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A study about (P,Q) and ϵ-{2,3,4} generalized inverses

THESIS

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Dedicatory

To my parents,
Eleazar and Sara.

To my brothers,
Andres and Giovanni.

Gratitudes

I want to gratitude, in a very special way, my parents who have always supported me unconditionally in all the decisions I have made in my life, they who have always given me everything a son can ask for and despite the shortcomings they suffered always tried to give me the best. I owe them everything I am and where I am now. That's why this work is dedicated to them.

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Introduction

The concept of generalized inverse in mathematics is a generalization of the notion of inverse of a matrix or function. It is a useful tool to solve systems of linear and nonlinear equations, as well as to study properties of linear or nonlinear operators. There are different types of generalized inverses, according to the type of relationship considered and the conditions imposed on the objects of study.

The concept as such has a long history dating back to the 19th century, when ill-posed linear problems of differential and integral equations were studied. One of the first to introduce a notion of generalized inverse was E. I. Fredholm [16], who in 1903 solved certain systems of functional equations using what he called a pseudo-inverse. For example, if f is a continuous function on $[0, 1]$ and K is a continuous kernel in the square $[0, 1] \times [0, 1]$, Fredholm's integral equation is:

$$f(x) = \int_0^1 K(x, y)g(y)dy + h(x),$$

where h is a known continuous function and g is the unknown function being found. Fredholm showed that if the kernel K satisfies certain conditions, then a pseudo-inverse of the form exists:

$$g(x) = \int_0^1 L(x, y)f(y)dy - m(x),$$

where L and m are continuous functions that depend on K and h. The pseudo-inverse of Fredholm allows finding the g solution from the given f function.

Later, in 1920, E.H. Moore was the first one to give an explicit definition of the generalized inverse of an arbitrary matrix. This definition was given in an abstract published in the Bulletin of the American Mathematical Society, which was either little noticed perhaps owing to Moore's very individualistic terminology and notation [31]. Another author who worked with the notion of generalized inverse for matrices was A. Bjerhammer in a paper published in 1951 [5].

The theory experienced a revival with the publication of a paper on matrix generalized inverses by R. Penrose in 1955 [34]. Penrose showed that for every finite rectangular matrix A, there exists a single matrix $A^\dagger$ such that it satisfies:

$$\begin{array}{ll} (1) \quad AA^\dagger A = A, & (2) \quad A^\dagger A A^\dagger = A^\dagger, \\ (3) \quad (AA^\dagger)^* = AA^\dagger, & (4) \quad (A^\dagger A)^* = A^\dagger A, \end{array}$$

where A^* denotes the conjugate transpose matrix of A. Among its applications we can find the solution by least squares of a system of linear equations, to define matrix partial orders such as the star order and in the study of general relativity. The equivalence of Penrose's definition of the generalized inverse to that of Moore was pointed out by R. Rado in 1956 [35]. So far, a large number of work have been done on generalized inverses concerning matrices as can be seen in [3] and [39].

Generalized inverses were studied, perhaps for the first time, in 1936 from the point of view of operator theory by Y. Tseng in his doctoral thesis [38], but it wasn't until 1949, in a series of important articles, that defined a generalized inverse for linear operators in Hilbert spaces. From this, the concept of generalized inverse has spread and generalized to various algebraic and topological contexts.

Generalized inverse theory is elegant and has numerous applications in areas such as linear algebra, functional analysis, matrix theory, optimization, control, statistics, physics, robotics (for example see [26], [3] and [14]).

A recent approach in the study of generalized inverses, from the point of view of operator theory, has been the block matrix representation of operator since this form allows a better understanding of the phenomenon to be considered, and simplify the demonstration of some results. This type of approach can be found in the following works [10], [21], [22], [23], among others.

Let $X \neq \{0\}$ is a Banach space, then $B(X)$ denoted the collection of all bounded linear operators of X in X, moreover $B(X)$ is a Banach algebra with identity. Usually when generalized inverses between Banach spaces are considered, work with operators T in $B(X)$.

In this type of spaces besides considering the topological properties obtained for the operator T, for being X a Banach space, also have very typical algebraic properties of algebra, which help to obtain a single generalized inverse of T.

However, despite the great advantages of working with a Banach algebra we can find certain limitations when considering the block matrix representation of operator T, since in this type of approach the domain and codomain decompositions of T, with respect to the direct sum of the corresponding subspaces, turns out to be very similar in both cases or in some others the same. Therefore, if we consider $T \in B(X, Y)$, with X and Y different Banach spaces, we can guarantee that is minor the relationship between the decompositions for X and Y with respect to T.

We will consider the definition used in [22] by Kantún-Montiel for a ring R with identity 1 which is as follows: Let $p, q \in \mathcal{R}$ be idempotents and $a \in \mathcal{R}$. An element $b \in \mathcal{R}$ is an image-kernel (p,q)-inverse of a if

$$bab = b, \;\; baR = pR, \;\; (1 - ab)R = qR.$$

Later, in 2014 this inverse was studied by Mosić et al. [32], where an algebraic characterization was given without the explicit use of ideals of the ring. The above to define a new generalized inverse in the context of bounded linear operators between Banach spaces. This inverse will be characterized as an outer inverse with range and nullspace prescribed. The idea of prescribing was considered by Djordjević and Wei in [12]. However, some of these ideas can be traced back at least to Bott-Duffin in 1953 [6]. Drazin himself considered that Bott-Duffin's ideas have to be recognized as a direct precursor of the (b, c)-inverse with b, c elements in a semigroup and, consenquently, of most other known types of uniquely defined outer generalized inverses [13].

