a continuum using the latter result. It will be convenient to have the next terminology.

Definition 2.8. *If $\mathcal{D}$ is a partition of a space X, then any subset of X which is a union of a subcollection of $\mathcal{D}$ is said to be $\mathcal{D}$-**saturated**.*

We combine the definitions of a upper semicontinuous partiton and a set $\mathcal{D}$-saturated.

Theorem 2.9. *[59, Proposition 3.7] Let (X,τ) be a topological space and $\mathcal{D}$ be a patition of X. Then, $\mathcal{D}$ is a upper semicontinuous partition if and only if for each $D \in \mathcal{D}$, $U \in \tau$ and $D \subset U$, then there exists $V \in \tau$ such that $D \subset V \subset U$ is $\mathcal{D}$-saturated.*

The path is clear: we have to prove that the partition given in 2.2 is upper semicontinuous for $HS^n_m(X)$.

Lemma 2.10. *Let X be a normal space and $F \subset X$ be a closed subset of X. If $\mathcal{D} = \{F\} \cup \{\{x\} : x \in X - F\}$, then $\mathcal{D}$ is an upper semicontinuous partition.*

Proof. Let $D \in \mathcal{D}$ and U be an open subset of X such that $D \subset U$. By definition of $\mathcal{D}$, we obtain two cases:

Case 1. If $D = \{x\}$, for some $x \in X - F$, we have that D and F are closed subsets of X. Since X is normal, there exist U_1, U_2 pairwise disjoint open subsets of X such that $D \subset U_1$ and $F \subset U_2$. Let $V = U \cap U_1$. Hence, we have that $D \subset V \subset U$ and notice that $V = \bigcup_{x \in V} \{x\}$. Thus, V is $\mathcal{D}$-saturated.

Case 2. If $D = F$, let $V = U$ and $U = \{F\} \cup \bigcup_{x \in U - F} \{x\}$. Hence, V is $\mathcal{D}$-saturated.

From these cases, applying Theorem 2.9, the result follows. □

The main result of this section follows from here.

Corollary 2.11. *If X is a continuum and $m, n \in \mathbb{N}$ with $m \leq n$, then $HS^n_m(X)$ is a continuum.*

Proof. Let $\mathcal{D} = \{F_m(X)\} \cup \{\{A\} : A \in C_n(X) - F_m(X)\}$. Since $C_n(X)$ is a normal space and $F_m(X)$ is closed in 2^X, we have that $\mathcal{D}$ is an upper semicontinuous partition by Lemma 2.10. By Corollary 2.7, it follows that $HS^n_m(X)$ is a continuum. □

2.2 General properties

In this section, all of the results we present extend those given in the literature for the case $n = m$, see [50], [51] and [52]. Moreover, Theorems 2.13, 2.17–2.25 were published in [24].

Our first result follows immediatly from the fact that $C_n(X)$ is a continuous image of Cantor fan for any $n \in \mathbb{N}$ (see [49, Theorem 2.7]) and Remark 2.5.

Theorem 2.12. *[50, Theorem 3.3] If X is a continuum and $n, m \in \mathbb{N}$ such that $m \leq n$, then $HS_m^n(X)$ is uniformly pathwise connected.*

The following result extends [51, Theorem 4.1].

Theorem 2.13. *Let X be a continuum and $n, m, s \in \mathbb{N}$ with $m \leq s < n$. Then, $HS_m^s(X)$ can be embedded in $HS_m^n(X)$.*

Proof. Let $i_{s,n} : C_s(X) \to C_n(X)$ be the inclusion function, $q_X^{(s,m)} : C_s(X) \to HS_m^s(X)$ and $q_X^{(n,m)} : C_n(X) \to HS_m^n(X)$ be quotient functions. We denote $q_X^{(s,m)}(F_m(X)) = F_X^{(s,m)}$ and $q_X^{(n,m)}(F_m(X)) = F_X^{(n,m)}$. Since $\{\{A\} : A \in C_n(X) - F_m(X)\} \cup \{F_m(X)\}$ and $\{\{B\} : B \in C_s(X) - F_m(X)\} \cup \{F_m(X)\}$ are partitions of $C_n(X)$ and $C_s(X)$, respectively; then $i_{s,n}$ is a relation-preserving and continuous. Now, let $h_{s,n} : HS_m^s(X) \to HS_m^n(X)$ be given by

$$h_{s,n}(\mathcal{A}) = \begin{cases} F_X^{(n,m)}, & if\ \mathcal{A} = F_X^{(s,m)}; \\ q_X^{(n,m)}(i_{s,m}((q_X^{(s,m)})^{-1}(\mathcal{A}))), & if\ \mathcal{A} \neq F_X^{(s,m)}. \end{cases}$$

Notice that $h_{s,n}$ is a continuous function by Theorem 1.51. Moreover, as $h_{s,n}$ is defined, it is clear that $h_{s,n}$ is a one-to-one function. Since the spaces are compact, $h_{s,n}$ is an embedding. □

Now we prove an important result about the relationship between the dimension of the hyperspaces $C_n(X)$ and $HS_m^n(X)$.

Lemma 2.14. *[15, Lemma 3.1] If X is a finite dimensional continuum and $n \in \mathbb{N}$, then* $\dim[F_n(X)] \leq n \cdot \dim[X]$.

Lemma 2.15. *[64, Theorem 10.3] If X is a finite dimensional continuum and A is a nowhere dense subset of X,* $\dim[X] = \dim[X - A]$.

Theorem 2.16. *If X is a finite-dimensional continuum and $n, m \in \mathbb{N}$ such that $n \geq m$, then* $\dim[C_n(X)] < \infty$ *if and only if* $\dim[HS_m^n(X)] < \infty$. *Moreover, if either* $\dim[C_n(X)] < \infty$ *or* $\dim[HS_m^n(X)] < \infty$, *then* $\dim[C_n(X)] = \dim[HS_m^n(X)]$.

Proof. Suppose that $\dim(C_n(X))$ is finite. Since $C_n(X) - F_m(X)$ is homeomorphic to $HS_m^n(X) - \{F_X^m\}$. By Lemma 1.47, we have that $\dim[HS_m^n(X)] = \dim[HS_m^n(X) - \{F_X^m\}]$. Observe that $C_n(X) = (C_n(X) - F_m(X)) \cup F_m(X)$ and since $F_m(X)$ is a nowhere dense subset of $C_n(X)$, by Lemma 2.15 it follows that $\dim[C_n(X)] = \dim[C_n(X) - F_m(X)]$. Therefore, $\dim[C_n(X)] = \dim[HS_m^n(X)]$.

On the other hand, suppose that $\dim[HS_m^n(X)]$ is finite. By Lemma 1.47, $\dim[HS_m^n(X)] = \dim[HS_m^n(X) - \{F_X^m\}]$. Since $HS_m^n(X) - \{F_X^m\}$ is homeomorphic to $C_n(X) - F_m(X)$, we have that $\dim[HS_m^n(X)] = \dim[C_n(X) - F_m(X)]$. Suppose that $\dim[C_n(X)]$ is infinite. By Lemma 2.14, $\dim[F_m(X)] \leq m \cdot \dim[X]$. Notice that $C_n(X) = (C_n(X) - F_m(X)) \cup F_m(X)$, it follows that $\dim[HS_m^n(X)]$ is infinite, a contradiction. Hence, $\dim[C_n(X)]$ is infinite. Since $F_m(X)$ is nowhere dense in $C_n(X)$, by Lemma 2.15 $\dim[C_n(X)] = \dim[C_n(X) - F_m(X)]$, and the result follows. □

Theorem 2.17. *If X is a continuum and $n, m \in \mathbb{N}$ such that $m \leq n$, then $HS_m^n(X)$ contains an n-cell.*

Proof. By Theorem 1.65, $C_n(X)$ contains an n-cell, denoted by $\mathcal{M}$. Moreover, since $C_n(X) - F_m(X)$ is a dense open subset of $C_n(X)$, we have that $((C_n(X) - F_m(X)) \cap \mathcal{M} \neq \emptyset$. By Lemma 1.57, there exists an n-cell $\mathcal{N}$ such that $\mathcal{N} \subset C_n(X) - F_m(X)$. Thus, $HS_m^n(X)$ has an n-cell. □

Using the fact that the continuum $C_n(X)$ contains a $2n$-cell (see [48, Theorem 3.5]) whenever X contains n pairwise disjoint decomposable subcontinua, the next result is stated.

Theorem 2.18. *[50, Theorem 3.8] If X is a continuum that contains n pairwise disjoint decomposable subcontinua and $n, m \in \mathbb{N}$ such that $m \leq n$, then $HS_m^n(X)$ contains a $2n$-cell.*

Next result extends [50, Theorem 4.1].

Theorem 2.19. *Let X be a continuum. Then, $HS_m^n(X)$ has property (b) for any $n, m \in \mathbb{N}$ with $m \leq n$.*

Proof. Let $\mathcal{A} \in HS_m^n(X)$. If $\mathcal{A} = F_X^m$, then $(q_X^{(n,m)})^{-1}(\mathcal{A}) = F_m(X)$ which is a connected subset of $C_n(X)$. On the other hand, if $\mathcal{A} \neq F_X^m$, by Remark 2.5, then $(q_X^{(n,m)})^{-1}(\mathcal{A})$ is a one-point set. Hence, $(q_X^{(n,m)})^{-1}(\mathcal{A})$ is a connected subset of $C_n(X)$. Therefore, $q_X^{(n,m)}$ is a monotone function. By Theorem 1.66, $C_n(X)$ has property (b). Since $q_X^{(n,m)}(C_n(X)) = HS_m^n(X)$ and Theorem 1.56, we conclude that $HS_m^n(X)$ has the property (b). □

An immediate application of Theorem 2.19 is now given.

Theorem 2.20. *Let X be a continuum and $n, m \in \mathbb{N}$ with $m \leq n$. Then, $HS_m^n(X)$ is unicoherent.*

Proof. Applying Theorem 2.19 and Theorem 1.41, the result follows. □

The following result extends [50, Theorem 4.2]. We present its proof since it is important to emphasize the changes from the case $n = m$ to $n < m$.

Theorem 2.21. *If X is a continuum and $n, m \in \mathbb{N}$ such that $m \leq n$, then $HS_m^n(X)$ is colocally connected.*

Proof. Case $n = m = 1$ is already proved in Theorem 4.1 from [18]. Suppose $n \geq 2$ and let $\mathcal{A} \in HS_m^n(X)$. We are going to consider three cases:

Case 1. $\mathcal{A} = F_X^m$. For any $\varepsilon > 0$, let $\mathcal{U}_\varepsilon = B_H(F_m(X), \varepsilon)$. Notice that $\{q_X^{(n,m)}(\mathcal{U}_\varepsilon) \colon \varepsilon > 0\}$ forms a base of open sets about F_X^m. Fix $\varepsilon > 0$. Let $\mathcal{B} \in HS_m^n(X) - q_X^{(n,m)}(\mathcal{U}_\varepsilon)$. Thus, $(q_X^{(n,m)})^{-1}(\mathcal{B}) \in C_n(X) - \mathcal{U}_\varepsilon$. By Theorem 1.62, there exists an order arc $\alpha : [0, 1] \to C_n(X)$ such that $\alpha(0) = (q_X^{(n,m)})^{-1}(\mathcal{B})$ and $\alpha(1) = X$ and $\alpha([0, 1]) \subset C_n(X) - \mathcal{U}_\varepsilon$. Notice that $q_X^{(n,m)} \circ \alpha : [0, 1] \to HS_m^n(X)$ is an arc from $\mathcal{B}$ to $q_X^{(n,m)}(X)$ satisfying $(q_X^{(n,m)} \circ \alpha)([0, 1]) \subset HS_m^n(X) - q_X^{(n,m)}(\mathcal{U}_\varepsilon)$, which implies that this space is arcwise connected.

Case 2. $\mathcal{A} = q_X^{(n,m)}(X)$. For any $\varepsilon > 0$, let $\mathcal{U}_\varepsilon = B_H(X, \varepsilon)$. Observe that $\{q_X^{(n,m)}(\mathcal{U}_\varepsilon) \colon \varepsilon > 0\}$ forms a base of open sets about $q_X^{(n,m)}(X)$. Fix $\varepsilon > 0$. Let $\mathcal{B} \in HS_m^n(X) - q_X^{(n,m)}(\mathcal{U}_\varepsilon)$. Thus, $(q_X^{(n,m)})^{-1}(\mathcal{B}) \in C_n(X) - \mathcal{U}_\varepsilon$. Let $D \in F_m((q_X^{(n,m)})^{-1}(B))$. By Theorem 1.62, there exists an order arc $\alpha : [0, 1] \to C_n(X)$ such that $\alpha(0) = D$ and $\alpha(1) = (q_X^{(n,m)})^{-1}(\mathcal{B})$. Moreover, $\alpha([0, 1]) \subset C_n(X) - \mathcal{U}_\varepsilon$. Hence, $q_X^{(n,m)} \circ \alpha : [0, 1] \to C_n(X)$ is an arc such that $(q_X^{(n,m)} \circ \alpha)(0) = F_X^m$, $(q_X^{n,m} \circ \alpha)(1) = D$ and $(q_X^{(n,m)} \circ \alpha)([0, 1]) \subset HS_m^n(X) - q_X^{(n,m)}(\mathcal{U}_\varepsilon)$. Therefore, $HS_m^n(X) - q_X^{(n,m)}(\mathcal{U}_\varepsilon)$ is an arcwise connected space.

Case 3. $\mathcal{A} \in HS_m^n(X) - \{F_x^m, q_X^{(n,m)}(X)\}$. For any $\varepsilon > 0$, let $\mathcal{U}_\varepsilon = B_H((q_X^{n,m})^{-1}(\mathcal{A}), \varepsilon)$. Thus, $\{q_X^{(n,m)}(\mathcal{U}_\varepsilon) \colon \varepsilon > 0\}$ forms a base of open sets about $\mathcal{A}$. Fix $\varepsilon > 0$ such that $q_X^{(n,m)}(\mathcal{U}_\varepsilon) \cap \{F_X^m, q_X^{(n,m)}(X)\} = \emptyset$. Let $\mathcal{B} \in HS_m^n(X) - q_X^{(n,m)}(\mathcal{U}_\varepsilon)$. If $(q_X^{(n,m)})^{-1}(\mathcal{B}) \not\subset (q_X^{(n,m)})^{-1}(\mathcal{A})$, by Theorem 1.62 there exists an order arc $\alpha : [0,1] \to C_n(X)$ such that $\alpha(0) = (q_X^{(n,m)})^{-1}(\mathcal{B})$ and $\alpha(1) = X$. Thus, $\alpha([0,1]) \subset C_n(X) - B_H((q_X^{(n,m)})^{-1}(\mathcal{A}), \varepsilon)$. Hence, $q_X^{(n,m)} \circ \alpha$ is an arc from $\mathcal{B}$ to $q_X^{(n,m)}(X)$ such that $q_X^{(n,m)} \circ \alpha \subset HS_m^n(X) - \mathcal{U}_\varepsilon$, as desired.

On the other hand, suppose that $(q_X^{(n,m)})^{-1}(\mathcal{B}) \subset (q_X^{(n,m)})^{-1}(\mathcal{A})$. Let $D \in F_m((q_X^{(n,m)})^{-1}(\mathcal{B}))$. By Theorem 1.62, there exists an order arc $\beta : [0,1] \to C_n(X)$ such that $\beta(0) = D$ and $\beta(1) = (q_X^{(n,m)})^{-1}(\mathcal{B})$. Thus, $\beta([0,1])$ is contained in $C_n(X) - B_H((q_X^{(n,m)})^{-1}(\mathcal{A}), \varepsilon)$. Hence, $q_X^{(n,m)} \circ \beta : [0,1] \to HS_m^n(X)$ is an arc from F_X^m to $\mathcal{B}$ and $(q_X^{(n,m)} \circ \beta)([0,1]) \subset HS_m^n(X) - q_X^{(n,m)}(\mathcal{U}_\varepsilon)$. Therefore, the last space is arcwise connected. □

Since colocal connectedness implies aposyndetic, we have the next result:

Corollary 2.22. *If X is a continuum and $n, m \in \mathbb{N}$ such that $m \leq n$, then $HS_m^n(X)$ is aposyndetic.*

From this, we can prove the following result, for the case $n = m$ see [50, Corollary 4.4].

Theorem 2.23. *If X is a continuum and $n, m \in \mathbb{N}$ such that $m \leq n$, then $HS_m^n(X)$ is finitely aposyndetic.*

Proof. By Theorem 2.20, $HS_m^n(X)$ is unicoherent. By Corollary we have that 2.22, $HS_m^n(X)$ is aposyndetic. Finally, by Lemma 1.40, any aposyndetic unicoherent continuum is finitely aposyndetic. □

To conclude this section, consider the next definition and a result that follows from there.

Definition 2.24. *If X is a finite dimensional continuum, we say that X is a* **Cantor manifold** *whenever $A \subset X$ is such that* $\dim[A] \leq \dim[X] - 2$, *then $X - A$ is connected.*

Theorem 2.25. *Let $n, m \in \mathbb{N}$ be such that $n \geq m$. If $C_n(X)$ is a finite dimensional Cantor such that* $\dim[C_n(X)] \geq n + 2$, *then $HS_m^n(X)$ is a finite dimensional Cantor manifold.*

Proof. Let $k = \dim[HS^n_m(X)]$. Proceeding by contradiction, suppose that $HS^n_m(X)$ is not a Cantor manifold. Hence, there exists $\mathcal{A}$ subset of $HS^n_m(X)$ such that $\dim[\mathcal{A}] \leq k-2$ and $HS^n_m(X) - \mathcal{A}$ is not connected. Furthermore, suppose also that $\mathcal{A}$ is closed in $HS^n_m(X)$ (see [66, Theorem 1.4]). Thus, there exist $\mathcal{D}, \mathcal{E}$ nonempty open subsets of $HS^n_m(X)$ such that $HS^n_m(X) - \mathcal{A} = \mathcal{D} \cup \mathcal{E}$. Hence, $C_n(X) - q_X^{-1}(\mathcal{A}) = q_X^{-1}(\mathcal{D}) \cup q_X^{-1}(\mathcal{E})$, where $q_X^{-1}(\mathcal{D})$ and $q_X^{-1}(\mathcal{E})$ are disjoint open subsets of $C_n(X)$. In order to reach a contradiction, we will see that $\dim[q_X^{-1}(\mathcal{A})] \leq k-2$. In this way, $C_n(X)$ is not a Cantor manifold. Consider two cases. If $F^m_X \notin \mathcal{A}$, then $q_X^{-1}(\mathcal{A})$ is homeomorphic to $\mathcal{A}$. Since the notion of dimension is an invariant topological property, it follows that $\dim[q_X^{-1}(\mathcal{A})] \leq k-2$ and the contradiction follows.

Consider the remaining case: $F^m_X \in \mathcal{A}$. Notice that $C(X) \subset C_n(X)$ and, by Theorem 2.16, we have that $\dim[C_n(X)] = \dim[HS^n_m(X)] = k < \infty$. Thus, $\dim[X] = 1$ by Lema 1.67. Hence, $\dim[F_m(X)] \leq m$. Observe that $q_X^{-1}(\mathcal{A}) = q_X^{-1}(\mathcal{A} - \{F^m_X\}) \cup q_X^{-1}(\{F^m_X\}) = q_X^{-1}(\mathcal{A} - \{F^m_X\}) \cup F_m(X)$. Since $\dim[F_m(X)] \leq m \leq n \leq k-2$, and $\dim[q_X^{-1}(\mathcal{A} - \{F^m_X\})] \leq \dim[\mathcal{A}] \leq k-2$, by Theorem 1.46, we conclude that $\dim[q_X^{-1}(\mathcal{A})] \leq k-2$. □

Corollary 2.26. *$HS^n_m([0,1])$ and $HS^n_m(S^1)$ are Cantor manifolds with dimension $2n$, where $n \geq m$.*

Proof. Case $n = m$ is already proved in [50, Corollary 3.10].

Suppose that $m < n$, Theorem 1.68 we have that $C_n([0,1])$ and $C_n(\mathcal{S}^1)$ are $2n$-dimensional Cantor manifolds. Since $n \geq 2$ and $2n \geq n+2$, the result follows by Theorem 2.25. □

In order to prove Lemma 2.27, we recall the following result about Cantor manifolds and the dimension on their points.

Theorem 2.27. *[35, A) p. 93] An n-dimension Cantor manifold has dimension n at each of its points.*

Using Corollary 2.26, the following useful result is given.

Lemma 2.28. *If X is a graph, then*

$$\dim_{F^m_X}[HS^n_m(X)] \leq \dim_{q_X^{(n,m)}(X)}[HS^n_m(X)].$$

Proof. Let $x \in X$. We will consider two cases.

Case 1. $\mathrm{ord}_X(x) \leq 2$.

By Theorem 1.15, X is an arc or a simple closed curve. Since, by Theorem 2.26, $HS_m^n([0,1])$ and $HS_m^n(\mathcal{S}^1)$ are $2n$-dimensional Cantor manifolds, both spaces are dimensionally homogeneous by Theorem 2.27. Hence, in both cases we have that $\dim_{q_X^{(n,m)}(X)}[HS_m^n(X)] = \dim_{F_X^m}[HS_m^n(X)]$.

Case 2. $\text{ord}_X(x) \geq 3$.

By Theorem 1.69, $\dim_X[C_n(X)] = \dim[C_n(X)]$. Since $q_X^{(n,m)}|_{C_n(X)-F_m(X)}$ is a homeomorphism, we have that $\dim_X[C_n(X)] = \dim_{q_X^{(n,m)}(X)}[HS_m^n(X)]$. Hence,

$$\dim[HS_m^n(X)] = \dim[C_n(X)] = \dim_X[C_n(X)] = \dim_{q_X^{(n,m)}(X)}[HS_m^n(X)].$$

Therefore, since F_X^m is another point in $HS_m^n(X)$, we have that

$$\dim_{F_X^m}[HS_m^n(X)] \leq \dim_{q_X^{(n,m)}(X)}[HS_m^n(X)].$$

□

2.3 Local connectedness

Theorem 2.29. *[60, Theorem 1.13]. The following statements are equivalent:*

(i) X *is a locally connected continuum,*

(ii) $C(X)$ *is a locally connected continuum,*

(iii) 2^X *is a locally connected continuum,*

(iv) $C_n(X)$ *is a locally connected continuum.*

Consider the next definition.

Definition 2.30. *Let S be a topological space and $p \in S$. We say that S is* **connected im-kleinnen** *at p if every neighborhood of p contains a connected neighborhood of p.*

Observation 2.31. *It is important to notice that there is no loss of generality if we assume the first neighborhood to be open in S. However, there is loss of generality if we assume the connected neighborhood to be open in S.*

The following result will be useful throughout this work.

Theorem 2.32. *[59, Theorem 5.22] A topological space S is connected im-kleinnen at every point of S if and only if it is locally connected at every point of S.*

Now we prove a more general result.

Lemma 2.33. *Let X be a continuum and $n, m \in \mathbb{N}$ with $n \geq m$. Then $C_n(X) - F_m(X)$ is locally connected if and only if X is a locally connected continuum.*

Proof. First suppose that X is locally connected. By Theorem 2.29, $C_n(X)$ is a locally connected continuum. Since $C_n(X) - F_m(X)$ is open in $C_n(X)$, the result follows. For the reciprocal, case $n = m = 1$ is already proved in [60, Theorem 1.208.2]. Now, suppose that $n \geq 2$ and $n \geq m$. Let $x \in X$ and U be an open neighborhood of x. Then there exists $\varepsilon > 0$ such that $\text{cl}(B_d(x, \varepsilon_0)) \subset U$. Let K be a component of $\text{cl}(B_d(x, \varepsilon_0))$ containing x. Thus, K is a nondegenerate subcontinuum of X satisfying $x \in K \subset U$. Notice that $K \in \langle U \rangle$. Moreover, $K \in \langle U \rangle - F_m(X)$, as K is a nondegenerate subcontinuum. Since $C_n(X) - F_m(X)$ is locally connected, there exists $\mathcal{V}$ a connected open subset of $C_n(X) - F_m(X)$ such that $K \in \mathcal{V} \subset \text{cl}_{C_n(X)}(\mathcal{V}) \subset \langle U \rangle - F_m(X)$. Since $\mathcal{V}$ is open, there exists $\varepsilon > 0$ such that $B_{H_d}(K, \varepsilon) \cap C_n(X) \subset \mathcal{V}$. Observe that $K \in \text{cl}_{C_n(X)}(\mathcal{V}) \cap C(X)$. Notice that $C = \bigcup(\text{cl}_{C_n(X)}(\mathcal{V}))$ is a subcontinuum of X. Let $y \in B_d(x, \varepsilon)$. Hence, we have that $K \cup \{y\} \in B_{H_d}(K, \varepsilon) \cap C_n(X) \subset \mathcal{V}$ and $y \in C$. Then, $x \in \text{int}_X(C) \subset U$, so that, X is connected im-kleinnen at x. Since x is an arbitrary point, X is connected im-kleinnen at every point which implies that X is locally connected, see Theorem 2.32. □

Using this result we are able to prove an equivalence of local connectedness between the continuum and the hyperspace.

Theorem 2.34. *A continuum X is locally connected is and only if $HS_m^n(X)$ is locally connected for every $n \geq m$.*

Proof. Suppose that X is a locally connected continuum. By Theorem 2.29, $C_n(X)$ is a locally connected continuum for every $n \in \mathbb{N}$. Since $q_X^{(n,m)}$ is a continuous function, we have that $HS_m^n(X)$ is locally connected. On the other hand, suppose that $HS_m^n(X)$ is locally connected. Notice that $HS_m^n(X) -$

$\{F_X^m\}$ is locally connected as $\{F_X^m\}$ is a closed subset of $HS_m^n(X)$. Since $q_m^n|_{C_n(X)-F_m(X)}: C_n(X) - F_m(X) \to HS_m^n(X) - \{F_X^m\}$ is a homeomorphism, it follows that $C_n(X) - F_m(X)$ is locally connected. By Lemma 2.33, X is locally connected. □

Theorem 2.35. *A locally connected continuum X is a graph if and only if $HS_m^n(X)$ is finite-dimensional.*

Proof. Suppose X is a graph. By Theorem 2.35, $C_n(X)$ is finite-dimensional. Since $\dim[HS_m^n(X)] = \dim[C_n(X)]$, $HS_m^n(X)$ is finite-dimensional.

On the other hand, if X is a locally connected continuum such that $HS_m^n(X)$ is finite-dimensional, it follows that $C_n(X)$ is also finite-dimensional. By Theorem 2.35, X is a graph. □

2.4 Contractibility

In this section, Definition 1.28 will be used. For further information about this property, see [44, 56, p. 360].

Theorem 2.36. *[51, Theorem 3.1] If X is an absolute retract and $n, m \in \mathbb{N}$ such that $n \geq m$, then $F_m(X)$ is a strong deformation retract of $C_n(X)$.*

A result about contractibility in the hyperspace $HS_m^n(X)$ is given. This result extends [51, Theorem 5.2].

Theorem 2.37. *Let X be a continuum and let $n, m \in \mathbb{N}$ such that $n \geq m$. If $F_m(X)$ is a strong deformation retract of $C_n(X)$, then $HS_m^n(X)$ is contractible.*

Proof. Since $F_m(X)$ is a strong deformation retract of $C_n(X)$, there exists a map $H : C_n(X) \times [0,1] \to C_n(X)$ such that $H(A,0) = A$, $H(A,1) = r(A)$ and $H(B,t)) = B$ for each $A \in C_n(X)$, each $B \in F_m(X)$ and $t \in [0,1]$, where $r : C_n(X) \to F_m(X)$ is a retraction.

Consider the function $G : HS_m^n(X) \times [0,1] \to HS_m^n(X)$ defined by

$$G(\mathcal{A},t) = \begin{cases} q_X^{(n,m)}(H((q_X^{(n,m)})^{-1}(\mathcal{A}),t)), & \mathcal{A} \neq F_X^m \\ F_X^m, & \mathcal{A} = F_X^m \end{cases}$$

Note that $G(\mathcal{A},0) = \mathcal{A}$ and $G(\mathcal{A},1) = F_X^m$ and G is continuous by [17, Theorem 4.3, p.126]. Therefore, $HS_m^n(X)$ is contractible. □

Using these results, the following Corollary is given.

Corollary 2.38. *If X is an absolute rectract and $n, m \in \mathbb{N}$ such that $n \geq m$, then $HS_m^n(X)$ is contractible.*

Proof. By Theorem 2.36, $F_m(X)$ is a strong deformation retract of $C_n(X)$. Hence, using Theorem 2.37 it follows that $HS_m^n(X)$ is contractible. □

In a similar way Theorem 2.37 was proved, the following result is established, see [51, Theorem 5.1].

Theorem 2.39. *If X is a contractible continuum and $n, m \in \mathbb{N}$ such that $n \geq m$, then $HS_m^n(X)$ is contractible.*

The concept of shape of Borsuk will be used now, see [9] for more information regarding this subject.

Lemma 2.40. *[11, Theorem 25.2] Let $A, B \subset \mathcal{Q}$ be such that they have the same shape in the sense of Borsuk. Then, $\mathcal{Q} - A$ is homeomorphic to $\mathcal{Q} - B$.*

An important result that will be used later on this paper is now given.

Theorem 2.41. *If X is a contractible locally connected continuum without free arcs, then $HS_m^n(X)$ is homeomorphic to the Hilbert cube for any $n, m \in \mathbb{N}$ with $m \leq n$. In particular, $HS_m^n(X)$ is homeomorphic to $C_n(X)$.*

Proof. Since X is a locally connected continuum without free arcs, $C_n(X)$ is homeomorphic to the Hilbert cube (for case $n = 1$ see [15, Theorem 4.1]). As X is contractible, we have that $F_m(X)$ is contractible, by Theorem 1.79. Hence, $F_m(X)$ has the shape of a point in the sense of Borsuk, [9, Theorem 5.5 p. 28]. By Theorem 2.40, $C_n(X) - F_m(X)$ is homeomorphic to $\mathcal{Q} - \{p\}$ for some $p \in \mathcal{Q}$. Since $C_n(X) - F_m(X)$ is homeomorphic to $HS_m^n(X) - \{F_X^m\}$, it follows that $\mathcal{Q} - \{p\}$ is homeomorphic to $HS_m^n(X) - \{F_X^m\}$. Thus, $HS_m^n(X)$ is homeomorphic to $\mathcal{Q}$. □

The next result follows easily.

Corollary 2.42. *Let $n, m \in \mathbb{N}$ with $m \leq n$, then $HS_m^n(\mathcal{Q})$ is homeomorphic to $\mathcal{Q}$.*

2.5 Indecomposability

In this section, we give a local topological property that holds for the case $n = m$ but fails for the case $m < n$, see Theorem 2.48. First recall the following result:

Theorem 2.43. *[52, Theorem 7.1.10] If X is a continuum and $n \in \mathbb{N}$, then $HS_n^n(X)$ is locally arcwise connected at $q_X^{(n,n)}(X)$ and F_X^n. Moreover, any neighborhood of F_X^n contains simple closed curves passing through F_X^n.*

Regarding the second part of Theorem 2.43, it is possible to generalize it as follows.

Theorem 2.44. *If X is a continuum and $n, m \in \mathbb{N}$ with $m \leq n$, then any neighborhood of F_X^m contains simple closed curves contained in $q_X^{(n,m)}(C(X))$ passing through F_X^m.*

Proof. Let $\mathcal{V}$ be an open subset of $HS_m^n(X)$ containing F_X^m. Consider $x \in X$. Since $(q_X^{(n,m)})^{-1}(\mathcal{V})$ is an open subset of $C_n(X)$ containing $F_m(X)$, there exists $\varepsilon > 0$ such that $B_{C_n(X)}(\{x\}, \varepsilon) \subset (q_X^{(n,m)})^{-1}(\mathcal{V})$. By Theorem 1.24, there exists a nondegenerate subcontinuum $\mathcal{C}$ of $C_n(X)$ such that $\{x\} \in \mathcal{C} \subset B_{C_n(X)}(\{x\}, \varepsilon)$ and choose $\{y\} \in \mathcal{C}$ such that $y \neq x$. By Theorem 1.62, there exist order arcs $\alpha_1, \alpha_2 : [0,1] \to C_n(X)$ such that $\alpha_1(0) = \{x\}$ and $\alpha_1(1) = \mathcal{C}$, $\alpha_2(0) = \{y\}$ and $\alpha_2(1) = \mathcal{C}$. Since α_1 and α_2 begin in $C(X)$, then $\alpha_1([0,1]) \cup \alpha_2([0,1]) \subset C(X)$ by Theorem 1.64 and $\alpha_1([0,1]) \cup \alpha_2([0,1]) \subset (q_X^{(n,m)})^{-1}(\mathcal{V})$. Since $\alpha_1([0,1])$ and $\alpha_2([0,1])$ are arcwise connected, we may obtain an arc $\mathcal{A}$, contained in $\alpha_1([0,1]) \cup \alpha_2([0,1])$, with end points in $F_m(X)$. Hence, $q_X^{(n,m)}(\mathcal{A})$ is a simple closed curve in $\mathcal{V}$ that contains F_X^m. □

Theorem 2.45. *If X is a continuum and $n, m \in \mathbb{N}$ with $m \leq n$, then $HS_m^n(X)$ is locally arcwise connected at $q_X^{(n,m)}(X)$.*

Proof. By Theorem 1.71, $C_n(X)$ is locally arcwise connected at X. Since $q_X^{(n,m)}|_{C_n(X)-F_m(X)}$ is a homeomorphism, $HS_m^n(X)$ is locally arcwise connected at $q_X^{(n,m)}(X)$. □

The following result extends [50, Lemma 6.1].

Lemma 2.46. *If X is a decomposable continuum and $n, m \in \mathbb{N}$ with $m \leq n$, then $C_n(X) - (\{X\} \cup F_m(X))$ is arcwise connected.*

Proof. The result for case $n = m = 1$ follows from [18, Theorem 3.3], whereas the proof for case $n = m > 1$ is given in [50, Lemma 6.1]. Suppose that $m < n$. According to the definition of arcwise conectedness, it is enough to prove that for each $A \in F_n(X) - F_m(X)$, there exists an arc $\alpha : [0, 1] \to C_n(X) - (\{X\} \cup F_m(X))$ such that $\alpha(0) = A$ and $\alpha(1) \in C_n(X) - (\{X\} \cup F_n(X))$. Let $A \in F_n(X) - F_m(X)$ and $p \in A$. Since X is a decomposable continuum, there exists $Z \in C(X)$ such that $p \in Z$ and $Z \neq X$. As $C(X)$ is arcwise connected, there exists an order arc $\beta : [0, 1] \to C(X)$ such that $\beta(0) = \{p\}$ and $\beta(1) = Z$. Let $\alpha : [0, 1] \to C_n(X)$ be defined by $\alpha(t) = \beta(t) \cup A$. Notice that $\alpha(0) = A$, $\alpha(1) = Z \cup A$ and $\alpha(t) \in C_n(X) - (\{X\} \cup F_m(X))$, for each $t \in [0, 1]$. Hence, it is possible to join an element in $F_n(X) - F_m(X)$ to an element in $C_n(X) - (\{X\} \cup F_n(X))$ and the result follows. □

The notion of (n, m)-fold hyperspace suspension allows for a characterization of indecomposable continua as follows.

Theorem 2.47. *Let X be a continuum and $n, m \in \mathbb{N}$ with $m \leq n$.*

(a) If X is decomposable, then $HS^n_m(X) - \{q^{(n,m)}_X(X)\}$ and $HS^n_m(X) - \{q^{(n,m)}_X(X), F^m_X\}$ are arcwise connected.

(b) X is indecomposable if and only if $HS^n_m(X) - \{q^{(n,m)}_X(X)\}$ is not arcwise connected, if $m < n$.

(c) X is indecomposable if and only if $HS^n_m(X) - \{q^{(n,m)}_X(X), F^m_X\}$ is not arcwise connected.

Proof. (a) First we prove that $HS^n_m(X) - \{q^{(n,m)}_X(X)\}$ is arcwise connected. By Theorem 1.72, $C_n(X) - \{X\}$ is arcwise connected. It is enough to prove that for some element $\mathcal{A} \in HS^n_m(X) - \{q^{(n,m)}_X(X), F^m_X\}$, there exists an arc with end points $\mathcal{A}$ and F^m_X, so that for each pair of points from $HS^n_m(X) - \{q^{(n,m)}_X(X), F^m_X\}$, we can join them with an arc passing through F^m_X. Let $\mathcal{A} \in HS^n_m(X) - \{q^{(n,m)}_X(X), F^m_X\}$. Then, $(q^{(n,m)}_X)^{-1}(\mathcal{A}) \in C_n(X) - (\{X\} \cup F_m(X))$. Since $C_n(X) - \{X\}$ is arcwise connected, there exists an arc α in $C_n(X) - \{X\}$ with end points $(q^{(n,m)}_X)^{-1}(\mathcal{A})$ and some $T \in F_m(X)$ such that $\alpha \cap F_m(X) = \{T\}$. Hence, $q^{(n,m)}_X(\alpha)$ is an arc in $HS^n_m(X) - \{q^{(n,m)}_X(X)\}$ joining $\mathcal{A}$ and F^m_X. Thus, any two points in $HS^n_m(X) - \{q^{(n,m)}_X(X)\}$ can be joined with an arc passing through F^m_X. On the other hand, by Theorem

2.46, $C_n(X) - (\{X\} \cup F_m(X))$ is arcwise connected. Hence, $q_X^{(n,m)}(C_n(X) - (\{X\} \cup F_m(X))) = HS_m^n(X) - \{q_X^{(n,m)}(X), F_X^m\}$ is arcwise connected.

(b) By Theorem 1.59, there are infinitely many composants of X. Let $\mathcal{K}_1, \ldots, \mathcal{K}_n$, $\mathcal{L}_1, \ldots, \mathcal{L}_n$ be pairwise different composants of X. Then, $\langle \mathcal{K}_1, \ldots, \mathcal{K}_n \rangle_n$ and $\langle \mathcal{L}_1, \ldots, \mathcal{L}_n \rangle_n$ are different arc components of $C_n(X) - \{X\}$, due to Theorem 1.77. Moreover, since $\mathcal{K}_1, \ldots, \mathcal{K}_n$, $\mathcal{L}_1, \ldots, \mathcal{L}_n$ are pairwise different subsets of X, it follows that $\langle \mathcal{K}_1, \ldots, \mathcal{K}_n \rangle_n \cap F_m(X) = \emptyset$ and $\langle \mathcal{L}_1, \ldots, \mathcal{L}_n \rangle_n \cap F_m(X) = \emptyset$, as $m < n$. Hence, $q_X^{(n,m)}(\langle \mathcal{K}_1, \ldots, \mathcal{K}_n \rangle_n)$ and $q_X^{(n,m)}(\langle \mathcal{L}_1, \ldots, \mathcal{L}_n \rangle_n)$ are two distinct arc components of $HS_m^n(X) - \{q_X^{(n,m)}(X)\}$. On the other hand, the converse holds by (a).

(c) The case $m = n$ is proved in [50, Theorem 6.2]. Suppose that $m < n$. Let $\mathcal{K}_1, \ldots, \mathcal{K}_n, \mathcal{L}_1, \ldots, \mathcal{L}_n$ be pairwise distinct arc components of X. By Theorem 1.77, $\langle \mathcal{K}_1, \ldots, \mathcal{K}_n \rangle_n$ and $\langle \mathcal{L}_1, \ldots, \mathcal{L}_n \rangle_n$ are two distinct arc components of $C_n(X) - \{X\}$. Moreover, by Theorem 1.78, $\langle \mathcal{K}_1, \ldots, \mathcal{K}_n \rangle_n - F_m(X)$ and $\langle \mathcal{L}_1, \ldots, \mathcal{L}_n \rangle_n - F_m(X)$ are distinct arc components of $C_n(X) - (\{X\} \cup F_m(X))$. Hence, $q_X^{(n,m)}(\langle \mathcal{K}_1, \ldots, \mathcal{K}_n \rangle_n - F_m(X))$ and $q_X^{(n,m)}(\langle \mathcal{L}_1, \ldots, \mathcal{L}_n \rangle_n - F_m(X))$ are two distinct arc components of $HS_m^n(X) - (\{q_X^{(n,m)}(X), F_X^m\})$. On the other hand, the converse holds by (a). □

The next result states that the local arcwise conectedness fails in our hyperspace if $m < n$.

Theorem 2.48. *If X is an indecomposable continuum and $n, m \in \mathbb{N}$ with $m < n$, then $HS_m^n(X)$ is not locally arcwise connected at F_X^m.*

Proof. Proceeding by contradiction, suppose that $HS_m^n(X)$ is locally arcwise connected at F_X^m. Let $\mathcal{U}$ be an open subset of $HS_m^n(X)$ such that $F_X^m \in \mathcal{U}$ and $q_X^{(n,m)}(X) \notin \mathcal{U}$. There exists an open arcwise connected set $\mathcal{V}$ of $HS_m^n(X)$ such that $F_X^m \in \mathcal{V} \subset \mathcal{U}$. As X is indecomposable, let $\mathcal{K}_1, \ldots, \mathcal{K}_n$ be pairwise distinct composants of X. Since $\mathcal{V}$ is open, there exist $x_i \in \mathcal{K}_i$ such that $\mathcal{A} = q_X^{(n,m)}(\{x_1, \ldots, x_n\}) \in \mathcal{V}$. Therefore, there exists an arc α with end points F_X^m and $\mathcal{A}$. By Theorem 1.77 and the fact that $q_X^{(n,m)}|_{C_n(X)-(F_m(X)\cup\{X\})}$ is a homeomorphism, it follows that $q_X^{(n,m)}(\langle \mathcal{K}_1, \ldots, \mathcal{K}_n \rangle)$ is an arc component of $HS_m^n(X) - \{q_X^{(n,m)}(X), F_X^m\}$ and $\mathcal{A} \in q_X^{(n,m)}(\langle \mathcal{K}_1, \ldots, \mathcal{K}_n \rangle)$. Thus $\alpha \subset q_X^{(n,m)}(\langle \mathcal{K}_1, \ldots, \mathcal{K}_n \rangle)$, which is a contradiction since $\langle \mathcal{K}_1, \ldots, \mathcal{K}_n \rangle \cap F_m(X) = \emptyset$. □

Theorem 2.49. *If X is an indecomposable continuum having the property of Kelley and $n, m \in \mathbb{N}$ with $m \leq n$, then* *(a)* $q_X^{(n,m)}(X)$ *and* F_X^m *are the only points at which* $HS_m^n(X)$ *is locally arcwise connected if* $n = m$.

(b) $q_X^{(n,m)}(X)$ *is the only point at which* $HS_m^n(X)$ *is locally arcwise connected if* $m < n$.

Proof. First we prove that $HS_m^n(X)$ is not locally arcwise connected at any point distinct from $q_X^{(n,m)}(X)$ and F_X^m. If $\mathcal{A} \in HS_m^n(X) - \{q_X^{(n,m)}(X), F_X^m\}$, then $(q_X^{(n,m)})^{-1}(\mathcal{A}) \in C_n(X) - (F_m(X) \cup \{X\})$. Since X is the only point at which $C_n(X)$ is locally connected by Theorem 1.73, $C_n(X)$ is not locally connected at $(q_X^{(n,m)})^{-1}(\mathcal{A})$. Hence, $C_n(X)$ is not locally arcwise connected at $(q_X^{(n,m)})^{-1}(\mathcal{A})$. Therefore, $HS_m^n(X)$ is not locally arcwise connected at $\mathcal{A}$.

(a) If $n = m$, using Theorem 2.43, $q_X^{(n,m)}(X)$ and F_X^m are the only points at which $HS_m^n(X)$ is locally arcwise connected.

(b) If $m < n$, by Theorem 2.48 and Theorem 2.45, $q_X^{(n,m)}(X)$ is the only point at which $HS_m^n(X)$ is locally arcwise connected. □

Theorem 2.50. *Let* X *be a continuum,* $n, m \in \mathbb{N}$ *such that* $n \geq m$ *and* $A \in C_n(X)$. *Then,* $C_n(X) - \{A\}$ *is not arcwise connected if and only if* $HS_m^n(X) - \{q_X^{(n,m)}(A), F_X^m\}$ *is not arcwise connected.*

Proof. Suppose that $C_n(X) - \{A\}$ is not arcwise connected. We will consider two cases.

Case 1. $X \neq A$.

Let $\mathcal{C}_1, \mathcal{C}_2$ be two different arcwise components of $C_n(X) - \{A\}$ and let $B \in C_n(X) - A$ such that $B \in \mathcal{C}_1$ and $X \in \mathcal{C}_2$. Thus, every arc joining B with X must pass through A. Assume $HS_m^n(X) - \{q_X^{(n,m)}(A), F_X^m\}$ is arcwise conneceted. Then, there exists an arc $\alpha : [0, 1] \to HS_m^n(X) - \{q_X^{(n,m)}(A), F_X^m\}$ such that $\alpha(0) = q_X^{(n,m)}(B)$ and $\alpha(1) = q_X^{(n,m)}(X)$. Since $q_X^{(n,m)}$ is a homeomorphism between $C_n(X) - F_m(X)$ and $HS_m^n(X) - \{F_X^m\}$, it follows that $((q_X^{(n,m)})^{-1} \circ \alpha)([0, 1])$ is an arc in $C_n(X) - (\{A\} \cup F_m(X))$ joining B and X, which is a contradiction. Therefore, $HS_m^n(X) - \{q_X^{(n,m)}(A), F_X^m\}$ is not arcwise connected.

Case 2. $X = A$.

By Theorem 1.75, X is indecomposable. Let $\mathcal{K}_1, \ldots, \mathcal{K}_n, \mathcal{T}_1, \ldots, \mathcal{T}_n$ pairwise different composants of X. By Theorem 1.77, $\langle \mathcal{K}_1, \ldots, \mathcal{K}_n \rangle_n$ and $\langle \mathcal{T}_1, \ldots, \mathcal{T}_n \rangle_n$ are arcwise components of $C_n(X) - \{X\}$. Notice that $\langle \mathcal{K}_1, \ldots, \mathcal{K}_n \rangle_n$ and $\langle \mathcal{T}_1, \ldots, \mathcal{T}_n \rangle_n$ are subsets of $C_n(X) - (\{X\} \cup F_m(X))$.

Thus, $C_n(X) - (\{X\} \cup F_m(X)$ is not arcwise connected. Hence, $HS_m^n(X) - \{q_X^{(n,m)}(X), F_X^m\}$ is not arcwise connected.

Now suppose that $HS_m^n(X) - \{q_X^{(n,m)}(A), F_X^m\}$ is not arcwise connected. Allow us to consider two cases.

Case 1. $X \neq A$.

Suppose that $C_n(X) - \{A\}$ is arcwise connected. Notice that $C_n(X) - (\{A\} \cup F_m(X))$ is not arcwise connected. Let $\mathcal{C}_1, \mathcal{C}_2$ be two arcwise components of $C_n(X) - (\{A\} \cup F_m(X))$. Let $B \in C_n(X) - F_m(X)$ be such that $B \in \mathcal{C}_1$ and $X \in \mathcal{C}_2$. Thus, every arc joining B with X intersects $F_m(X)$. Since $C_n(X) - F_m(X)$ is arcwise connected, there exists an arc $\alpha \subset C_n(X) - F_m(X)$ joining B with X such that $\alpha \cap F_m(X) = \emptyset$. This is a contradiction, which implies that $C_n(X) - \{A\}$ is not arcwise connected.

Case 2. $X = A$.

Notice that $C_n(X) - (\{X\} \cup F_m(X))$ is not arcwise connected. By Theorem 2.46, X is an indecomposable continuum. By Theorem 1.75, $C_n(X) - \{X\}$ is not arcwise connected, as desired. □

As a consequence of Theorem 2.50, the next result follows.

Corollary 2.51. *If X is a continuum, $n, m \in \mathbb{N}$ with $m \leq n$ and $\mathcal{A} \in HS_m^n(X) - \{F_X^m\}$ is such that $HS_m^n(X) - \{F_X^m, \mathcal{A}\}$ is not arcwise connected, then $(q_X^{(n,m)})^{-1}(\mathcal{A}) \in C(X)$.*

To conclude this section, allow us to state next result which extends [52, Theorem 7.9.6].

Theorem 2.52. *Let X be a hereditarily indecomposable continuum, and let $n, m, s, t \in \mathbb{N}$ with $s \geq t$ and $m \leq n$. If Y is a continuum such that there exists a homeomorphism $h : HS_m^n(X) \to HS_t^s(Y)$, with $h(F_X^m) = F_Y^t$ then Y is homeomorphic to X.*

Proof. By Theorem 2.45, $HS_t^s(Y)$ is locally arcwise connected at $q_Y^{(s,t)}(Y)$. By Theorem 1.38, X has property of Kelley. Using Theorem 2.49, $q_X^{(n,m)}(X)$ and F_X^m are the only points at which $HS_m^n(X)$ is locally arcwise connected or $q_X^{(n,m)}(X)$ is the only point at which $HS_m^n(X)$ is locally arcwise connected. In either case, since $h(F_X^m) = F_Y^t$, it follows that $h(q_X^{(n,m)}(X)) = q_Y^{(s,t)}(Y)$.

Let $g : C_n(X) - F_m(X) \to C_s(Y) - F_t(Y)$ be the homeomorphism defined by

$$g(A) = \left(q_Y^{(s,t)}\right)^{-1}\left(h\left(q_X^{(n,m)}(A)\right)\right).$$

Notice that $g(X) = Y$. Allow us to prove that $g(C(X) - F_1(X)) \subset C(Y) - F_1(Y)$.

Let $\mathcal{A} \in HS_m^n(X)$ be such that $(q_X^{(n,m)})^{-1}(\mathcal{A}) \in C(X) - (\{X\} \cup F_1(X))$. Then, by Theorem 1.76, $C_n(X) - \{(q_X^{(n,m)})^{-1}(\mathcal{A})\}$ is not arcwise connected. Hence, by Theorem 2.50, we obtain that $HS_m^n(X) - \{\mathcal{A}, F_X^m\}$ is not arcwise connected. Since h is a homeomorphism, $HS_t^s(Y) - \{h(\mathcal{A}), F_Y^t\}$ is not arcwise connected. By Theorem 2.51 and since $(q_X^{(n,m)})^{-1}(\mathcal{A}) \in C(X) - (\{X\} \cup F_1(X))$ and $h(F_X^m) = F_Y^t$, it follows that $(q_Y^{(s,t)})^{-1}(h(\mathcal{A})) \in C(Y) - F_1(Y)$.

Under analogous arguments as the final part of the proof [51, Theorem 7.1], the result follows. □

An immediate question regarding Theorem 2.52 is asked: can the property $h(F_X^m) = F_Y^t$ be omitted from the hypothesis?

2.6 Absolute (n,m)-fold hyperspace suspensions

Inside the (n,m)-hyperespace suspension of a continuum X we distinguish two important points: F_X^m and $q_X^{(n,m)}(X)$. For any pair of points p, q in X, we wish to find a continuum whose (n,m)-hyperespace suspension is determined by p and q.

Definition 2.53. *A continuum X is said to be an absolute (n,m)-fold hyperspace suspension provided that for each pair of different points p and q of X, there exists a continuum $Y(p,q)$ such that (X,p,q) is homeomorphic to $(HS_m^n(Y(p,q)), q_{Y(p,q)}^{(n,m)}(Y(p,q)), F_{Y(p,q)}^m)$.*

Theorem 2.54. *If a continuum X is an absolute (n,m)-fold hyperspace suspension, for some $n, m \in \mathbb{N}$ with $m \geq n$, then X is a unicoherent locally connected continuum.*

Proof. Let $p, q \in X$ be two arbitrary points. By Definition 2.53, there exists a continuum $Y(p,q)$ such that (X,p,q) is homeomorphic to $(HS_m^n(Y(p,q)), q_{Y(p,q)}^{(n,m)}(Y(p,q)), F_{Y(p,q)}^m)$. Let $h : X \to HS_m^n(Y(p,q))$ be a homeomorphism such that $h(p) = q_{Y(p,q)}^{(n,m)}(Y(p,q)$ and $h(q) = F_{Y(p,q)}^m$. By Theorem 2.45, $HS_m^n(Y(p,q))$ is locally arcwise connected at $q_{Y(p,q)}^{(n,m)}(Y(p,q))$. Hence, it is also locally connected at $q^{(n,m)}$. Since h is a homeomorphism, X is locally connected at p. As p is an arbitrary point in X, we have that X is locally connected. By Theorem 2.19, $HS_m^n(Y(p,q))$ has the property (b). Thus, $HS_m^n(Y(p,q))$ is unicoherent. Therefore, X is unicoherent. □

For the next result, recall that a space is *dimensionally homogeneous* if the dimension of X at any point remains the same.

Theorem 2.55. *If a continuum X is an absolute (n, m)-fold hyperspace suspension, then X is dimensionally homogeneous.*

Proof. By Theorem 2.54, X is locally connected. Let p, q be two arbitrary points of X. Since X is an absolute (n, m)-fold hyperspace suspension, there exists a continuum $Y(p,q)$ and a homeomorphism $h : X \to HS_m^n(Y(p,q))$ such that $h(p) = q_{Y(p,q)}^{(n,m)}(Y(p,q))$ and $h(q) = F_{Y(p,q)}^m$. Thus, $HS_m^n(Y(p,q))$ is a locally connected continuum. By Theorem 2.34, $Y(p,q)$ is locally connected. We will consider two cases.

Case 1. $\dim[X]$ is finite.

By Theorem 2.35, $Y(p,q)$ is a graph. Since h is a homeomorphism and by Lemma 2.28, $\dim_q[X] \leq \dim_p[X]$. Since p and q are arbitrary points, we can exchange their order in the homeomorphism h, so that $h(q) = q_{Y(p,q)}^{(n,m)}(Y(p,q))$ and $h(p) = F_{Y(p,q)}^m$. Again, by Lemma 2.28, $\dim_q[X] \leq \dim_p[X]$. Therefore, X is dimensionally homogeneous.

Case 2. $\dim[X]$ is not finite.

Then, $\dim[HS_m^n(Y(p,q))]$ is not finite. Since Y is locally connected and $HS_m^n(Y(p,q))$ is not finite-dimensional, by Theorem 2.35, Y is not a graph. By Theorem 2.16, $\dim[C_n(Y(p,q))]$ is not finite. By [62, Theorem 2.9], it follows that $\dim_{Y(p,q)}[C_n(Y(p,q)]$ is not finite. Since $q_{Y(p,q)}^{(n,m)}|_{C_n(Y(p,q))-F_m(Y(p,q))}$ is a homeomorphism, we have that $\dim_{q_{Y(p,q)}^{(n,m)}(Y(p,q))}[HS_m^n(Y(p,q))]$ is not finite. Since h is a homeomorphism, $\dim_p[X]$ is not finite. Moreover, p is an arbitrary point of X so that X is dimensionally homogeneous. □

The last result of this section relates to a $\mathcal{Q}$-manifold, also known as *Hilbert manifold*. Recall that a space Z is a $\mathcal{Q}$-manifold if every point $z \in Z$ has a neighborhood homeomorphic to the Hilbert cube $\mathcal{Q}$.

Theorem 2.56. *Let X be an infinite-dimensional continuum and $n, n \in \mathbb{N}$ with $m \leq n$. If X is an absolute (n, m)-fold hyperspace suspension, then X is an unicoherent $\mathcal{Q}$-manifold.*

Proof. By Theorem 2.54, X is a unicoherent locally connected continuum. Let $p, q \in X$. Since X is an absolute (n,m)-fold hyperspace suspension, there exists a continuum $Y(p,q)$ such that (X,p,q) is homeomorphic to $(HS_m^n(Y(p,q)), F_{Y(p,q)}^m, q_{Y(p,q)}^{(n,m)}(Y(p,q)))$. Notice that $HS_m^n(Y(p,q))$ is locally connected. By Theorem 2.34, $Y(p,q)$ is locally connected. Consequently, $C_n(Y(p,q))$ is locally connected, by Theorem 2.29. Suppose that there exists a free arc α in $Y(p,q)$. By Theorem 1.69, $C_n(\alpha)$ is a $2n$-dimensional subspace of $C_n(Y(p,q))$ and there exists a point $A \in C_n(\alpha) - F_m(\alpha)$ such that $\dim_A[C_n(Y(p,q))] = 2n$. Thus, $\dim_{q_{Y(p,q)}^{(n,m)}}[HS_m^n(Y(p,q)] = 2n$. On the other hand, $\dim[X]$ is not finite and X is dimensionally homogeneous, by Theorem 2.55. Hence, $\dim[HS_m^n(Y(p,q))]$ is not finite and $HS_m^n(Y(p,q))$ is dimensionally homogeneous, a contradiction. Thus, $Y(p,q)$ does not have free arcs. By Theorem 1.74, $C_n(Y(p,q))$ is a Hilbert cube. This implies that $Y(p,q)$ has a Hilbert cube neighborhood $\mathcal{W}$ in $C_n(Y(p,q))$ such that $\mathcal{W} \cap F_m(Y(p,q)) = \emptyset$. Therefore, $q_{Y(p,q)}^{(n,m)}(Y(p,q))$ has a Hilbert cube neighborhood in $HS_m^n(p,q)$. Since (X,p,q) is homeomorphic to $(HS_m^n(Y(p,q)), F_{Y(p,q)}^m, q_{Y(p,q)}^{(n,m)}(Y(p,q)))$, p has a Hilbert cube neighborhood in X. Thus, X is a $\mathcal{Q}$-manifold. □

Chapter 3

Uniqueness of the (n, m)-hyperspace suspension

In this chapter, we present new results about uniqueness of hyperspaces for $HS_m^n(X)$, see Theorems 3.22, 3.25-3.31. In fact, these results were published in [23]. To start this chapter, recall the following definition.

Definition 3.1. *Let X be a continuum, $n, m \in \mathbb{N}$, with $n \geq m$, Λ be a class of continua, and $\mathcal{H}(X) \in \{2^X, C_n(X),\ F_n(X), HS_m^n(X)\}$. The class of continua Λ is said to be $\mathcal{H}$-*determined *if X is homeomorphic to Y when $\mathcal{H}(X)$ is homeomorphic to $\mathcal{H}(Y)$, for any two continua X and Y in Λ (as in [12, p. 247]).*

A well-known result about this concept for the class of finite graphs is written.

Theorem 3.2. *[6, Theorem 3.3] The class of finite graphs is HS_m^n-determined, for each $n, m \in \mathbb{N}$ with $m \geq n$.*

3.1 Classes of continua with unique hyperspace

Definition 3.3. *Let X be a continuum and $n, m \in \mathbb{N}$ with $n \geq m$. The continuum X has* **unique hyperspace** *$K(X) \in \{2^X, C_n(X), F_n(X), HS_m^n(X)\}$, whenever the following implication holds: if Y is a continuum and $\mathcal{H}(X)$ is homeomorphic to $\mathcal{H}(Y)$, then X is homeomorphic to Y.*

Our goal is to find collections of continua having this property for the hyperspace $HS_m^n(X)$.

Theorem 3.4. *[6, Theorem 3.6] If X if a finite graph, then X has unique hyperspace $HS_m^n(X)$ for each $n, m \in \mathbb{N}$ with $n \leq m$.*

We will consider a more complex collection of continua, using the next definitions.

Definition 3.5. *Given a continuum X, let*

$$\mathcal{G}(X) = \{x \in X \colon x \text{ has a neighborhood in } X \text{ which is a finite graph}\},$$

and let

$$\mathcal{P}(X) = X - \mathcal{G}(X).$$

A continuum X is said to be **almost meshed** *whenever the set $\mathcal{G}(X)$ is a dense subset of X. An almost meshed continuum X is* ***meshed*** *when X has a basis of neighborhoods $\mathfrak{B}$ such that $U - \mathcal{P}(X)$ is connected, for each element $U \in \mathfrak{B}$.*

Example 3.6. *(a) The universal dendrite is an almost meshed continuum.*

(b) Let $X = ([-1,1] \times \{0\}) \cup \left(\bigcup_{n \in \mathbb{N}} \left(\{\frac{1}{n}\} \times \left[0, \frac{1}{n}\right]\right)\right)$. Under the usual topology in $\mathbb{R}^2$, X is a continuum. Notice that $\mathcal{P}(X) = \{(0,0)\}$. Thus, $\mathcal{G}(X)$ is a dense subset of X and therefore, X is almost meshed. Moreover, if U is any open neighborhood of X containing $(0,0)$, it follows that $U - \mathcal{P}(X)$ is not connected. Hence, X is not meshed.

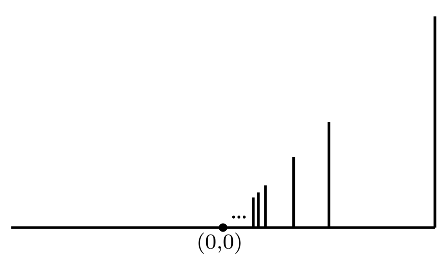


Figure 3.1: Almost meshed but not meshed

(c) *For each* $n \in \mathbb{N}$*, define* $A_n = \{(x, 2^{-n+1}) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$ *and* $A_0 = \{(x, 0) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$*. Now, for each* $n \in \mathbb{N} \cup \{0\}$ *and* m *integer such that* $0 \leq m \leq 2^{n+1}$*, let* $B_{n,m} = \{(m2^{-n+1}, y) \in \mathbb{R}^2 : 0 \leq y \leq 2^{-n}\}$*. Finally, we consider*

$$X = \left(\bigcup_{n\in\mathbb{N}} A_n\right) \cup \left(\bigcup_{n\in\mathbb{N}} \left(\bigcup_{m=0}^{2^{n+1}} B_{n,m}\right)\right).$$

In this way, X *is a continuum. Observe that* $\mathcal{P}(X) = A_0$*. It is clear that* X *is almost meshed. On the other hand, if* $x \in X$ *and* U *is an open neighborhood of* x *in* X*, it is easy to see that* $U - \mathcal{P}(X)$ *is connected. Thus,* X *is meshed.*

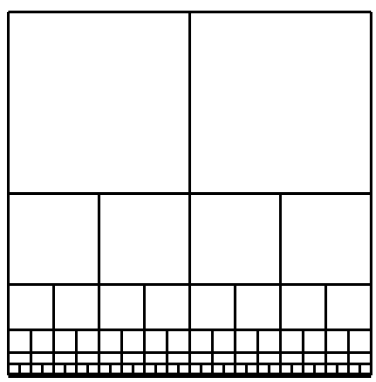


Figure 3.2: Meshed continuum

An immediate result from Definition 3.5 is given.

Theorem 3.7. *[20, Lemma 2] If* X *is a meshed continuum, then* X *is a locally connected continuum.*

Our attention will focus on these collections of continua. Given a continuum X and $n, m \in \mathbb{N}$ with $m \leq n$, let

$$\begin{aligned}
\mathcal{L}_n(X) &= \{A \in C_n(X) : A \text{ has a neighborhood in } C_n(X) \text{ which is a } 2n\text{-cell}\},\\
\mathcal{D}_n(X) &= \{A \in C_n(X) : A \notin \mathcal{L}_n(X) \text{ and } A \text{ has a basis of neighborhoods}\\
&\qquad \mathfrak{B} \text{ in } C_n(X) \text{ such that for each } \mathcal{U} \in \mathfrak{B}, \dim[\mathcal{U}] = 2n\\
&\qquad \text{and } \mathcal{U} \cap \mathcal{L}_n(X) \text{ is arcwise connected}\},
\end{aligned}$$

$$\mathcal{HL}^n_m(X)=\{A\in HS^n_m(X)\colon A \text{ has a neighborhood in } HS^n_m(X) \text{ which is a } 2n\text{-cell}\},$$
$$\mathcal{HD}^n_m(X)=\{A\in HS^n_m(X)\colon A\notin \mathcal{HL}^n_m(X) \text{ and } A \text{ has a basis of neighborhoods } \mathfrak{B} \text{ in } HS^n_m(X) \text{ such that for each } \mathcal{U}\in\mathfrak{B},\ \dim[\mathcal{U}]=2n \text{ and } \mathcal{U}\cap\mathcal{HL}^n_m(X) \text{ is arcwise connected}\},$$
$$\mathcal{HE}^n_m(X)=\{A\in HS^n_m(X) : \dim_A[HS^n_m(X)]=2n\}.$$

Let $J\in X$. Consider the following sets:

$$\mathfrak{A}_R(X)=\{J\subset X\colon J \text{ is a cycle in } X\},$$
$$\mathfrak{A}_E(X)=\{J\subset X\colon J \text{ is a maximal free arc with an end point } p \text{ of } J \text{ such that } p\in J^\circ\},$$
$$\mathfrak{A}_S(X)=\{J\subset X\colon J \text{ is a maximal free arc in } X\}\cup\mathfrak{A}_R(X).$$

Notice that we have the following remark.

Remark 3.8. *Let X be a continuum and $n,m\in\mathbb{N}$ with $m\le n$. Then*

$$q_X^{(n,m)}(\mathcal{D}_n(X)-F_m(X))=\mathcal{HD}^n_m(X)-\{F^m_X\}.$$

Now we present some very useful results that will be used throughout this section.

Theorem 3.9. *[20, Theorem 4] Let X be a locally connected continuum, $n\in\mathbb{N}$ and $A\in C_n(X)$. Then the following are equivalent.*

- *$dim_A[C_n(X)]$ is finite,*
- *there exists a finite graph D contained in X such that $A\subset D_0$,*
- *$A\cap\mathcal{P}(X)=\emptyset$.*

Theorem 3.10. *[20, Lemma 8] For a locally connected continuum X, let $\{J_k\}_{k\in\mathbb{N}}$ be a sequence of pairwise different elements of $\mathfrak{A}_S(X)$ and $x_k\in J_k$, for each $k\in\mathbb{N}$. If $lim x_m=x$ for some $x\in X$, then $lim J_m=\{x\}$ (in $C(X)$).*

Theorem 3.11. *[20, Lemma 10] Let X be a locally connected continuum, and let J be a free arc. Then there exists a $K\in\mathfrak{A}_S(X)$ such that $J\subset K$.*

Theorem 3.12. *[20, Lemma 11] Let X be a locally connected continuum and $A\in C_n(X)$. Then $\dim_A[C_n(X)]\ge 2n$ and, if $dim_A[C_n(X)]=2n$, then there exist $k\in\mathbb{N}$ and elements $J_1,\dots,J_k\in\mathfrak{A}_S(X)$ such that $A\in\langle J_1^\circ,\dots,J_k^\circ\rangle_n$, where each component of A is contained in some J_i°.*

Theorem 3.13. *[20, Lemma 28] Let X be a locally connected continuum and $n \geq 3$. Then $\Gamma_n(X) = \{A \in C_n(X) \colon A$ is connected and there exists a $J \in \mathfrak{A}_S(X)$ such that $A \subset J^\circ\} = \mathfrak{B}(X)$.*

The following is a efficient characterization for points in $\mathcal{P}(X)$.

Theorem 3.14. *[30, Theorem 4.1] Let X be an almost meshed locally connected continuum. Then $x \in \mathcal{P}(X)$ if and only if there exists a sequence of pairwise different elements contained in $\mathfrak{A}_S(X)$ which converges to $\{x\}$.*

Next result extends [29, Lemma 2.4].

Lemma 3.15. *Let X be a locally connected continuum and $n, m \in \mathbb{N}$ with $m \leq n$. If $A \in C_n(X) - F_m(X)$ and $A \cap R(X) \neq \emptyset$, then*

$$\dim_{q_X^{(n,m)}(A)}[HS_m^n(X)] \geq 2n + 1$$

.

Proof. By Remark 2.5, it follows that $\dim_{q_X^{(n,m)}(A)}[HS_m^n(X)] = \dim_A[C_n(X)]$. If $\dim_A[C_n(X)]$ is not finite, the result follows. Suppose that $\dim_A[C_n(X)]$ is finite. Conditions on the hypothesis allow us to use Theorem 3.9 so that there exists a finite graph G such that $A \subset \text{int}_X(G)$. Observe that $\dim_A[C_n(X)] = \dim_A[C_n(G)]$ as A is contained in $\text{int}_X(G)$. Since $A \cap R(X) \neq \emptyset$, clearly $A \cap R(G) \neq \emptyset$. Thus, by Theorem 1.69, $\dim_A[C_n(G)] \geq 2n+1$. By Remark 2.5, the result follows from here. □

The following result extends [29, Lemma 2.8].

Lemma 3.16. *Let X be a locally connected continuum such that $R(X) \neq \emptyset$ and $n, m \in \mathbb{N}$ with $m \leq n$. Then $\dim[\mathcal{U}] \geq 2n+1$, for each neighborhood $\mathcal{U}$ of F_X^m in $HS_m^n(X)$.*

Proof. Let $\mathcal{U}$ be an open neighborhood of F_X^m in $HS_m^n(X)$ and $\mathcal{V} = (q_X^{(n,m)})^{-1}(\mathcal{U})$. Then, $\mathcal{V}$ is an open subset of $C_n(X)$. Let $p \in R(X)$. Clearly, $\{p\} \in \mathcal{V}$. Consequently, there exists $r > 0$ such that $B_{C_n(X)}(\{p\}, r) \subset \mathcal{V}$. Let C be the component of $B_X(p, r)$ containing p. Since C is an open connected subset of X, it follows that C is arcwise connected. Hence, there exists an arc A such that $p \in A \subset B_X(p, r)$. Notice that $A \in \mathcal{V}$. Thus, $q_X^{(n,m)}(A) \in \mathcal{U}$. Therefore, by Lemma 3.15, $\dim_{q_X^{(n,m)}(A)}[\mathcal{U}] \geq 2n+1$. Therefore, the result follows. □

Next two results extend [45, Lemma 3.6] and [45, Lemma 3.7], respectively.

Lemma 3.17. *Let X be a locally connected continuum such that $R(X) \neq \emptyset$, $n, m \in \mathbb{N}$ with $m \leq n$ and $n \geq 3$. Then $\mathcal{HD}_m^n(X) = \{q_X^{(n,m)}(A) \in HS_m^n(X) : A \in C(X) - F_1(X)$ and $A \cap [R(X) \cup \mathcal{P}(X)] = \emptyset\}$.*

Proof. Given $B \in \mathcal{HD}_m^n(X)$, there exists $A \in C_n(X)$ such that $B = q_X^{(n,m)}(A)$. Since $R(X) \neq \emptyset$, by Lemma 3.16 and the definition of $\mathcal{HD}_m^n(X)$, we conclude that $B \neq F_X^m$; thus, $A \notin F_1(X)$. Moreover, by Remark 3.8, $A \in \mathcal{D}_n(X)$. According to Theorem 3.13, $A \in C(X) - F_1(X)$ and $A \subset \mathrm{int}_X(J)$, for some $J \in \mathfrak{A}_S(X)$. This implies that A does not intersect the end points of J. Thus, $A \cap [R(X) \cup \mathcal{P}(X)] = \emptyset$.

On the other hand, in order to prove the opposite inclusion, let $A \in C(X) - F_1(X)$ be such that $A \cap [R(X) \cup \mathcal{P}(X)] = \emptyset$. As we want to prove that $q_X^{(n,m)}(A) \in \mathcal{HD}_m^n(X)$, by Remark 3.8, it will be enough to prove that $A \in \mathcal{D}_n(X)$. By Theorem 3.9, there exists a finite graph G contained in X such that $A \subset \mathrm{int}_X(G)$. Since $A \cap R(X) = \emptyset$, it follows that $A \cap R(G) = \emptyset$. Thus, there exists a free arc L in G such that $A \subset \mathrm{int}_G(L)$. Since $A \subset \mathrm{int}_X(G)$ and $A \subset \mathrm{int}_X(L)$, we may assume that $L \subset \mathrm{int}_X(G)$. This implies that L is a free arc in X using the relative topology. By Theorem 3.11, there exists $J \in \mathfrak{A}_S(X)$ such that $L \subset J$. Therefore, by Theorem 3.13, $A \in \mathcal{D}_n(X)$. $\square$

Before one of the main results of this section, allow us to present a useful lemma.

Lemma 3.18. *Let X be a locally connected continuum such that $R(X) \neq \emptyset$ and let $n, m \in \mathbb{N}$ with $m \leq n$.*

(a) If $n \geq 3$, then the components of $\mathcal{HD}_m^n(X)$ are the sets of the form $q_X^{(n,m)}((\langle \mathrm{int}_X(J) \rangle \cap C(X)) - \{F_X^m\}$, where $J \in \mathcal{A}_S(X)$.

(b) The components of $\mathcal{HE}_m^n(X)$ are the sets of the form

$$q_X^{(n,m)}(\langle J_1^\circ, \ldots, J_k^\circ \rangle_n) - \{F_X^m\},$$

where $J_1, \ldots, J_k \in \mathcal{A}_S(X)$ and $n \geq k$.

Proof. (a) Notice that for each $A \in C(X) - F_1(X)$ such that $A \cap [R(X) \cap \mathcal{P}(X)] = \emptyset$, there exists $J \in \mathfrak{A}_S(X)$ such that $A \subset \mathrm{int}_X(J)$ (Theorem 3.13).

By Lemma 3.17, it follows that $\mathcal{HD}_m^n(X) = \bigcup\{q_X^{(n,m)}(\langle \mathrm{int}_X(J)\rangle \cap C(X)) - \{F_X^m\} : J \in \mathcal{A}_S(X)\}$. By order arcs in $C(X)$, the sets $q_X^{(n,m)}(\langle \mathrm{int}_X(J)\rangle \cap C(X)) - \{F_X^m\}$ are arcwise connected and, therefore, connected. Moreover, the sets $q_X^{(n,m)}(\langle \mathrm{int}_X(J)\rangle \cap C(X)) - \{F_X^m\}$ are open in $\mathcal{HD}_m^n(X)$ and pairwise disjoint. We conclude that they are the components of $\mathcal{HD}_m^n(X)$.

(b) By Lemma 3.16, $F_X^m \notin \mathcal{HE}_m^n(X)$. Given $B \in \mathcal{HE}_m^n(X)$, there exists $A \in C_n(X) - F_m(X)$ such that $B = q_X^{(n,m)}(A)$. Notice that $\dim_A[C_n(X)] = \dim_B[HS_m^n(X)] = 2n$. According to Theorem 3.12, there exist $J_1, \ldots, J_k \in \mathcal{A}_S(X)$, with $m \leq n$, such that $A \in \langle J_1^\circ, \ldots, J_k^\circ\rangle_n - F_m(X)$. This implies that $\mathcal{HE}_m^n(X) \subset \bigcup\{q_X^{(n,m)}(\langle J_1^\circ, \ldots, J_k^\circ\rangle_n) - \{F_X^m\} : J_1, \ldots, J_k \in \mathcal{A}_S(X)\}$. In order to prove the other inclusion, let $A \in \langle J_1^\circ, \ldots, J_k^\circ\rangle_n - F_m(X)$. Thus, $A \cap [R(X) \cup \mathcal{P}(X)] = \emptyset$, as $A \subset \bigcup J_i^\circ$. By Theorem 3.9, there exists a finite graph G contained in X such that $A \subset \mathrm{int}_X(G)$. Since $A \cap R(X) = \emptyset$, we have that $A \cap R(G) = \emptyset$. Hence, by Theorem 1.69, $\dim_A[C_n(G)] = 2n$. Since $\dim_{q_X^{(n,m)}(A)}[HS_m^n(X)] = \dim_A[C_n(X)] = \dim_A[C_n(G)]$, $q_X^{(n,m)}(A) \in \mathcal{HE}_m^n(X)$. Therefore, $\mathcal{HE}_m^n(X) = \bigcup\{q_X^{(n,m)}(\langle J_1^\circ, \ldots, J_k^\circ\rangle_n) - \{F_X^m\} : J_1, \ldots, J_k \in \mathcal{A}_S(X)\}$. Notice that the sets $q_X^{(n,m)}(\langle J_1^\circ, \ldots, J_k^\circ\rangle_n) - \{F_X^m\}$ are arcwise connected and, therefore, connected. Moreover, the sets $q_X^{(n,m)}(\langle J_1^\circ, \ldots, J_k^\circ\rangle_n) - \{F_X^m\}$ are open in $\mathcal{HE}_m^n(X)$ and pairwise disjoint. In conclusion, they are the components of $\mathcal{HE}_m^n(X)$. □

Given a continuum X, a nonempty closed subset Z of X, and $n, m \in \mathbb{N}$ with $m < n$, let

$$C_n(X, Z) = \{A \in C_n(X)\colon A \cap Z \neq \emptyset\} \text{ and}$$

$$C(X, Z) = C_1(X, Z).$$

If $J \in \mathfrak{A}_S(X)$, let

$$\mathcal{K}_m^n(J, X) = \mathrm{cl}_{C_n(X)}(\langle J^\circ\rangle_n) - F_m(X).$$

If $x \in J \cap ((R(X) \cap \mathcal{G}(X)) \cup \mathcal{P}(X))$, let

$$\mathcal{K}_{x,J} = \{A \in \mathcal{K}_m^n(J, X)\colon A \cap (R(X) \cup \mathcal{P}(X)) = \{x\}\}.$$

Consider the following Remark that closes the gap between the concepts of meshed (almost meshed) and framed (almost framed) continua.

Remark 3.19. *Let X be a locally connected continuum. Then X is almost framed if and only if X is almost meshed. Moreover, X is framed if and only if X is meshed distinct to a simple closed curve.*

Theorem 3.20. *Let X and Y be almost meshed locally connected continua, $n,m \in \mathbb{N} - \{1,2\}$ with $m \leq n$ and let $h\colon HS_m^n(X) \to HS_m^n(Y)$ be a homeomorphism. Suppose that for each $J \in \mathfrak{A}_S(X)$ there exists $J_h \in \mathfrak{A}_S(Y)$ such that $h(q_X^{(n,m)}(\langle J^\circ \rangle_n \cap C(X)) - \{F_X^m\}) \subset q_Y^{(n,m)}(\langle (J_h)^\circ \rangle_n)$ and $\mathfrak{A}_S(Y) = \{J_h\colon J \in \mathfrak{A}_S(X)\}$. Then*

(A) $h(q_X^{(n,m)}(\langle J^\circ \rangle_n) - \{F_X^m\}) = q_Y^{(n,m)}(\langle (J_h)^\circ \rangle_n) - \{F_Y^m\}$, for each $J \in \mathfrak{A}_S(X)$.

(B) $h^{-1}(q_Y^{(n,m)}(\langle (J_h)^\circ \rangle_n \cap C(Y)) - \{F_Y^m\}) \subset q_X^{(n,m)}(\langle J^\circ \rangle_n) - \{F_X^m\}$, for each $J \in \mathfrak{A}_S(X)$.

(C) The association $J \to J_h$ is a bijection between $\mathfrak{A}_S(X)$ and $\mathfrak{A}_S(Y)$.

(D) $h(F_X^m) = F_Y^m$.

If we also suppose that

(1) $J \in \mathfrak{A}_R(X)$ if and only if $J_h \in \mathfrak{A}_R(Y)$ and

(2) $J \in \mathfrak{A}_E(X)$ if and only if $J_h \in \mathfrak{A}_E(Y)$, then X is homeomorphic to Y.

Proof. We may assume that X and Y are not finite graphs, see Theorem 3.4. To prove (A), let $A \in \mathfrak{A}_S(X)$ and take two different subarcs A, B of J°. Hence, $h(q_X^{(n,m)}(A)), h(q_X^{(n,m)}(A)) \in h(q_X^{(n,m)}(\langle J^\circ \rangle_n) - \{F_X^m\})$. Suppose that $h(q_X^{(n,m)}(A)) \neq F_X^m$. By Theorem 3.18 (b), we have that $h(q_X^{(n,m)}(\langle J^\circ \rangle_n) - \{F_X^m\})$ is a component of $\mathcal{HE}_m^n(Y)$. By hypothesis, $h(q_X^{(n,m)}(A)) \in q_Y^{(n,m)}(\langle J_h^\circ \rangle_n) - \{F_Y^{(n,m)}\}$. Therefore, $h(q_X^{(n,m)}(A)) \in (q_Y^{(n,m)}(\langle J_h^\circ \rangle_n) - \{F_Y^{(n,m)}\}) \cap h(q_X^{(n,m)}(\langle J^\circ \rangle_n) - \{F_X^m\})$. Since this two sets are components of $\mathcal{HE}_m^n(Y)$, we can conclude that $h(q_X^{(n,m)}(A)) \in (q_Y^{(n,m)}(\langle J_h^\circ \rangle_n) - \{F_Y^{(n,m)}\}) = h(q_X^{(n,m)}(\langle J^\circ \rangle_n) - \{F_X^m\})$. This proves (A). Observe that (B) follows from (A).

Allow us to prove (C). Let $J, L \in \mathcal{A}_S(X)$ be such that $J_h = L_h$. By (A), it follows that $h(q_X^{(n,m)}(\langle J^\circ \rangle_n) - \{F_X^m\}) = h(q_X^{(n,m)}(\langle L^\circ \rangle_n) - \{F_X^m\})$. Hence, $q_X^{(n,m)}(\langle J^\circ \rangle_n) - \{F_X^m\} = q_X^{(n,m)}(\langle L^\circ \rangle_n) - \{F_X^m\}$. We will prove that $J = L$. To do so, let A be an arc such that $A \subset J^\circ$, then $q_X^{(n,m)}(A) \in q_X^{(n,m)}(\langle L^\circ \rangle_n) - \{F_X^m\}$, so there exists $B \in \langle L^\circ \rangle_n - \{F_X^m\}$ such that $q_X^{(n,m)}(A) = q_X^{(n,m)}(B)$. Consequently, $A = B \in \langle J^\circ \rangle_n \cap \langle L^\circ \rangle_n$ and $\emptyset \neq A \subset J^\circ \cap L^\circ$. Then, $J = L$.

Finally, we prove (D). In order to do so, allow us to prove

$$\{F_X^m\} = \bigcap \{\mathrm{cl}_{HS_m^n(X)}(q_X^{(n,m)}(\langle J^\circ \rangle_n) - \{F_X^m\}\colon J \in \mathfrak{A}_S(X)\} \tag{3.1}$$

Let $J \in \mathfrak{A}_S(X)$ and $p \in J^\circ$. Notice that we can approximate to p using elements in $\langle J^\circ\rangle_n - F_m(X)$. Thus, $\{p\} \in \mathrm{cl}_{C_n(X)}(J^\circ\rangle_n - F_m(X))$. Applying the quotient function, we have that $F_X^m \in \mathrm{cl}_{HS_m^n(X)}(q_X^{(n,m)}(\langle J^\circ\rangle_n) - \{F_X^m\})$. On the other hand, let $B \in \bigcap\{\mathrm{cl}_{HS_m^n(X)}(q_X^{(n,m)}(\langle J^\circ\rangle_n) - \{F_X^m\} \colon J \in \mathfrak{A}_S(X)\}$. Fix $J_0 \in \mathfrak{A}_S(X)$. Hence, it is possible to find a sequence $\{A_k\}_{k\in\mathbb{N}}$ contained in $\langle J_0^\circ\rangle_n$ such that $\{q_X^{(n,m)}(A_k)\}_{k\in\mathbb{N}}$ converges to B. Since 2^X is a continuum, we may assume that $\{A_k\}_{k\in\mathbb{N}}$ converges to some $A \in \langle J_0^\circ\rangle_n$. As $B \neq F_X^m$ and $q_X^{(n,m)}(A) = B$, we have that $A \in C_n(J_0) - F_m(J_0)$. Thus, A has a nondegenerate component and $A \subset J_0$. It follows that $A \cap J_0 \neq \emptyset$. Since $R(X) \neq \emptyset$, there exists $J \in \mathfrak{A}_S(X)$ such that $J \neq J_0$. Under analagous arguments, there exists $D \in C_n(J) - F_m(J)$ such that $q_X^{(n,m)}(D) = B$. Since $B \neq F_X^m$ and by Remark 2.5, we have that $A = D$. Thus, $J_0^\circ \cap J \neq \emptyset$, which is a contradiction and 3.1 is now proved. Note that

$$\begin{aligned} h(\{F_X^m\}) &= \bigcap\{\mathrm{cl}_{HS_m^n(Y)}(h(q_Y^{(n,m)}(\langle J^\circ\rangle_n) - \{F_X^m\}) \colon J \in \mathfrak{A}_S(X)\} \\ &= \bigcap\{\mathrm{cl}_{HS_m^n(Y)}(h(q_Y^{(n,m)}(\langle J_h^\circ\rangle_n) - \{F_Y^m\}) \colon J_h \in \mathfrak{A}_S(Y)\} \\ &= \{F_Y^m\}. \end{aligned}$$

This proves (D). Let $A \in C_n(X) - F_m(X)$. Then, $h(q_X^{(n,m)}(A)) \neq F_Y^m$ and there exists a unique $D_A \in C_n(Y) - F_m(Y)$ such that $h(q_X^{(n,m)}(A)) = q_Y^{(n,m)}(D_A)$. Let

$$g\colon C_n(X) - F_m(X) \to C_n(Y) - F_m(Y) \text{ be given by } g = (q_Y^{(n,m)})^{-1} \circ h \circ q_X^{(n,m)}.$$

In this way, observe that g is a homeomorphism by how it is defined and the fact that h is a homeomorphism. Notice that $g(A) = D_A$ and

$$\begin{aligned} \dim_A[C_n(X)] &= \dim_{q_X^{(n,m)}(A)}[HS_m^n(X)] \\ &= \dim_{h(q_X^{(n,m)}(A))}[HS_m^n(Y)] \\ &= \dim_{g(A)}[C_n(Y)]. \end{aligned}$$

In order to prove that X and Y are homeomorphic, we show several Claims that will be helpful to reach this goal.

Claim 1. Let $J \in \mathfrak{A}_S(X)$. The following statements are satisfied.

(a) $\mathcal{K}_m^n(J_h, Y) = \{g(A) \in C_n(Y) - F_m(Y) \colon A \in \mathcal{K}_m^n(J, X)\}$,

(b) $\{\dim_A[C_n(X)]\colon A \in \mathcal{K}^n_m(J,X)\} = \{\dim_B[C_n(Y)]\colon B \in \mathcal{K}^n_m(J_h,Y)\}$,
(c) $|J \cap \mathcal{P}(X)| = |J_h \cap \mathcal{P}(Y)|$, and
(d) $|J \cap R(X) \cap \mathcal{G}(X)| = |J_h \cap R(Y) \cap \mathcal{G}(Y)|$.

Proof of Claim 1. (a) Let $A \in \mathcal{K}^n_m(J,X)$. Thus, there exists a sequence $\{A_k\}_{k\in\mathbb{N}}$ contained in $\langle J^\circ\rangle_n - F_m(X)$ converging to A. Since h is a homeomorphism and by Remark 2.5, we have that $h(q^{(n,m)}_X(\langle J^\circ\rangle_n) - F_m(X)) = q^{(n,m)}_Y(\langle J^\circ_h\rangle_n) - \{F^m_Y\}$. Consequently, for each $k \in \mathbb{N}$, we have that $D_{A_k} \in \langle J^\circ_h\rangle_n - F_m(Y)$. Thus, $D_A \in \mathcal{K}^n_m(J_h,Y)$. This proves the first inclusion. The opposite inclusion can be proved analogously.

(b) follows from assertion (a) since topological dimension is preserved under homeomorphisms. Now we prove assertions (c) and (d). First, notice that

$$|J \cap \mathcal{P}(X)| = 0 \textit{ if and only if } |J_h \cap \mathcal{P}(Y)| = 0. \tag{3.2}$$

To prove this, let $J \in \mathfrak{A}_S(X)$ be such that $|J \cap \mathcal{P}(X)| = 0$. Suppose that $|J_h \cap \mathcal{P}(Y)| \leq 1$. Let $q \in J_h \cap \mathcal{P}(Y)$ and $A \in \mathcal{K}_{q,J_h}$. Notice that $\dim_A[C_n(Y) - F_m(Y)]$ is not finite as A contains points from $\mathcal{P}(X)$. Hence, $\dim_{g^{-1}(A)}[C_n(X) - F_m(X)]$ is not finite, which contradicts $J \cap \mathcal{P}(X) = \emptyset$. The converse follows analogously. Now we consider some cases in order to complete the proof of (c) and (d).

Case 1. $J \in \mathfrak{A}_E(X) \cup \mathfrak{A}_R(X)$.

By (1) and (2), $J_h \in \mathfrak{A}_E(Y) \cup \mathfrak{A}_R(Y)$. If $|J \cap \mathcal{P}(X)| = 0$, then in either case $J \in \mathfrak{A}_R(X)$ or $J \in \mathfrak{A}_E(X)$ it follows that $|J \cap R(X) \cap \mathcal{G}(X)| = 1$. By (3.2), we have that $|J_h \cap \mathcal{P}(Y)| = 0$, which implies that $|J_h \cap R(Y) \cap \mathcal{G}(Y)| = 1$. Now, if $|J \cap \mathcal{P}(X)| = 1$, then $|J \cap R(X) \cap \mathcal{G}(X)| = 0$. By (3.2), we have that $|J_h \cap \mathcal{P}(Y)| = 1$, which implies that $|J_h \cap R(Y) \cap \mathcal{G}(Y)| = 0$. This proves (c) and (d) for this case.

Case 2. J is an arc and $J \notin \mathfrak{A}_E(X)$.

By (1) and (2), J_h is an arc and $J_h \notin \mathfrak{A}_E(Y)$. If $|J \cap \mathcal{P}(X)| = 0$, then we have that $|J \cap R(X) \cap \mathcal{G}(X)| = 2$, as J is an arc which is not extreme. By (3.2), it follows that $|J_h \cap \mathcal{P}(Y)| = 0$, which implies that $|J_h \cap R(Y) \cap \mathcal{G}(Y)| = 2$.

Case 2.1 $|J \cap \mathcal{P}(X)| = 1$.

Then, since $J \notin \mathcal{A}_E(X)$, it follows that $|J \cap R(X) \cap \mathcal{G}(X)| = 1$. By (3.2), we have that $|J_h \cap \mathcal{P}(Y)| \geq 1$. Suppose that $|J_h \cap \mathcal{P}(Y)| = 2$. Thus, $E(J_h) \subset \mathcal{P}(Y)$. Let $A \in \mathcal{K}^n_m(J,X)$ be such that $A \cap R(X) \neq \emptyset$ and $A \cap \mathcal{P}(X) = \emptyset$. Hence, using Lemma 3.15, $\dim_A[(C_n(X) - F_m(X)]$ is finite and greater than $2n$; as g is a homeomorphism, it follows that $\dim_{g(A)}[C_n(Y) - F_m(Y)]$ is finite and greater than $2n$. Since $A \in \mathcal{K}_m^n(J, X)$, note that $g(A) \not\subset (J_h)^\circ$. Thus, $g(A) \cap E(J_h) \neq \emptyset$, which implies that $\dim_{g(A)}[C_n(Y) - F_m(Y)]$ is not finite, leading to a contradiction. Therefore, $|J_h \cap \mathcal{P}(Y)| = 1$ and $|J_h \cap R(Y) \cap \mathcal{G}(Y)| = 1$.

Case 2.2 $|J \cap \mathcal{P}(X)| = 2$.

Then $|J \cap R(X) \cap \mathcal{P}(X)| = 0$. By (3.2), we have that $|J_h \cap \mathcal{P}(Y)| \geq 1$. If $|J_h \cap \mathcal{P}(Y)| = 1$, similarly as Case 2.1, it follows that $|J \cap \mathcal{P}(X)| = 1$ which is a contradiction. Hence, $|J_h \cap \mathcal{P}(Y)| = 2$, which implies that $|J_h \cap R(Y) \cap \mathcal{G}(Y)| = 0$. This proves (c) and (d) for Case 2.

This completes the proof of Claim 1.

Claim 2. If $J \in \mathfrak{A}_S(X)$ and $x \in J \cap \mathcal{P}(X)$, then $\mathcal{K}_{x,J}$ is arcwise connected.

Proof of Claim 2. Let $A \in \mathcal{K}_{x,J}$ be such that $x \in A \cap (R(X) \cup \mathcal{P}(X))$.

Case 1 J is an arc.

Hence, there exists a subarc L such that $A \subset L$ and $L \cap (R(X) \cup \mathcal{P}(X)) = \{x\}$. By Theorem 1.62, there exists a map $\alpha : [0, 1] \to C_n(L)$ such that $\alpha(0) = A$, $\alpha(1) = L$, and $\alpha(s) \subset \alpha(t)$, if $s \leq t$. Since $x \in L$, it is clear that $Im(\alpha) \subset \mathcal{K}_{x,J}$ and $\alpha(s) \cap (R(X) \cup \mathcal{P}(X)) = \{x\}$ for each $s \in [0, 1]$. Since the set $\{N \colon N$ is a subarc of J and $N \cap (R(X) \cup \mathcal{P}(X)\}$ is arcwise connected, we conclude that $\mathcal{K}_{x,J}$ is arcwise connected.

Case 2 J is a cycle.

We identify J with the unit circle $\mathcal{S}^1$ in $\mathbb{R}^2$ and x with the point (1,0). Let $e : [0, 1] \to \mathcal{S}^1$ be defined by $e(t) = (\cos(2\pi t, \sin(2\pi t)$, and let $A = A_1 \cup \cdots \cup A_r$ where A_i is a component of A for each $i \in \{1, \ldots, n\}$. Suppose without loss of generality that $x \in A_1$. If $r < n$ or ($r = n = 1$ and A = J) then for each $s \in (0, 1]$, the set $B(s) = e(\{sx \colon x \in e^{-1}(A)\})$ belongs to $\mathcal{K}_{x,J}$. Observe that the association $s \mapsto B(s)$ is continuous and $B(1/2) \subset e([0, 1/2])$. Since $e([0, 1/2]) \in \mathcal{K}_{x,J}$ and $e([0, 1/2])$ is a subarc of J, the proof can be completed in a similar way as Case 1. If $r = n > 1$ (or $r = n = 1$ and A_1 is a subarc of J), note that $A \in \mathcal{K}_{x,J}$. Thus, x is not in the interior manifold of A. Hence, we may assume that there exists $s_0 \in [0, 1)$ such that $A \in e([0, s_0])$. Since $e([0, s_0]) \in \mathcal{K}_{x,J}$ and $e([0, x_0])$ is a subarc of J, again, we can complete the proof analogously as Case 1. From here, the result follows.

Next result shows one of the main steps in the construction of the homeomorphism between X and Y.

Claim 3. There exists a bijection between $R(X) \cap \mathcal{G}(X)$ and $R(Y) \cap \mathcal{G}(Y)$.

Proof of Claim 3. Given $v \in R(X) \cap \mathcal{G}(X)$, let $J \in \mathfrak{A}_S(X)$ be such that $v \in J$. Let $A \in \mathcal{K}_{v,J}$. Notice that $A - \{v\} \subset J^\circ$. Since $v \in \mathcal{G}(X)$, we have that $A \subset \mathcal{G}(X)$ and thus, $A \cap \mathcal{P}(X) = \emptyset$. By Theorem 3.9, there exists a finite graph G_1 contained in X such that $A \subset G_1^\circ$. By (a) of Claim 1, we have that $g(A) \in \mathcal{K}_m^n(J_h, Y)$. As $\dim_A[C_n(X)]$ is finite, it follows that $\dim_{g(A)}[C_n(Y)]$ is finite. Thus, $g(A) \cap \mathcal{P}(Y) = \emptyset$. By Theorem 3.9, there exists a finite graph G_2 contained in Y such that $g(A) \subset G_2^\circ$. Hence, we have that

$$\dim_A[C_n(G_1)] = \dim_A[C_n(X) - F_m(X)] =$$

$$\dim_{g(A)}[C_n(Y) - F_m(Y)] = \dim_{g(A)}[C_n(G_2)].$$

By Theorem 1.69, $\dim_A[C_n(G_1)] > 2n$. Thus, $\dim_{g(A)}[C_n(G_2)] > 2n$, which implies that $g(A) \cap R(Y) \cap \mathcal{G}(Y) \neq \emptyset$. Suppose that $|g(A) \cap R(Y) \cap \mathcal{G}(Y)| = 2$. Notice that $\dim_{g(A)}[C_n(G_2)] = \max\{\dim_B[C_n(Y)] \text{ is finite}\colon B \in \mathcal{K}_m^n(J_h, Y)\}$. Thus, $\dim_A[C_n(G_1)] = \max\{\dim_A[C_n(X)] \text{ is finite}\colon A \in \mathcal{K}_m^n(J, X)\}$ and it follows that $|A \cap R(X) \cap \mathcal{G}(X)| = 2$, a contradiction since $A \in \mathcal{K}_{x,J}$. Therefore, $|g(A) \cap R(Y) \cap \mathcal{G}(Y)| = 1$. Let $v_h(A) \in g(A) \cap R(Y) \cap \mathcal{G}(Y)$. Notice that $v_h(A) \in J_h$. We will prove that $v_h(A)$ does not depend on the choice of A noe the choice of J. Let $L \in \mathfrak{A}_S(X)$, $E \in \mathcal{K}_{v,L}$ and $v_h(A) = v_h(E)$. Since $(J \cup L) \cap \mathcal{P}(X) = \emptyset$, there exists a finite graph D in X such that $(J \cup L) \subset D^\circ$. Thus, $\mathcal{K}_n(J, X) = \mathcal{K}_n(J, D)$, $\mathcal{K}_n(L, X) = \mathcal{K}_n(L, D)$, $J \cap R(X) = J \cap R(D)$ and $L \cap R(X) = L \cap R(D)$. Let A_1 be a subarc of J such that $A_1 \neq J$ and $v \in A_1$. From here, it is clear that $A_1 \in \mathcal{K}_{v,J}$, In a similar way, there exists $E_1 \in \mathcal{K}_{v,L}$ such that E_1 is connected. By Claim 2, we have that $\mathcal{K}_{v,J}$ and $\mathcal{K}_{v,L}$ are arcwise connected. Hence, we can find continuous functions $\alpha_A : [0, 1] \to \mathcal{K}_{v,J}$ and $\alpha_E : [0, 1] \to \mathcal{K}_{v,L}$ such that $\alpha_A(0) = A$ and $\alpha_A(1) = A_1$, $\alpha_E(0) = E_1$ and $\alpha_E(1) = E$. We can combine this functions to get another continuous function $\alpha_0 : [0, 1] \to C(A_1 \cup E_1)$ such that $\alpha_0(0) = A_1$ and $\alpha_0(1) = E_1$ and, for each $t \in [0, 1]$, $\alpha_0(t) \cap R(X) = \{v\}$ and $\alpha_0(t) \notin F_m(X)$. Define $\alpha : [0, 1] \to C(A_1 \cup E_1) \cup \mathcal{K}_{v,J} \cup \mathcal{K}_{v,L}$ by

$$\alpha(t) = \begin{cases} \alpha_A(3t) & \text{if } 0 \leq t \leq 1/3, \\ \alpha_0(3t-1) & \text{if } 1/3 \leq t \leq 2/3, \\ \alpha_E(3t-2) & \text{if } 2/3 \leq t \leq 1. \end{cases}$$

Observe that $\alpha(0) = A$, $\alpha(1) = E$, α is continuous and for each $t \in [0, 1]$, $\alpha(t) \cap R(X) = \{v\}$. Now, let $j_0 = \mathrm{ord}_X(v)$. By Theorem 1.69, for each $t \in [0, 1]$, we have that

$$\begin{aligned}2n < 2n + (j_0 - 2) &= \dim_{\alpha(t)}[C_n(X)] \\ &= \dim_{q_X^{(n,m)}(\alpha(t))}[HS_m^n(X)] \\ &= \dim_{h(q_X^{(n,m)}(\alpha(t)))}[HS_m^n(Y)] \\ &= \dim_{(q_Y^{(n,m)})^{-1}(h(q_X^{(n,m)}(\alpha(t))))}[C_n(Y)].\end{aligned}$$

Let $T = \{t \in [0,1] \colon v_h(A) \in (q_Y^{(n,m)})^{-1}(h(q_X^{(n,m)}(\alpha(t))))\}$. We have to prove that $T = [0,1]$. Notice that T is a closed subset of $[0,1]$ and $0 \in T$. Proceeding by contradiction, suppose that $T \neq [0,1]$ and let R be a component of $[0,1] - T$. By the continuity of g, $t_0 = \inf(R) \in T$ and there exists a sequence $\{r_k\}_{k\in\mathbb{N}}$ contained in R converging to t_0. Now suppose that $\mathrm{ord}_Y(v_h(A)) = j \geq 3$. By Theorem 1.69, $(q_Y^{(n,m)})^{-1}(h(q_X^{(n,m)}(\alpha(t))))$ intersects $R(Y)$, for each $t \in [0,1]$. As $R(Y)$ is a finite set, it is possible to assume that $w \in R(Y) \cap (q_Y^{(n,m)})^{-1}(h(q_X^{(n,m)}(\alpha(r_k))))$ for each $k \in \mathbb{N}$. Hence, $w \in (q_Y^{(n,m)})^{-1}(h(q_X^{(n,m)}(\alpha(t_0))))$. Once again, by Theorem 1.69 and since $v_h(A)$ is the only ramification point in $(q_Y^{(n,m)})^{-1}(h(q_X^{(n,m)}(\alpha(0))))$, we conclude that $j = j_0$. As $r_1 \notin T$, it follows that $v_h(A) \notin (q_Y^{(n,m)})^{-1}(h(q_X^{(n,m)}(\alpha(r_1))))$. Thus, $v_h(A) \neq w$. Let $\mathrm{ord}_Y(w) = i_0 \geq 3$. As $v_h(A), w \in (q_Y^{(n,m)})^{-1}(h(q_X^{(n,m)}(\alpha(t_0))))$, using Theorem 1.69, we reach a contradiction. Therefore, $T = [0,1]$. From here, it is implied that $v_h(A)$ is the only ramification point of $(q_Y^{(n,m)})^{-1}(h(q_X^{(n,m)}(\alpha(t_0))))$, for each $t \in [0,1]$. Particularly, when $t = 1$, we have that $v_h(A) = v_h(E)$. Finally, we have that $v_h(A)$ does not depend on A and, in fact, it does not depend on the choice of J. In this way, we have a function

$$\varphi_1 \colon R(X) \cap \mathcal{G}(X) \to R(Y) \cap \mathcal{G}(Y)$$

given by $\varphi_1(v) = v_h$, for each point $v \in R(X) \cap \mathcal{G}(X)$. Notice that φ_1 satisfies the following property: if $v \in R(X) \cap \mathcal{G}(X) \cap J$ and $A \in \mathcal{K}_{v,J}$, then $g(A) \cap R(Y) \cap \mathcal{G}(Y) = \{v_h\}$ and $v_h \in J_h$.

By (B) and (C), X and Y satisfy symmetric conditions. Thus, we may define a function $\phi \colon R(Y)\cap\mathcal{G}(Y) \to R(X)\cap\mathcal{G}(X)$ through a similar procedure satisfying the property: if $w \in R(Y) \cap \mathcal{G}(Y) \cap J_h$ and $B \in \mathcal{K}_{w,J_h}$, then $g^{-1}(B) \cap R(X) \cap \mathcal{G}(X) = \{\phi(w)\}$ and $\phi(w) \in J$. Allow us to prove that φ_1 is a bijection. Let $v \in R(X) \cap \mathcal{G}(X)$, $J \in \mathfrak{A}_S(X)$ be such that $v \in J$ and $A \in \mathcal{K}_{v,J}$. Notice that $g(A) \cap R(Y) \cap \mathcal{G}(Y) = \{\varphi_1(v)\}$ and $\varphi_1(v) \in J_h$. Moreover, $g(A) \in \mathcal{K}_{\varphi_1(v),J_h}$. Hence, $A \cap R(X) \cap \mathcal{G}(X) = \{\phi(\varphi_1(v))\}$ and $\phi(\varphi_1(v)) \in J$. Since $A \in \mathcal{K}_{v,J}$, we have that $v = \phi(\varphi_1(v))$. Analogously, it is possible to conclude that $v = \varphi_1(\phi(v))$. Therefore, φ_1 is a bijection. This completes the proof of Claim 3.

Notice that the following two conditions are equivalent:
(i) $x \in R(X) \cap \mathcal{G}(X) \cap J$, where $J \in \mathfrak{A}_S(X)$,
(ii) $\varphi_1(x) \in R(Y) \cap \mathcal{G}(Y) \cap J_h$, where $J_h \in \mathfrak{A}_S(Y)$.
Moreover, for each $J \in \mathfrak{A}_S(X)$, by (A),

$$g(J) \subset J_h. \tag{3.3}$$

Claim 4. Let $x \in \mathcal{P}(X)$. If $\{J_k\}_{k\in\mathbb{N}}$ is a sequence of pairwise different elements contained in $\mathfrak{A}_S(X)$ which converges to $\{x\}$, then $\{g(J_k)\}_{k\in\mathbb{N}}$ converges to a one point set in $C_n(Y)$. Moreover, $\{(J_k)_h\}_{k\in\mathbb{N}}$ converges to exactly the same one point set as $\{g(J_k)\}_{k\in\mathbb{N}}$.

Proof of Claim 4. It will be enough to prove that every convergent subsequence of $\{g(J_k)\}_{k\in\mathbb{N}}$ converges to exactly the same element since Y is a continuum. Let $\{g(J_{k_l})\}_{l\in\mathbb{N}}$ be a subsequence of $\{g(J_k)\}_{k\in\mathbb{N}}$ which converges to $T \in C_n(Y)$. Let $y \in T$. There exists a sequence $\{t_{k_l}\}_{l\in\mathbb{N}}$ where $t_{k_l} \in g(J_{k_l})$ for each $l \in \mathbb{N}$, which converges to y. By (3.3) and Theorem 3.10, we have that $\{(J_{k_l})_h\}_{l\in\mathbb{N}}$ converges to $\{y\}$. Thus, $T = \{y\}$. Hence, $\{g(J_{k_l})\}_{l\in\mathbb{N}}$ converges to $\{y\}$. From this, every convergent subsequence of $\{g(J_k)\}_{k\in\mathbb{N}}$ converges to a one point set. Allow us to prove that the original sequence converges to the same one point set. Let $\{g(J_{k_s})\}_{s\in\mathbb{N}}$ be another subsequence of $\{g(J_k)\}_{k\in\mathbb{N}}$ which converges to $\{w\}$, for some $w \in Y$. We will prove that $y = w$. Suppose that $y \neq w$. Let U_y and U_w be disjoint open subsets of Y such that $y \in U_y$ and $w \in U_w$. Thus, by the convergence, there exists $N \in \mathbb{N}$ such that $g(J_{k_l}) \in \langle U_y\rangle_n - F_m(Y)$ and $g(J_{k_s}) \in \langle U_w\rangle_n - F_m(Y)$, for each l, $s \geq N$. Let $\mathcal{U}_y = g^{-1}(\langle U_y\rangle_n - F_n(Y))$ and $\mathcal{U}_w = g^{-1}(\langle U_w\rangle_n - F_n(Y))$. Notice that $\mathcal{U}_y$ and $\mathcal{U}_w$ are disjoint open subsets of $C_n(X)$ such that $J_{k_l} \in \mathcal{U}_y$ and $J_{k_s} \in \mathcal{U}_w$, for each l, $s \geq N$. This is a contradiction, which proves that $y = w$. Therefore, $\{g(J_k)\}_{k\in\mathbb{N}}$ converges to $\{y\}$. Since $g(J_k) \subset (J_k)_h$ for each $k \in \mathbb{N}$, by Theorem 3.10, we have that $\{(J_k)_h\}_{k\in\mathbb{N}}$ converges to $\{y\}$. This completes the proof of Claim 4.

Now we focus on $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.

Claim 5. There exists a bijection φ_2 from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$.

Proof of Claim 5. Let $x \in \mathcal{P}(X)$. By Theorem 3.14, there exists a sequence of pairwise different elements $\{J_k\}_{k\in\mathbb{N}}$ contained in $\mathfrak{A}_S(X)$ which converges to $\{x\}$. By Claim 4, there exists $y \in Y$ such that $\{g(J_k)\}_{k\in\mathbb{N}}$ converges to $\{y\}$. Moreover, $\{(J_k)_h\}_{k\in\mathbb{N}}$ converges to $\{y\}$. By Theorem 3.14, we have that $y \in \mathcal{P}(Y)$. Let $\{L_k\}_{k\in\mathbb{N}}$ be another sequence contained in $\mathfrak{A}_S(X)$ which converges to $\{x\}$. Allow us to see that this association is well defined. By Claim 4, $\{g(L_k)\}_{k\in\mathbb{N}}$ converges to $\{w_1\}$ for some $w_1 \in Y$. Moreover, $\{(L_k)_h\}_{k\in\mathbb{N}}$ converges to $\{w_1\}$. In order to prove that $y = w_1$, we consider two cases:

Case 5.1. $\{J_k : k \in \mathbb{N}\} \cap \{L_k : k \in \mathbb{N}\}$ is finite.

Then, there exists $N \in \mathbb{N}$ such that $\{J_k : k \geq N\} \cap \{L_k : k \geq N\} = \emptyset$. Let $S_{2k+1-N} = J_k$ and $S_{2k-N} = L_k$, for each $k \geq N$ and let $I = \{k \in \mathbb{N} : k \geq N\}$. Notice that $\{S_k\}_{k\in I}$ is composed by pairwise different elements in $\mathfrak{A}_S(X)$. Observe that $\{S_k\}_{k\in I}$ converges to $\{x\}$. By Claim 4, $\{g(S_k)\}_{k\in I}$ converges to $\{z\}$, for some $z \in Y$. Since $\{g(J_k)\}_{k\in\mathbb{N}}$ and $\{g(L_k)\}_{k\in\mathbb{N}}$ are subsequences of $\{g(S_k)\}_{k\in I}$, we conclude that $y = z = w_1$.

Case 5.2. $\{J_k : k \in \mathbb{N}\} \cap \{L_k : k \in \mathbb{N}\}$ is infinite.

Thus, we may extract a sequence $\{T_k\}_{k\in\mathbb{N}}$ contained in $\{J_k : k \in \mathbb{N}\} \cap \{L_k : k \in \mathbb{N}\}$, which is a subsequence of both $\{J_k\}_{k\in\mathbb{N}}$ and $\{L_k\}_{k\in\mathbb{N}}$. Since $\{T_k\}_{k\in\mathbb{N}}$ is a sequence of pairwise different elements in $\mathfrak{A}_S(X)$, by Claim 4 we have that $\{g(T_k)\}_{k\in\mathbb{N}}$ converges to $\{z\}$, for some $z \in Y$. Since $\{g(T_k)\}_{k\in\mathbb{N}}$ is a subsequence of both $\{g(J_k)\}_{k\in\mathbb{N}}$ and $\{g(L_k)\}_{k\in\mathbb{N}}$, we have that $z = w_1 = y$.

From both cases, we have that given $x \in \mathcal{P}(X)$, there exists a unique point $y_x \in \mathcal{P}(Y)$ such that if $\{J_k\}_{k\in\mathbb{N}}$ is a sequence of pairwise different elements contained in $\mathfrak{A}_S(X)$ which converges to $\{x\}$, then $\{(J_k)_h\}_{k\in\mathbb{N}}$ converges to $\{y_x\}$. Now, we define a function:

$$\varphi_2 : \mathcal{P}(X) \to \mathcal{P}(Y)$$

by $\varphi_2(x) = y_x$, for each point $x \in \mathcal{P}(X)$. From (B) and (C), X and Y satisfy symmetric conditions. Thus, we define a function $\varphi_2' : \mathcal{P}(Y) \to \mathcal{P}(X)$ satisfying the following property: if $y \in \mathcal{P}(Y)$ and $\{(J_k)_h\}_{k\in\mathbb{N}}$ is a sequence of pairwise different elements in $\mathfrak{A}_S(Y)$ which converges to $\{y\}$, then $\{J_k\}_{k\in\mathbb{N}}$ converges to $\{(\varphi_2')(y)\}$. By properties of φ_2 and φ_2', it follows that $(\varphi_2)^{-1} = \varphi_2'$. This completes the proof of Claim 5.

Claim 6. If $J \in \mathfrak{A}_S(X)$ and $x \in J \cap \mathcal{P}(X)$, then $\varphi_2(x) \in J_h \cap \mathcal{P}(Y)$. Moreover, if J is an arc and $x \in E(J) \cap \mathcal{P}(X)$, then $\varphi_2(x) \in E(J_h) \cap \mathcal{P}(Y)$.

Proof of Claim 6. Let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of arcs in $\mathcal{K}_{x,J}$ which converges to $\{x\}$. By (A) and Lemma 3.18 (a), we obtain $h(q_X^{(n,m)}(\langle J^\circ\rangle_n \cap C(X)) - \{F_X^m\}) = q_Y^{(n,m)}(\langle (J_h)^\circ\rangle_n \cap C(Y)) - \{F_Y^m\}$. Thus, $g(\langle J^\circ\rangle_n \cap C(X)) - F_m(X) = (\langle (J_h)^\circ\rangle_n \cap C(Y)) - F_m(Y)$. Given $k \in \mathbb{N}$, notice that $A_k \in$

$\mathrm{cl}_{C_n(X)}((\langle J^\circ\rangle_n\cap C(X))-F_m(X))$. Hence, $g(A_k)\in \mathrm{cl}_{C_n(Y)}((\langle (J_h)^\circ\rangle_n\cap C(Y))-F_m(Y))$.

In order to prove that $\{g(A_k)\}_{n\in\mathbb{N}}$ converges to $\{\varphi_2(x)\}$, first we show that $\{g(A_k)\}_{k\in\mathbb{N}}$ converges in $C(Y)$ and, moreover, in $F_1(Y)$. To do so, we will prove that every convergent subsequence of $\{g(A_k)\}_{k\in\mathbb{N}}$ converges to exactly the same element of $C(Y)$. Let $\{g(A_{k_l})\}_{l\in\mathbb{N}}$ be a subsequence of $\{g(A_k)\}_{k\in\mathbb{N}}$ which converges to some $T\in C(Y)$. Suppose that $T\notin F_1(Y)$. Hence, $g^{-1}(T)\notin F_1(X)$; however, $\{A_{k_l}\}_{l\in\mathbb{N}}$ converges to $g^{-1}(T)$ which contradicts the convergence to $\{x\}$. Thus, $T=\{y\}$, for some $y\in Y$. Let $\{g(A_{k_s})\}_{s\in\mathbb{N}}$ be another subsequence of $\{g(A_k)\}_{k\in\mathbb{N}}$ which converges to $\{w\}$, for some $w\in Y$. Following analogous arguments as in the proof of Claim 4, we prove that $y=w$. Thus, $\{g(A_k)\}_{k\in\mathbb{N}}$ converges to $\{y\}$. Hence, $y\in J_h$. By Theorem 3.14, there exists a sequence $\{J_k\}_{k\in\mathbb{N}}$ of pairwise different elements contained in $\mathfrak{A}_S(X)$ which converges to $\{x\}$. By Claim 4 and the definition of φ_2, $\{g(J_k)\}_{k\in\mathbb{N}}$ converges to $\{\varphi_2(x)\}$. Let $S_{2k+1}=J_k$ and $S_{2k}=A_k$ for each $k\in\mathbb{N}$. Thus, $\{S_k\}_{k\in\mathbb{N}}$ converges to $\{x\}$. Following analogous arguments to those used to prove that $\{g(A_k)\}_{k\in\mathbb{N}}$ converges to $\{y\}$, we can conclude that $\{g(S_k)\}_{k\in\mathbb{N}}$ converges to $\{w_1\}$, for some $w_1\in Y$. Thus, $y=w_1=\varphi_2(x)$. Hence, $\varphi_2(x)\in J_h$. This completes the proof of Claim 6.

Let $\varphi_3:(R(X)\cap\mathcal{G}(X))\cup\mathcal{P}(X)\to(R(Y)\cap\mathcal{G}(Y))\cup\mathcal{P}(Y)$ be a function defined as follows:

$$\varphi_3(x)=\begin{cases}\varphi_1(x) & \text{if } x\in R(X)\cap\mathcal{G}(X),\\ \varphi_2(x) & \text{if } x\in\mathcal{P}(X).\end{cases}$$

Notice that φ_3 is a bijection. Using φ_3 we will find a homeomorphism between $\mathcal{G}(X)\cup\mathcal{P}(X)$ and $\mathcal{G}(Y)\cup\mathcal{P}(Y)$.

Claim 7. There exists a homeomorphism φ from $\mathcal{G}(X)\cup\mathcal{P}(X)$ to $\mathcal{G}(Y)\cup\mathcal{P}(Y)$ such that $\varphi|_{(R(X)\cap\mathcal{G}(X))\cup\mathcal{P}(X)}=\varphi_3$.

Proof of Claim 7. Let $J\in\mathfrak{A}_S(X)$. We consider five cases.

Case 7.1. $|J\cap R(X)\cap\mathcal{G}(X)|=2$.

By the number of ramification points in J, it is implied that J is an arc. Let $x,z\in E(J)$. Then $\{x,z\}=J\cap R(X)\cap\mathcal{P}(X)$. Since J_h is an arc with end points $\varphi_1(x)$ and $\varphi_1(z)$, we can consider a homeomorphism $\varphi_J\colon J\to J_h$ such that $\varphi_J(x)=\varphi_1(x)$ and $\varphi_J(z)=\varphi_1(z)$.

Case 7.2. $|J\cap\mathcal{P}(X)|=2$.

Analogously as Case 7.1, notice that J and J_h are arcs. Let $E(J) = \{x, z\}$. By Claim 6, we have that $E(J_h) = \{\varphi_2(x), \varphi_2(z)\}$. Thus, there exists a homeomorphism $\varphi_J : J \to J_h$ such that $\varphi_J(x) = \varphi_2(x)$ and $\varphi_J(z) = \varphi_2(z)$.

Case 7.3. $|J \cap R(X) \cap \mathcal{G}(X)| = 1$ and $|J \cap \mathcal{P}(X)| = 1$.

Notice that J is an arc. Let $x \in J \cap R(X) \cap \mathcal{G}(X)$ and $z \in J \cap \mathcal{P}(X)$. Since J_h is an arc with end points $\varphi_1(x)$ and $\varphi_2(z)$, we may consider a homeomorphism $\varphi_J : J \to J_h$ such that $\varphi_J(x) = \varphi_1(x)$ and $\varphi_J(z) = \varphi_2(z)$.

Case 7.4. $|J \cap R(X) \cap \mathcal{G}(X)| = 1$ and $J \cap \mathcal{P}(X) = \emptyset$.

Assuming that $J \cap R(X) \cap \mathcal{G}(X) = \{x\}$, we know that $J_h \cap R(Y) \cap \mathcal{G}(Y) = \{\varphi_1(x)\}$. In either case, if J is an arc or a cycle, by (1) and (2) we have that J_h is an arc or a cycle, respectively. Thus, there exists a homeomorphism $\varphi_J : J \to J_h$ such that $\varphi_J(x) = \varphi_1(x)$.

Case 7.5. $|J \cap \mathcal{P}(X)| = 1$ and $J \cap R(X) \cap \mathcal{G}(X) = \emptyset$.

Let $x \in J \cap \mathcal{P}(X)$. By Claim 6, $\varphi_1(x) \in J_h \cap \mathcal{P}(Y)$. If J is an arc or a cycle, by (1) and (2) we have that J_h is an arc or a cycle, respectively. Thus, we can take a homeomorphism $\varphi_J\colon J \to J_h$ such that $\varphi_J(x) = \varphi_1(x)$.

By Theorem 1.49, we can take a common extension

$$\varphi'\colon \bigcup \mathfrak{A}_S(X) \to \bigcup \mathfrak{A}_S(Y)$$

of the functions φ_J, for each $J \in \mathfrak{A}_S(X)$. Notice that $\varphi'(J) = J_h$, for each $J \in \mathfrak{A}_S(X)$. Moreover, φ' is an homeomorphism.

Now, we define φ from $\mathcal{G}(X) \cup \mathcal{P}(X)$ onto $\mathcal{G}(Y) \cup \mathcal{P}(Y)$, for each $x \in X$, by

$$\varphi(x) = \begin{cases} \varphi_2(x) & \text{if } x \in \mathcal{P}(X), \\ \varphi'(x) & \text{if } x \in \bigcup \mathfrak{A}_S(X). \end{cases}$$

We are going to prove that φ is a map. In order to prove this assertion, let $x \in \mathcal{G}(X)$ and let $\{x_m\}_{k\in\mathbb{N}}$ be a sequence of elements of X which converges to x. Since $\mathcal{G}(X)$ is an open subset of X we may assume that $x_m \in \mathcal{G}(X)$, for each $m \in \mathbb{N}$. Since φ' is continuous over $\mathcal{G}(X)$, we have that φ is continuous at x. Let $x \in \mathcal{P}(X)$. We have to consider three cases.

Case i. $\{x_k\}_{k\in\mathbb{N}}$ is a sequence of pairwise different elements of $\mathcal{G}(X)$ which converges to x.

First, suppose that $\{x_k\}_{k\in\mathbb{N}}$ is contained in $J_1 \cup \cdots \cup J_r$, for some $J_1, \ldots, J_r \in \mathfrak{A}_S(X)$. Since $J_1 \cup \cdots \cup J_r$ is a closed subset of X, we have that $x \in J_1 \cup \cdots \cup J_r$. Notice that $\varphi(x_k) = \varphi'(x_k) \in (J_1)_h \cup \cdots \cup (J_r)_h$, for each $k \in \mathbb{N}$ and $\varphi(x) = \varphi'(x)$. Since φ' is continuous, we have that $\{\varphi(x_k)\}_{k\in\mathbb{N}}$ converges to $\varphi(x)$.

Finally, suppose that there exist pairwise different elements $J_k \in \mathfrak{A}_S(X)$ such that $x_k \in J_k$, for each $k \in \mathbb{N}$. By Theorem 3.10, we have that $\{J_k\}_{k\in\mathbb{N}}$ converges to $\{x\}$. By the definition of φ_2, we have that $\{(J_k)_h\}_{k\in\mathbb{N}}$ converges to $\{\varphi_2(x)\}$. Since $\varphi(x_k) \in (J_k)_h$, for each $k \in \mathbb{N}$, we have that $\{\varphi(x_k)\}_{k\in\mathbb{N}}$ converges to $\varphi_2(x)$.

Case ii. $\{x_k\}_{k\in\mathbb{N}}$ is a sequence of pairwise different elements of $\mathcal{P}(X)$ which converges to x with $x_k \neq x$, for each $k \in \mathbb{N}$.

Let $\{J_l^k\}_{l\in\mathbb{N}}$ be a sequence of pairwise different elements of $\mathfrak{A}_S(X)$ which converges to $\{x_k\}$, for each $k \in \mathbb{N}$. By definition of φ_2, we have that $\{(J_l^k)_h\}_{l\in\mathbb{N}}$ converges to $\{\varphi_2(x_k)\}$, for each $k \in \mathbb{N}$. Let $\{(J_{s_k}^k)_h\}_{k\in\mathbb{N}}$ be a sequence of pairwise different elements in $\mathfrak{A}_S(Y)$ such that $H_{d'}((J_{s_k}^k)_h, \varphi(x_k)) < \frac{1}{k}$, for each $k \in \mathbb{N}$, where d' is the metric of Y. By Theorem 3.10, $\{(J_{s_k}^k)_h\}_{k\in\mathbb{N}}$ converges to $\{y\}$, for some $y \in Y$. Notice that $\{\varphi_2(x_k)\}_{k\in\mathbb{N}}$ converges to y. We may suppose that $y \neq \varphi_2(x_k)$, for each $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $r_k = \min\{H_d(\{x\},\{x_m\}), H_{d'}(\{\varphi_2(x_m)\},\{y\})\}$. Let $l_k \in \mathbb{N}$ be such that $H_d(\{x_k\}, J_{l_k}^k) < r_k$ and $H_{d'}((J_{l_k}^k)_h, \{\varphi_2(x_k)\}) < r_k$. Then

$$H_d(\{x\}, J_{l_k}^k) \leq H_d(\{x\},\{x_k\}) + H_d(\{x_k\}, J_{l_k}^k) \leq 2H_d(\{x\},\{x_k\})$$

and

$$\begin{aligned} H_{d'}(\{y\}, (J_{l_k}^k)_h) &\leq H_{d'}(\{y\},\{\varphi_2(x_k)\}) + H_{d'}((J_{l_k}^k)_h, \{\varphi_2(x_k)\}) \\ &\leq 2H_{d'}(\{y\},\{\varphi_2(x_k)\}). \end{aligned}$$

From this, $\{J_{l_k}^k\}_{k\in\mathbb{N}}$ converges to $\{x\}$ and $\{(J_{l_k}^k)_h\}_{k\in\mathbb{N}}$ converges to $\{y\}$. By definition of φ_2, $\{(J_{l_k}^k)_h\}_{k\in\mathbb{N}}$ also converges to $\{\varphi_2(x)\}$. Thus, $\varphi_2(x) = y$. Therefore, $\{\varphi(x_k)\}_{k\in\mathbb{N}}$ converges to $\varphi_2(x)$.

Case iii. $\{x_k\}_{k\in\mathbb{N}}$ is a sequence of pairwise different elements of $\mathcal{G}(X) \cup \mathcal{P}(X)$ which converges to x with $x_k \neq x$, for each $k \in \mathbb{N}$.

By Case i and Case ii, $\{\varphi(x_k)\}_{k\in\mathbb{N}}$ converges to $\varphi(x)$.

Finally, if $\{x_k\}_{k\in\mathbb{N}}$ is a sequence in X which converges to $x \in \mathcal{P}(X)$, then, by Case i, Case ii and Case iii, we have that $\{\varphi(x_k)\}_{k\in\mathbb{N}}$ converges to $\varphi(x)$. Hence, φ is a continuous function between continua. Therefore, X is homeomorphic to Y. $\square$

Remark 3.21. *Let X be a continuum and $J \in \mathfrak{A}_S(X)$. Using [41, Example 2] and [41, Example 3], we obtain the following models of $\langle J^\circ\rangle_n \cap C(X) - F_1(X)$,*

Case 1. *$J \in \mathfrak{A}_E(X)$.*

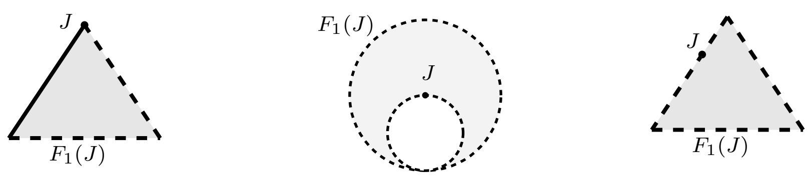
Then, J is an arc with end points p and q such that $p \in J^\circ$. Notice that $\langle J^\circ\rangle_n \cap C(X) - F_1(X) = \langle J - \{q\}\rangle_n \cap C(X) - F_1(J)$. Thus, $\langle J^\circ\rangle_n \cap C(X) - F_1(X)$ is a 2-manifold with manifold boundary.

Case 2. *$J \in \mathfrak{A}_R(X)$.*

There exists $p \in J$ such that $J^\circ = J - \{p\}$. Thus, $\langle J^\circ\rangle_n \cap C(X) = C(J) - \{A \in C(J) \colon p \in A\}$. Hence, $\langle J^\circ\rangle_n \cap C(X) - F_1(X)$ is a 2-manifold without manifold boundary.

Case 3. *J is an arc such that $J \notin \mathfrak{A}_E(X)$.*

Then, $J^\circ = J - \{p, q\}$ where p and q are the end points of J. Thus, $\langle J^\circ\rangle_n \cap C(X) - F_1(X) = \langle J - \{p, q\}\rangle_n \cap C(X) - F_1(J)$. Hence, $\langle J^\circ\rangle_n \cap C(X) - F_1(X)$ is a 2-manifold without manifold boundary.



(a) Case 1 (b) Case 2 (c) Case 3

If X and Y are almost meshed locally connected continua such that $HS_m^n(X)$ and $HS_m^n(Y)$ are homeomorphic ($n \in \mathbb{N} - \{1, 2\}$ and $m \in \mathbb{N} - \{1\}$), then X and Y must be homeomorphic, as the following result states:

Theorem 3.22. *The members of the class of almost meshed locally connected continua are HS_m^n-determined, for each $n \in \mathbb{N} - \{1, 2\}$, $m \in \mathbb{N} - \{1\}$ and $m < n$.*

Proof. Let X and Y be two almost meshed locally connected continua and let $h : HS_m^n(X) \to HS_m^n(Y)$ be a homeomorphism. Since the definition of $\mathcal{HL}_m^n(X)$ is given in terms of topological properties, $h(\mathcal{HL}_m^n(X)) = \mathcal{HL}_m^n(Y)$. This implies that $h(\mathcal{HD}_m^n(X)) = \mathcal{HD}_m^n(Y)$. By Lemma 3.18 (a), we have that for each $J \in \mathfrak{A}_S(X)$, there exists $J_h \in \mathfrak{A}_S(Y)$ such that $h(q_X^{(n,m)}(\langle \mathrm{int}_X(J)\rangle_n \cap C(X)) - \{F_X^{(n,m)}\}) = q_Y^{(n,m)}(\langle \mathrm{int}_Y(J_h)\rangle_n \cap C(Y)) - \{F_Y^m\}$. Since we have

$q_Y^{(n,m)}(\langle \mathrm{int}_Y(J_h)\rangle_n \cap C(Y)) - \{F_Y^m\} \subset q_Y^{(n,m)}(\langle \mathrm{int}_Y(J_h)\rangle_n) - \{F_Y^m\}$, by Theorem 3.20 (C) and (D), the association $J \mapsto J_h$ between $\mathfrak{A}_S(X)$ and $\mathfrak{A}_S(Y)$ is a bijection, and $h(F_X^m) = F_Y^m$. Therefore, $g : C_n(X) - F_m(X) \to C_n(Y) - F_m(X)$ defined by $g = (q_Y^{(n,m)})^{-1} \circ h \circ q_X^{(n,m)}$ is a homeomorphism.

Claim 1. $J \in \mathfrak{A}_E(X)$ if and only if $J_h \in \mathfrak{A}_E(Y)$.

Proof of Claim 1. Let $J \in \mathfrak{A}_E(X)$. By Lemma 3.18 (a), it follows that $g(\langle J^\circ\rangle_n \cap C(X) - F_1(X)) = \langle J_h^\circ\rangle_n \cap C(Y) - F_1(Y)$. By Remark 3.21, we have that $J_h \in \mathfrak{A}_E(Y)$. The converse implication follows analogously.

Claim 2. $J \in \mathfrak{A}_R(X)$ if and only if $J_h \in \mathfrak{A}_R(Y)$.

Proof of Claim 2. Let $J \in \mathfrak{A}_R(X)$. Suppose that J_h is an arc. By Claim 1, $J_h \notin \mathfrak{A}_E(Y)$.

Case 1. $J \cap \mathcal{P}(X) = \emptyset$.

Let $p \in J \cap R(X)$. Notice that $\dim_{J_h}[C_n(Y) - F_m(Y)] = \dim_{g^{-1}(J_h)}[C_n(X) - F_m(X)]$, as g is a homeomorphism. Since $g(\mathrm{cl}_{C_n(X)-F_m(X)}(\langle J^\circ\rangle_n \cap C(X) - F_1(X))) = \mathrm{cl}_{C_n(Y)-F_m(Y)}(\langle J_h^\circ\rangle_n \cap C(Y) - F_1(Y))$, we have that $g^{-1}(J_h) \subset J$. Therefore, $\dim_{J_h}[C_n(Y) - F_m(Y)]$ is finite. Hence, $J_h \cap \mathcal{P}(Y) = \emptyset$. By Theorem 3.9, there exist finite graphs D, G such that $J_h \subset D^\circ$ and $J \subset G^\circ$, respectively. Let $q, r \in J_h \cap R(Y)$ and Q be a subarc of J_h with $q \in Q$ and $r \notin Q$. By Theorem 1.69, we have that

$$\begin{aligned}
2n + [\mathrm{ord}(q) - 2] &= \dim_Q[C_n(D)] \\
&= \dim_Q[C_n(X)] \\
&= \dim_Q[C_n(X)] \\
&= \dim_Q[C_n(X)] \\
&= \dim_Q[C_n(X) - F_m(X)] \\
&= \dim_{g^{-1}(Q)}[C_n(Y) - F_m(Y)] \\
&= \dim_{g^{-1}(Q)}[C_n(Y)] \\
&= \dim_{g^{-1}(Q)}[C_n(G)] \\
&= 2n + [\mathrm{ord}(p) - 2].
\end{aligned}$$

Hence, $\mathrm{ord}(p) = \mathrm{ord}(q)$. Analogously, we have that $2n + [\mathrm{ord}(q) - 2] + [\mathrm{ord}(r) - 2] = \dim_{J_h}[C_n(D)] = \dim_{g^{-1}(J_h)}[C_n(G)] = 2n + [\mathrm{ord}(p) - 2]$, which leads a contradiction. Therefore, $J_h \in \mathfrak{A}_R(X)$.

Case 2. $J \cap \mathcal{P}(X) \neq \emptyset$.

Let $p \in J \cap \mathcal{P}(X)$. Hence, $J_h \cap \mathcal{P}(Y) \neq \emptyset$. Moreover, $E(J_h) \cap \mathcal{P}(Y) \neq \emptyset$. Let $a, b \in E(J_h)$. Allow us to prove that $a, b \in \mathcal{P}(Y)$. Suppose that $a \in \mathcal{P}(Y)$ and $b \notin \mathcal{P}(Y)$. Since $J_h \notin \mathcal{A}_E(Y)$, we have that $b \in R(Y) \cap \mathcal{G}(X)$. Let Q be a proper subarc of J_h containing b. By Theorem 3.9, there exist a finite graph D such that $Q \subset D^\circ$. By Theorem 1.69, we have that

$$\begin{aligned}2n + [\operatorname{ord}(b) - 2] &= \dim_Q[C_n(D)] \\ &= \dim_Q[C_n(X)] \\ &= \dim_Q[C_n(X) - F_m(X)] \\ &= \dim_{g^{-1}(Q)}[C_n(Y) - F_m(Y)] \\ &= \dim_{g^{-1}(Q)}[C_n(Y)] \\ &= \dim_{g^{-1}(Q)}[C_n(h^{-1}(D)] = 2n.\end{aligned}$$

Observe that this leads to a contradiction, since $\operatorname{ord}(b) \geq 3$. Therefore, $b \in \mathcal{P}(X)$. By Theorem 3.14, there exist $\{L_k^a\}_{k\in\mathbb{N}}$ and $\{L_k^b\}_{k\in\mathbb{N}}$ sequences of pairwise different elements of $\mathfrak{A}_S(Y) - \{J_h\}$ converging to $\{a\}$ and $\{b\}$, respectively. Since $q_Y^{(n,m)}(\langle J_h^\circ, (L_k^a)^\circ\rangle_n) - F_m(Y)$ is a component of $HS_m^n(Y)$, for each $k \in \mathbb{N}$, by Lemma 3.18 part (b), there exist $M_1^k, M_2^k, \ldots, M_{r_k}^k \in \mathfrak{A}_S(X)$ such that $h(q_X^{(n,m)}(\langle (M_1^k)^\circ, \ldots, (M_{r_k}^k)^\circ\rangle_n - F_X^m) = q_Y^{(n,m)}(\langle (J_h)^\circ, (L_k^a)^\circ\rangle_n) - F_m(Y)$. In other words,

$$g^{-1}(\langle (J_h)^\circ, (L_k^a)^\circ\rangle_n - F_m(Y)) = \langle (M_1^k)^\circ, \ldots, (M_{r_k}^k)^\circ\rangle_n - F_m(X).$$

Thus,

$$\begin{aligned}&g^{-1}\left(\mathcal{K}_m^n(J_h, Y) \cap \operatorname{cl}_{C_n(Y)-F_m(Y)}\left[\bigcup_{k=1}^{\infty}\langle (J_h)^\circ, (L_k^a)^\circ\rangle_n - F_m(Y)\right]\right) \\ &= \mathcal{K}_m^n(J, X) \cap \operatorname{cl}_{C_n(X)-F_m(X)}\left[\bigcup_{k=1}^{\infty}\langle (M_1^k)^\circ, \ldots, (M_{r_k}^k)^\circ\rangle_n - F_m(X)\right].\end{aligned} \tag{3.4}$$

Similarly for b, there exist $N_1^k, N_2^k, \ldots N_{s_k}^k \in \mathfrak{A}_S(X)$ such that

$$\begin{aligned}&g^{-1}\left(\mathcal{K}_m^n(J_h, Y) \cap \operatorname{cl}_{C_n(Y)-F_m(Y)}\left[\bigcup_{k=1}^{\infty}\langle (J_h)^\circ, (L_k^b)^\circ\rangle_n - F_m(Y)\right]\right) \\ &= \mathcal{K}_m^n(J, X) \cap \operatorname{cl}_{C_n(X)-F_m(X)}\left[\bigcup_{k=1}^{\infty}\langle (N_1^k)^\circ, \ldots, (N_{s_k}^k)^\circ\rangle_n - F_m(X)\right].\end{aligned} \tag{3.5}$$

Let $\{B_k\}_{k\in\mathbb{N}}$ be a sequence contained in $\langle J_h^\circ\rangle_n \cap C(Y)$ converging to J_h and $\{D_k\}_{k\in\mathbb{N}}$ be a sequence such that $D_k \in \langle (L_k^a)^\circ\rangle_n \cap C(Y)$, for every $k \in \mathbb{N}$. Notice that $\{D_k\}_{k\in\mathbb{N}}$ converges to $\{a\}$. Thus, $B_k \cup D_k \in \langle J_h^\circ, (L_k^a)^\circ\rangle_n$ for every $k \in \mathbb{N}$ and $\{B_k \cup D_k\}_{k\in\mathbb{N}}$ converges to J_h. Let $C = g^{-1}(J_h)$ and $C_k = g^{-1}(B_k \cup D_k)$ for each $k \in \mathbb{N}$. Notice that $C_k \in \langle (M_1^k)^\circ, \ldots, (M_{r_k}^k)^\circ\rangle_n$.

Since $C \in K_n(J, X)$, there exists $c \in C$ such that $c \in J^\circ$. Moreover, there exists $\{c_k\}_{k\in\mathbb{N}}$ a sequence converging to c such that $c_k \in C_k$ for every $k \in \mathbb{N}$. Hence, there exists $N \in \mathbb{N}$ such that if $k \geq N$, then $c_k \in J^\circ$. Therefore, $c_k \in (M_l^k)^\circ$ for some $l_1 \in \{1, \ldots, r_k\}$ and $k \geq N$. Thus, $c_k \in (M_{l_1}^k)^\circ \cap J^\circ$ for $k \geq N$. Hence, we may suppose that $l_1 = 1$ and $M_1^k = J$ for each $k \geq N$. Moreover, we may suppose that this occurs for each $k \in \mathbb{N}$, so that (3.4) becomes

$$\begin{aligned} &g^{-1}\left(\mathcal{K}_m^n(J_h, Y) \cap \mathrm{cl}_{C_n(Y)-F_m(Y)}\left[\bigcup_{k=1}^{\infty}\langle (J_h)^\circ, (L_k^a)^\circ\rangle_n - F_m(Y)\right]\right) \\ &= \mathcal{K}_m^n(J, X) \cap \mathrm{cl}_{C_n(X)-F_m(X)}\left[\bigcup_{k=1}^{\infty}\langle (J)^\circ, \ldots, (M_{r_k}^k)^\circ\rangle_n - F_m(X)\right]. \end{aligned} \tag{3.6}$$

Analogously for b, (3.5) can be rewritten as follows

$$\begin{aligned} &g^{-1}\left(\mathcal{K}_m^n(J_h, Y) \cap \mathrm{cl}_{C_n(Y)-F_m(Y)}\left[\bigcup_{k=1}^{\infty}\langle (J_h)^\circ, (L_k^b)^\circ\rangle_n - F_m(Y)\right]\right) \\ &= \mathcal{K}_m^n(J, X) \cap \mathrm{cl}_{C_n(X)-F_m(X)}\left[\bigcup_{k=1}^{\infty}\langle (J)^\circ, \ldots, (N_{s_k}^k)^\circ\rangle_n - F_m(X)\right]. \end{aligned} \tag{3.7}$$

In (3.6) and (3.7), we need $r_k = n = s_k$. To do so, let $k \in \mathbb{N}$. If $r_k < n$, then $M_t^k = M_{r_k}^k$, for each $t \in \{r_k, \ldots, n\}$. Similarly, if $s_k < n$, then $N_j^k = N_{s_k}^k$, for each $j \in \{s_k, \ldots, n\}$. Allow us to prove that we may consider $\{M_l^k\}_{k\in\mathbb{N}}$ as a sequence of pairwise different elements for each $l \in \{2, \ldots, n\}$.

Let $l \in \{2, \ldots, n\}$ and let $d_k \in M_l^k$ for each $k \in \mathbb{N}$. Since X is a continuum, we may suppose that $\{d_k\}_{k\in\mathbb{N}}$ is a sequence converging to a point $d \in X$. By Theorem 3.10, we have that $\{M_l^k\}_{k\in\mathbb{N}}$ converges to $\{d\}$. Moreover, by Theorem 3.14, it follows that $d \in \mathcal{P}(X)$. Notice that the sequence $\{M_l^k \cap C_k\}$ converges to $\{d\} \cap C$. Since $M_l^k \cap C_k \neq \emptyset$, we have that $d \in C$. Thus, $d \in J$, as $C = g^{-1}(J_h)$, and it follows that $d = p$. Hence, $\{M_l^k\}_{k\in\mathbb{N}}$ converges to $\{p\}$, for each $l \in \{2, \ldots, n\}$. Analogously, $\{N_l^k\}_{k\in\mathbb{N}}$ converges to $\{p\}$.

Now, we try to prove that the right side of (3.6) and (3.7) are equal. Let $A \in \mathcal{K}_m^n(J, X) \cap \mathrm{cl}_{C_n(X)-F_m(X)}\left[\bigcup_{k=1}^{\infty}\langle (J)^\circ, \ldots, (M_{r_k}^k)^\circ\rangle_n - F_m(X)\right]$. Hence, there exists a sequence $\{A_k\}_{k\in\mathbb{N}}$ contained in $\bigcup_{k=1}^{\infty}\langle (J)^\circ, \ldots, (M_{r_k}^k)^\circ\rangle_n - F_m(X)$ converging to A. Let $\{D_k^l\}_{k\in\mathbb{N}}$ be a sequence converging to $\{p\}$ such that

$D_k^l \subset \langle (N_l^k)^\circ \rangle_n \cap C(N_l^k)$ for each $l \in \{2, \ldots, n\}$, and consider the sequence $B_k = A_k \cap J \cup (D_k^2 \cup \cdots \cup D_k^l)$ for each $k \in \mathbb{N}$. Since $A_k \cap J \subset J^\circ$, we have that $\{B_k\}_{k \in \mathbb{N}}$ is a sequence contained in $\bigcup_{k=1}^{\infty} \langle (J)^\circ, \ldots, (N_{s_k}^k)^\circ \rangle_n - F_m(X)$. Notice that $\{B_k\}_{k \in \mathbb{N}}$ converges to $(A \cap J) \cup \{p\}$. Moreover, since $p \in A \subset J$, we have that $\{B_k\}_{k \in \mathbb{N}}$ converges to A. This concludes the first inclusion. The second inclusion follows analogously. This contradicts the fact that the left side of (3.6) and (3.7) are not equal. Therefore, $J_h \in \mathfrak{A}_R(X)$.

Hence, we have proved (1) and (2) from Theorem 3.20, so we can conclude that X and Y are homeomorphic. □

For the next result, we need the following notation:

$$\mathcal{F}_n(X) = \{A \in C_n(X) \colon \dim_A[C_n(X)] < \infty\}.$$

Moreover, this result will be used immediatily:

Theorem 3.23. *[20, Theorem 5] For a locally connected continuum X, the following are equivalent.*

(i) X is meshed,

(ii) for each $n \in \mathbb{N}$, $\mathcal{F}_n(X)$ is dense in $C_n(X)$,

(iii) there exists an $n \in \mathbb{N}$ such that $\mathcal{F}_n(X)$ is dense in $C_n(X)$.

The class of meshed continua has the property of uniqueness for the (n, m)-fold hyperspace suspension. To see this assertion, we present the following two results:

Theorem 3.24. *Let X be a meshed continuum, $n, m \in \mathbb{N}$, with $m \leq n$. If Y is a continuum such that $HS_m^n(X)$ is homeomorphic to $HS_m^n(Y)$, then Y is a meshed continuum.*

Proof. Since X is a locally connected continuum Theorem 3.7, by Theorem 2.34, we have that $HS_m^n(X)$ is a locally connected continuum. Hence, $HS_m^n(Y)$ is a locally connected continuum. Again, by Theorem 2.34, we obtain that Y is a locally connected continuum.

Let $h \colon HS_m^n(X) \to HS_m^n(Y)$ be a homeomorphism, $A \in C_n(X)$ and $B \in C_n(Y)$ such that $h(q_X^{(n,m)}(A)) = F_Y^m$ and $h^{-1}(q_Y^{(n,m)}(B)) = F_X^m$. Let $\mathcal{K} = C_n(X) - (F_m(X) \cup \{A\})$ and $\mathcal{L} = C_n(Y) - (F_m(Y) \cup \{B\})$. Define

$g\colon \mathcal{K} \to \mathcal{L}$ as $g = (q_Y^{(n,m)})^{-1} \circ h \circ q_X^{(n,m)}|_{\mathcal{K}}$, which is a homeomorphism. Since $\mathcal{K}$ and $\mathcal{L}$ are open subsets of $C_n(X)$ and $C_n(Y)$, respectively ($F_m(X) \cup \{A\}$ and $F_m(Y) \cup \{B\}$ are closed subsets of $C_n(X)$ and $C_n(Y)$, respectively), we have that $\dim_W[C_n(X)] = \dim_W[\mathcal{K}] = \dim_{g(W)}[\mathcal{L}] = \dim_{g(W)}[C_n(Y)]$, for each $W \in \mathcal{K}$. This implies $g(\mathcal{F}_n(X) \cap \mathcal{K}) = \mathcal{F}_n(Y) \cap \mathcal{L}$. By Theorem 3.23, we obtain that $\mathcal{F}_n(X)$ is a dense subset of $C_n(X)$. Since $\mathcal{K}$ is an open subset of $C_n(X)$, we obtain that $\mathcal{F}_n(X) \cap \mathcal{K}$ is a dense subset of $\mathcal{K}$. Hence, $\mathcal{F}_n(Y) \cap \mathcal{L}$ is a dense subset of $\mathcal{L}$. Since $\mathcal{L}$ is a dense subset of $C_n(Y)$, we have that $\mathcal{F}_n(Y)$ is a dense subset of $C_n(Y)$. By Theorem 3.23, we have that Y is a meshed continuum. □

The following is the main result of this section.

Theorem 3.25. *Let X be a meshed continuum, $n, m \in \mathbb{N}$, with $m \leq n$. Then X has unique hyperspace $HS_m^n(X)$.*

Proof. Let Y be a continuum and let $h\colon HS_m^n(X) \to HS_m^n(Y)$ be a homeomorphism. By [29, Theorem 3.4], we obtain this result when $m = n$. Suppose that $m < n$. Since X is a meshed continuum, by Theorem 3.24, we have that Y is a meshed continuum. By Theorem 3.22, we obtain that Y is homeomorphic to X. Therefore, X has unique hyperspace $HS_m^n(X)$. □

3.2 A class of continua without unique hyperspace

The first result of this section is a consequence of Theorem 2.41, since any two contractible non-homeomorphic locally connected continua without free arcs X and Y satisfy that $HS_m^n(X)$ and $HS_m^n(Y)$ are Hilbert cubes.

Theorem 3.26. *If X is a contractible locally connected continuum without free arcs and $n, m \in \mathbb{N}$ such that $m \leq n$, then X does not have unique hyperspace $HS_m^n(X)$.*

In [31, Theorem 3.7] a class of almost meshed locally connected continua that does not have unique hyperspace $HS_n^n(X)$ is presented. Now, we show a class of almost meshed locally connected continua that does not have unique hyperspace $HS_m^n(X)$. Most of the arguments in the proof of our main result are similar those on the proof of the main result in [31, Theorem 3.7];

this means that most of the arguments that work for $HS_n^n(X)$ will work for $HS_m^n(X)$. It is essential to write the proof of this result with the appropriate notation in this new hyperspace. The following results can be found published in [23].

The following notation helps to understand the proof of Theorem 3.29. Given a continuum X, a nonempty closed subset Z of X, and $n \in \mathbb{N}$, let

$$F_n(X, Z) = \{A \in F_n(X) \colon A \cap Z \neq \emptyset\}$$

and

$$C_n(X, Z) = \{A \in C_n(X) \colon Z \cap A \neq \emptyset\}.$$

Given two disjoint continua X and Z, and given points $x \in X$ and $z \in Z$, let $X \cup_x Z$ be the continuum obtained by attaching X to Z, identifying x to z.

Given a continuum X, $\varepsilon > 0$ and $A \in 2^X$, define $C_d(\varepsilon, p)$, to be the closed ball in X with radius ε about p, by $C_d(\varepsilon, p) = \{x \in X \colon d(p, x) \leq \varepsilon\}$. For the next results, if $\varepsilon > 0$, and locally connected continuum X with a convex metric d, define $\Phi_\varepsilon : 2^X \to 2^X$ as $\Phi_\varepsilon(A) = C_d(\varepsilon, A)$ for each $A \in 2^X$. By [42, Proposition 10.5], Φ_ε is a map which is within a distance ε from the identity function of 2^X. Furthermore, for every $A \in 2^X - F_m(X)$, we have that $\Phi_\varepsilon(C_n(X, A)) \subset C_n(X, A) - F_m(X, A)$, for $n, m \in \mathbb{N}$ with $m \leq n$.

Under this notation, consider the following results.

Theorem 3.27. *[20, Theorem 16] Let X be a locally connected continuum and R a nonempty closed subset of $\mathcal{P}(X)$. Then $C_n(X, R)$ is a Hilbert cube.*

Theorem 3.28. *[20, Theorem 18] Let X be a locally connected continuum and $p \in X$. Then there exists an uncountable family $\mathcal{D}$ of pairwise non-homeomorphic dendrites such that:*

(i) for each $D \in \mathcal{D}$, D does not contain free arcs,

(ii) the locally connected continuum $X \cup_p D$ is not homeomorphic to X, and

(iii) if $B \neq D$ are elements of $\mathcal{D}$, then $X \cup_p B$ and $X \cup_p D$ are not homeomorphic.

Theorem 3.29. *Let X be an almost meshed dendrite and $n \in \mathbb{N}$. Suppose that there exists a contractible closed subset B of $\mathcal{P}(X)$ and pairwise disjoint nonempty open subsets $U_1, \ldots, U_{n+1}$ of X such that:*

(a) $X - B = \bigcup_{i=1}^{n+1} U_i$,

(b) $B \subset \mathrm{cl}_X(U_i)$, *for each* $i \in \{1, 2, \ldots, n+1\}$.

Then X does not have unique hyperspace $HS^l_m(X)$, for every $m \leq l \leq n$.

Proof. Fix a point $b \in B$. By Theorem 3.28, there exists a dendrite D without free arcs and disjoint to X such that $Y = X \cup_b D$ is not homeomorphic to X. Notice that Y is a dendrite. Since $b \in \mathrm{cl}_X(U_i)$, for each $i \in \{1, \ldots, n+1\}$, we have that $C_l(Y)$ is homeomorphic to $C_l(X)$. In fact, the homeomorphism $h : C_l(X) \to C_l(Y)$ constructed in such proof satisfies $h(A) = A$, for each $A \in C_l(X) - C_l(X, B)$. In particular, $h(F_m(\mathcal{G}(X))) = F_m(\mathcal{G}(X))$ and since X is an almost meshed continuum, we obtain that

$$h(F_m(X)) = h(\mathrm{cl}_{C_n(X)} F_m(\mathcal{G}(X))) = \mathrm{cl}_{C_l(Y)} F_m(\mathcal{G}(X)) = F_m(X).$$

Let $\mathcal{A} = C_l(Y)/F_m(X)$ and $q^{(l,m)}_{X,Y} : C_l(Y) \to \mathcal{A}$ the quotient function. Since

$$q^{(l,m)}_X|_{C_l(X)-F_m(X)} \colon C_l(X) - F_m(X) \to HS^l_m(X) - \{F^m_X\},$$
$$h|_{C_l(X)-F_m(X)} \colon C_l(X) - F_m(X) \to C_l(Y) - F_m(X),$$

$$q^{(l,m)}_{X,Y}|_{C_l(Y)-F_m(X)} \colon C_l(Y) - F_m(X) \to \mathcal{A} - \{F^m_{X,Y}\}.$$

Then, $HS^l_m(X) - \{F^m_X\}$ is homeomorphic to $\mathcal{A} - \{F^m_{X,Y}\}$. Hence, $HS^l_m(X)$ is homeomorphic to $\mathcal{A}$. So, to prove this theorem, we only need to show that $\mathcal{A}$ is homeomorphic to $HS^l_m(Y)$.

Let $R = D \cup_b B$. We are going to prove that $q^{(l,m)}_Y(C_l(Y, R))$ and $q^{(l,m)}_{X,Y}(C_l(Y, R))$ are Hilbert cubes. By Theorem 3.27 $C_l(Y, R)$ is a Hilbert cube and by Theorem 1.54 (a), we have that $q^{(l,m)}_Y(C_l(Y, R))$ is homeomorphic to $C_l(Y, R)/F_m(Y, R)$ and $q^{(l,m)}_{X,Y}(C_l(Y, R))$ is homeomorphic to $C_l(Y, R)/F_m(X, B)$. By Theorem 1.52, it is sufficient to show that $F_m(Y, R)$ and $F_m(X, B)$ are contractible and that they are Z-sets of $C_l(Y, R)$.

Claim 1. $F_m(Y, R)$ and $F_m(X, B)$ are contractible.

Proof of Claim. Observe that B is a compact. Since B is a contractible subset of X, by Theorem 1.30, B is arcwise connected. Hence, B is a subcontinuum of X which also implies that B is a dendrite itself, due to Theorem1.20. Thus, R is a dendrite and a subcontinuum of Y. Notice that, by Theorem 1.21, each subcontinuum of Y is a strong deformation retract of Y, so there exists a map $H_1 \colon Y \times [0,1] \to Y$ such that $H_1(y,0) = y$, $H_1(y,1) \in R$ and $H_1(a,t) = a$, for each $y \in Y$, $a \in R$ and $t \in [0,1]$. Now, we define $G_1 : F_m(Y,R) \times [0,1] \to F_m(Y,R)$ by $G_1(E,t) = H_1(E \times \{t\})$. Clearly, G_1 is a map (because G_1 is the restriction of the composition of the function on $2^Y \times [0,1]$ given by $(E,t) \mapsto E \times \{t\} \in 2^{Y\times[0,1]}$ followed by the induced function $H_1^* : 2^{Y\times[0,1]} \to 2^Y$) and satisfies $G_1(A,0) = A$, for each $A \in F_m(Y,R)$, and $G_1(F_m(Y,R) \times \{1\}) = F_m(R)$. Since R is contractible Theorem 1.21, there exists $c \in R$ and a map $H_2 : R \times [0,1] \to R$ such that $H_2(a,0) = a$ and $H_2(a,1) = c$, for each $a \in R$. We define $G_2 : F_m(R) \times [0,1] \to F_m(R)$ by $G_2(E,t) = H_2(E \times \{t\})$, for each $E \in F_m(R)$ and $t \in [0,1]$. In a similar way, it can be proved that G_2 is a map. Since $G_2(G_1(E,1),0) = G_1(E,1)$ we can define the function $G : F_m(Y,R) \times [0,1] \to F_m(Y,R)$ by

$$G(E,t) = \begin{cases} G_1(E,2t), & \text{if } E \in F_m(R) \text{ and } t \in [0,\frac{1}{2}], \\ G_2(G_1(E,1),2t-1), & \text{if } E \in F_m(R) \text{ and } t \in [\frac{1}{2},1], \end{cases}$$

which is also a map. Moreover, G is a homotopy between the identity function of $F_m(Y,R)$ and a constant function. Thus, $F_m(Y,R)$ is contractible. Under similar arguments, it can be proved that $F_m(X,B)$ is contractible.

Since Y is locally connected, we can assume that the metric of Y, say d, is convex. Using the function Φ_ε, defined above, we have that $\Phi_\varepsilon(C_l(Y,R)) \subset C_l(Y,R) - F_m(Y,R)$. Hence, $F_m(Y,R)$ and $F_m(X,B)$ (which is a closed subset of $F_m(Y,R)$) are Z-sets of $C_l(Y,R)$. By Theorem 1.52, we have that

$$q_Y^{(l,m)}(C_l(Y,R)) \text{ and } q_{X,Y}^{(l,m)}(C_l(Y,R)) \text{ are Hilbert cubes.} \tag{3.8}$$

On the other hand, by Theorem 1.53, we have that

$$\mathrm{bd}_{\mathcal{A}}(q_{X,Y}^{(l,m)}(C_l(Y,R))) = q_{X,Y}^{(l,m)}(\mathrm{bd}_{C_l(Y)}(C_l(Y,R))) \tag{3.9}$$

and

$$\mathrm{bd}_{HS_m^l(Y)}(q_Y^{(l,m)}(C_l(Y,R))) = q_Y^{(l,m)}(\mathrm{bd}_{C_l(Y)}(C_l(Y,R))). \tag{3.10}$$

Moreover, by Theorem 1.54, there exists a natural homeomorphism $h_1 : q_{X,Y}^{(l,m)}(C_l(X)) \to q_Z^{(l,m)}(C_l(X))$ such that

$$h_1(q_{X,Y}^{(l,m)}(A)) = q_Y^{(l,m)}(A), \text{ for each } A \in C_l(X).$$

Since $\mathrm{bd}_{C_l(Y)}(C_l(Y,R)) \subset \mathrm{cl}_{C_l(Y)}(C_l(Y) - C_l(Y,R)) \subset C_l(X)$, we have that

$$h_1(q_{X,Y}^{(l,m)}(\mathrm{bd}_{C_l(Y)}(C_l(Y,R)))) = q_Y^{(l,m)}(\mathrm{bd}_{C_l(Y)}(C_l(Y,R))).$$

By equations (3.9) and (3.10), we have that

$$h_1(\mathrm{bd}_{\mathcal{A}}(q_{X,Y}^{(l,m)}(C_l(Y,R)))) = \mathrm{bd}_{HS_m^l(Y)}(q_Y^{(l,m)}(C_l(Y,R))).$$

Claim 2. The space $\mathrm{bd}_{HS_m^l(Y)}(q_Y^{(l,m)}(C_l(Y,R)))$ is a Z-set of $q_Y^{(l,m)}(C_l(Y,R))$.

Proof of Claim. Denote by η a metric for the hyperspace $HS_m^l(Y)$. Let $\varepsilon > 0$. Since $C_l(Y)$ is a compact subset of 2^Y, it is known that $q_Y^{(l,m)}$ is a uniformly continuous function. Hence, there exists $\delta > 0$ such that if $P, Q \in C_l(Y)$ with $H(P,Q) < \delta$, then $\eta(q_Y^{(l,m)}(P), q_Y^{(l,m)}(Q)) < \frac{\varepsilon}{2}$.

By [20, Theorem 22, Claim 2], there exists a continuous function

$$g_\delta : C_l(Y,R) \to C_l(Y,R) - \mathrm{bd}_{C_l(Y)}(C_l(Y,R))$$

such that $H(g_\delta(A), A) < \delta$ for all $A \in C_l(Y,R)$.

On the other hand, since regular sets are Z-sets of the Hilbert cube, there is a map

$$\gamma : q_Y^{(l,m)}(C_l(Y,R)) \to q_Y^{(l,m)}(C_l(Y,R)) - \{F_Y^m\}$$

with $\eta(\gamma(L), L) < \frac{\varepsilon}{2}$ for each $L \in q_Y^{(l,m)}(C_l(Y,R))$. Now, define $f = q_Y^{(l,m)}|_{C_l(Y) - F_m(Y)}$. By Theorem 1.53, it follows that

$$\mathrm{bd}_{HS_m^l(Y)}(q_Y^{(l,m)}(C_l(Y,R))) = q_Y^{(l,m)}(\mathrm{bd}_{C_l(Y)}(C_l(Y,R))).$$

Therefore, we may define the map

$$f_\varepsilon : q_Y^{(l,m)}(C_l(Y,R)) \to q_Y^{(l,m)}(C_l(Y,R)) - \mathrm{bd}_{HS_m^l(Y)}(q_Y^{(l,m)}(C_l(Y,R)))$$

as $f_\varepsilon(L) = q_Y^{(l,m)} \circ g_\delta \circ f^{-1} \circ \gamma(L)$ for each $L \in q_Y^{(l,m)}(C_l(Y,R))$.

Given $L \in q_Y^{(l,m)}(C_l(Y,R))$, we have that $H(g_\delta(f^{-1}(\gamma(L))), f^{-1}(\gamma(L))) < \delta$. Hence, $\eta(q_Y^{(l,m)}(g_\delta(f^{-1}(L))), q_Y^{(l,m)}(f^{-1}(\gamma(L)))) < \frac{\varepsilon}{2}$. Thus, $\eta(f_\varepsilon(L), \gamma(L)) < \frac{\varepsilon}{2}$. Since $\eta(\gamma(L), L) < \frac{\varepsilon}{2}$, it follows that $\eta(f_\varepsilon(L), L) < \varepsilon$. From here, the Claim is true.

With arguments analogous to those in the previous claim, we obtain that $\mathrm{bd}_{\mathcal{A}}(q_{X,Y}^{(l,m)}(C_l(Y,R)))$ is a Y-set of $q_{X,Y}^{(l,m)}(C_l(Y,R))$. Since

$$h_1|_{\mathrm{bd}_{\mathcal{A}}(q_{X,Y}^{(l,m)}(C_l(Y,R)))}\colon \mathrm{bd}_{\mathcal{A}}(q_{X,Y}^{(l,m)}(C_l(Y,R))) \to \mathrm{bd}_{HS_m^l(Y)}(q_Y^{(l,m)}(C_l(Y,R)))$$

is a homeomorphism and using sentence (3.8), by Anderson's Homogeneity Theorem 1.50 there exists a homeomorphism

$$h_2 : q_{X,Y}^{(l,m)}(C_l(Y,R)) \to q_Y^{(l,m)}(C_l(Y,R))$$

such that $h_2(A) = h_1|_{\mathrm{bd}_{\mathcal{A}}(q_{X,Y}^{(l,m)}(C_l(Y,R)))}(A)$, for each $A \in \mathrm{bd}_{\mathcal{A}}(q_{X,Y}^{(l,m)}(C_l\ (Y,R)))$. We define $h : \mathcal{A} \to HS_m^l(Y)$ by

$$h(A) = \begin{cases} h_1(A), & \text{if } A \in \mathrm{cl}_{\mathcal{A}}(\mathcal{A} - q_{X,Y}^{(l,m)}(C_l(Y,R))), \\ h_2(A), & \text{if } A \in q_{X,Y}^{(l,m)}(C_l(Y,R)). \end{cases}$$

Then h is a homeomorphism. Therefore X does not have unique hyperspace $HS_m^l(X)$. □

Example 3.30. *Allow us to present an illustrative example of an almost meshed dendrite that satisfies conditions on Theorem 3.29. In fact, consider the dendrite X from Example 3.6 (b) and attach $n+1$ arcs to the point (0,0) (which is the only point contained in $\mathcal{P}(X)$). In this way, X does not have unique hyperspace $HS_m^l(X)$, for every $m \leq l \leq n$.*

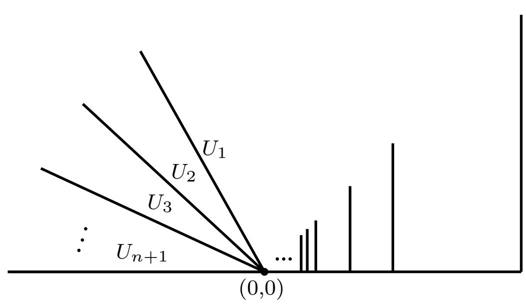


Figure 3.4: Caption

Theorem 3.31. *Let X be a dendrite that is not almost meshed. Then, for every $n, m \in \mathbb{N}$ with $m \leq n$, X does not have unique hyperspace $HS_m^n(X)$.*

Proof. Let $E = C_d(p, \varepsilon)$. Since X is a dendrite, we may assume that d is a convex metric. As X is not almost meshed, there exist $r > 0$ and $p \in \mathcal{P}(X)$ such that $p \in B(p, 2r) \subset \mathcal{P}(X)$. Since d is a convex metric, E is a subcontinuum of X. By Theorem 1.21, E is a dendrite, hence a contractible continuum. By Theorem 3.28, there exists a dendrite D without free arcs and disjoint to X such that $Y = X \cup_b D$ is not homeomorphic to X. Notice that Y is a dendrite. Let $\varepsilon > 0$ and define $\phi_\varepsilon : 2^X \to 2^X$ defined as $\phi_\varepsilon(A) = C_d(\varepsilon, A)$. If we define $f_\varepsilon : C_n(X, E) \to C_n(X, E) - F_m(E)$ by $f_\varepsilon)A = \phi_\varepsilon(A)$, it follows that $F_m(E)$ is a Z-set of $C_n(X, E)$ and $C_n(Y, E \cup D)$.

By [20, Theorem 22, Claim 2], we have that $\mathrm{bd}_{C_n(X)}(C_n(X, E)) \cup F_m(E)$ is a Z-set of $C_n(X, E)$ and $\mathrm{bd}_{C_n(Y)}(C_n(Y, E \cup D)) \cup F_m(E)$ is a Z-set of $C_n(Y, E \cup D)$. Furthermore, by Theorem 3.27 the identity map

$$id : \mathrm{bd}_{C_n(X)}(C_n(X, E)) \cup F_m(E) \to \mathrm{bd}_{C_n(Y)}(C_n(Y, E \cup D)) \cup F_m(E)$$

is a well-defined homeomorphism. By Theorem 3.27, it follows that $C_n(X, E)$ and $C_n(Y, E \cup D)$ are Hilbert cubes. Thus, by Theorem 1.50, this identity map may be extended to a homeomorphism $h' : C_n(X, E) \to C_n(Y, E \cup D)$. Define $h : C_n(X, E) \to C_n(Y, E \cup D)$ by

$$h(A) = \begin{cases} h'(A), & \text{if } A \in C_n(X, E), \\ A, & \text{if } A \in C_n(X) - C_n(X, E). \end{cases}$$

Since h' is a homeomorphism over $C_n(X, E)$ and h behaves as the identity function over $C_n(X) - C_n(X, E)$, we have that h is a homeomorphism, see Theorem 1.55. Notice that h is a homeomorphism such that $h(F_m(X)) = F_m(X)$.

Let $q_{X,Y}^{(n,m)} : C_n(Y) \to C_n(Y)/F_m(X)$ the quotient function, where $q_{X,Y}^{(n,m)}(F_m(X)) = \{F_{X,Y}^m\}$. Since $q_X^{(n,m)}|_{C_n(X)-F_m(X)}$, $h|_{C_n(X)-F_m(X)}$ and $q_{X,Y}^{(n,m)}|_{C_n(Y)-F_m(X)}$ are all homeomorphisms, we have that $HS_m^n(X) - \{F_X^m\}$ is homeomorphic to $C_n(Y)/F_m(X) - \{F_{X,Y}^m\}$. Hence, $HS_m^n(X)$ is homeomorphic to $C_n(Y)/F_m(X)$. In order to complete this proof, we shall prove that $C_n(Y)/F_m(X)$ is homeomorphic to $HS_m^n(Y)$. To do so, allow us to show that $q_Y^{(n,m)}(C_n(Y, E \cup D))$ and $q_{X,Y}^{(n,m)}(C_n(Y, E \cup D))$ are Hilbert cubes. Observe that $q_Y^{(n,m)}(C_n(Y, E \cup D))$ and $q_{X,Y}^{(n,m)}(C_n(Y, E \cup D))$ are homeomorphic to $C_n(Y, D)/F_m(Y, E)$ and $C_n(Y, E \cup D)/F_m(Y, E)$, respectively. By Claim 1 in Theorem 3.29, $F_m(Y, E)$ and $F_m(Y, E \cup D)$ are contractible.

By [42, Proposition 10.5], given $\varepsilon > 0$, the function $\Phi_\varepsilon : 2^Y \to 2^Y$ previously defined satisfies $\Phi_\varepsilon(C_n(Y, E \cup D)) \subset C_n(Y, E \cup D) - F_m(Y, E \cup D)$.

Thus, $F_m(Y, E \cup D)$ and $F_m(E)$ are Z-sets of $C_n(Y, E \cup D)$. By Theorem 1.53, it follows that $q_Y^{(n,m)}(C_n(Y, E \cup D))$ and $q_{X,Y}^{(n,m)}(C_n(Y, E \cup D))$ are Hilbert cubes, as we wanted.

Similarly as Claim 2 from Theorem 3.29 was proven, the following result is also true.

Claim. The space $\mathrm{bd}_{HS_m^n(Y)}(q_Y^{(n,m)}(C_n(Y, E \cup D)))$ is a Z-set of $q_Y^{(n,m)}(C_n(Y, E \cup D))$ and the set $\mathrm{bd}_{C_n(Y)/F_m(X)}(q_{X,Y}^{(n,m)}(C_n(Y, E \cup D)))$ is a Y-set of $q_{X,Y}^{(n,m)}(C_n(Y, E \cup D))$.

By Theorem 1.54, the function $f : q_{X,Y}^{(n,m)} \to q_Y^{(n,m)}(C_n(X))$ defined by $f(q_{X,Y}^{(n,m)}(A)) = q_X^{(n,m)}(A)$, for each $A \in C_n(X)$ is a homeomorphism. Therefore,

$$f(q^{(n,m)}(\mathrm{bd}_{C_n(Y)}(C_n(Y, E \cup D)))) = q_Y^{(n,m)}(\mathrm{bd}_{C_n(Z)}(C_n(Y, E \cup D)))$$

and it follows that

$$f(\mathrm{bd}_{C_n(Y)/F_m(X)}(q_{X,Y}^{(n,m)}(C_n(Y, E \cup D)))) = \mathrm{bd}_{HS_m^n(Y)}(q_{X,Y}^{(n,m)}(C_n(Y, E \cup D))).$$

Consequently, $f|_{\mathrm{bd}_{C_n(Y)/F_m(X)}(q_{X,Y}^{(n,m)}(C_n(Y,E\cup D)))}$ is a homeomorphism between Z-sets $\mathrm{bd}_{C_n(Y)/F_m(X)}(q_{X,Y}^{(n,m)}(C_n(Y, E \cup D)))$ and $\mathrm{bd}_{HS_m^n(Y)}(q_{X,Y}^{(n,m)}(C_n(Y, E \cup D)))$. By Theorem 1.50, there exists a homeomorphism

$$g : q_{X,Y}^{(n,m)}(C_n(Y, E \cup D)) \to q_Y^{(n,m)}(C_n(Y, E \cup D)),$$

such that $g(A) = f(A)$ for each $A \in \mathrm{bd}_{C_n(Y)/F_m(X)}(q_{X,Y}^{(n,m)}(C_n(Y, E \cup D)))$.

Now, define $\overline{h} : C_n(Y)/F_m(X) \to HS_m^n(Y)$ by

$$\overline{h}(A) = \begin{cases} f(A), & \text{if } A \in C_n(Y)/F_m(X) - q_{X,Y}^{(n,m)}(C_n(Y, E \cup D)), \\ g(A), & \text{if } A \in q_{X,Y}^{(n,m)}(C_n(Y, E \cup D)). \end{cases}$$

Hence, $\overline{h}$ is a homeomorphism and it follows that X does not have unique hyperspace $HS_m^n(X)$. □

Chapter 4

Conclusions

In this section, the authors would like to show the following works that emerge from this project:

- G. Hernández-Valdez, D. Herrera-Carrasco, M. de J. López, F. Macías-Romero, *Uniqueness of the* (n, m)*-fold hyperspace suspension for continua*, Topology Appl. 325 (2023) 108385, 23 pp.

- G. Hernández-Valdez, D. Herrera-Carrasco, M.J. López, F. Macías-Romero, *Properties of the* (n, m)*-fold hyperspace suspension of continua*, Rev. Integr. Temas Mat.,40 (2022), No. 2,159–168.

From the study of the (n, m)-fold hyperspace suspension of a continuum, its general properties and the uniqueness, we pose the following questions:

Question 4.1. *For what continua* X *does the natural embedding in the proof of Theorem 2.13 embed* $HS_m^s(X)$ *as a retract of* $HS_m^n(X)$ $(m \leq s < n)$*? In particular, what about the case when* X *is* $\mathcal{S}^1$*?*

Corollary 2.38 and Theorem 2.39 give a partial answer to the more general problem:

Question 4.2. *For what continua* X*, can* $HS_m^s(X)$ *be embedded in* $HS_m^n(X)$ *as a retract* $(m \leq s < n)$*?*

Question 4.3. *What continua* X *have the property that* $HS_m^n(X)$ *is contractible for each* $n, m \in \mathbb{N}$ *with* $m \leq n$*?*

Question 4.4. *If* X *is decomposable and* $n, m \in \mathbb{N}$ *with* $m < n$*, is* $HS_m^n(X)$ *locally arcwise connected at* F_X^m*?*

Question 4.5. *What continua X have the property that $HS_m^n(X)$ is pseudo-contractible for each $n, m \in \mathbb{N}$ with $m \leq n$?*

Related to Theorem 2.52,

Question 4.6. *Is Theorem 2.52 still true if we remove the assumption that $h(F_X^m) = F_Y^t$?*

Related to Theorem 3.22 the following question extends [31, Question 3.8].

Question 4.7. *Are the members of the class of almost meshed locally connected continua*

(1) HS_1^n-determined?, for $n \in \mathbb{N}$,

(2) HS_2^2-determined?

Related to Theorem 3.31, the following question is asked.

Question 4.8. *If X is a locally connected continuum that is not almost meshed, does X have unique hyperspace $HS_m^n(X)$ for each $n, m \in \mathbb{N}$ with $m \leq n$?*

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