Also we consider Campbell's idea: Let $A \in \mathcal{M}^{m\times n}$ (the space of rectangular arrays of size $m \times n$) and consider the equations:

$$(1) \;\; AXA - A = E_1, \qquad (2) \;\; XAX - X = E_2,$$
$$(3) \;\; AX - X^*A^* = E_3, \quad (4) \;\; XA - A^*X^* = E_4.$$

Where, X was thought as the computed estimate and E_i as error terms (see [9]). Then, he sought a solution to the equations system proposed.

So, if we consider $B(\mathcal{H},\mathcal{K})$ the set of bounded linear operators with $\mathcal{H}$ and $\mathcal{K}$ be a infinite dimension complex Hilbert spaces, then we define a new generalized inverse, called ϵ-{2,3,4} inverse. This inverse exists even if operator doesn't has closed range.

Therefore, the objectives of this thesis work are:

- Define a new generalized inverse for bounded linear operators between Banach spaces. Give a block matrix representation of the operator and its generalized inverse.

- Define a new generalized inverse for bounded linear operators between infinite-dimensional complex Hilbert spaces, where the operators considered does not necessarily have a closed range. Give a block matrix representation of the operator and its generalized inverse.

Using block matrix representations of an operator, we will have a detailed structure of its generalized inverse. In this way, we can approach parts of generalized inverse and modify it to do better, in order to obtain a unique inverse.

The thesis contains 4 chapters.
Chapter 1 contains basic notations and results that are used in the thesis work.
Chapter 2 presents the image-kernel (P,Q)-inverse study for bounded linear operators between Banach spaces. Section 2.2 defines the image-kernel (P,Q)-inverse of an operator (Definition 2.1). Theorem 2.2 characterizes the (P,Q)-inverse as an external inverse with prescribed range and nullspace. Another characterization of the (P,Q)-inverse is given in Theorem 2.3 and Proposition 2.4 gives a block matrix representation of the operator and its (P,Q)-inverse. Section 2.3 discusses the relationship of the (P,Q)-inverse to other generalized inverses that already exist, such as: group inverse, Drazin inverse, generalized Drazin inverse and Moore-Penrose inverse. Section 2.4 discusses the (P,Q)-inverse over Hilbert spaces.

Chapter 3 is devoted to the study of ϵ-{2,3,4} inverse for bounded linear operators on Hilbert spaces. Section 3.2 discusses the reduced minimum module of an operator and some of its results. Section 3.3 defines the ϵ-{2,3,4} inverse of an operator (Definition 3.5) and there are some immediate consequences from the definition. Section 3.4 has one of the main outcomes of this chapter, Theorem 3.7, where a characterization of the ϵ-{2,3,4} inverse is given. In Section 3.5 we give an estimate of the error in norm between the MP-inverse and an arbitrary ϵ-{2,3,4} inverse, Theorem 3.9. The Section 3.6 discusses the ϵ-{2,3,4} inverse of the module of an operator. The results necessary to prove the Theorem 3.15 are also given, which tells us that T has an ϵ-{2,3,4} inverse if and only if $|T|$ has one. The Section 3.8 mentions results over the spectrum of positive operators that have an ϵ-{2,3,4} inverse.
Finally, Chapter 4 gives a construction of the ϵ-{2,3,4} inverse for compact operators in $B(\mathcal{H})$.

The main results used in the development and presentation of this thesis are included in the following works:

1. E. Salgado-Matias, S.V. Djordjević and G. Kantún-Montiel, *The operator image-kernel (P,Q)-inverse.* Complex Analysis and Operator Theory. Submitted.

2. E. Salgado-Matias, S.V. Djordjević and G. Kantún-Montiel, *The ϵ-{2,3,4} inverse for bounded linear operators on Hilbert spaces.* Submitted.

Contents

Chapter 1

Preliminaries

In this chapter we present several results about the theory of operators and analysis. We make the observation that all the results of this chapter will not be given their demonstrations, but you can review the bibliography for it in [1], [20], [26], [27] and [33].

1.1 Basic definitions

We begin with some basic definitions.

Definition 1.1. *Let $\mathbb{F}$ a field. A linear space (or vector space) over $\mathbb{K}$ is a nonempty set X (whose elements are called vectors) satisfying the following:*

1. *A binary operation on X, called vector addition and denoted by $+$ such that $(X, +)$ is a group.*

2. *There is a binary operation $\mathbb{F} \times X$ into X, called scalar multiplication, that assigns to each scalar $\alpha \in \mathbb{K}$ and each vector $x \in X$, a vector αx in X, with the following properties. For all α and β in $\mathbb{F}$, and all vectors x and y in X:*

    - $1x = x$,
    - $\alpha(\beta x) = (\alpha \cdot \beta)x$,

- $\alpha(x+y) = \alpha x + \alpha y$,
- $(\alpha + \beta)x = \alpha x + \beta y$.

Let X a linear space over a field $\mathbb{F}$. A function

$$\|\cdot\| : X \to \mathbb{R}$$

is a norm over X, if satisfying the following conditions for all vectors $x, y \in X$ and all scalars $\alpha \in \mathbb{F}$:

- $\|x\| \geq 0$,
- $\|x\| > 0$ if $x \neq 0$,
- $\|\alpha x\| = |\alpha|\|x\|$,
- $\|x + y\| \leq \|x\| + \|y\|$.

A linear space X endowed with a norm is called normed space. A Banach space is a complete normed space, i.e., a Banach space is a normed space that is complete as a metric space with respect to the metric generated by the norm.

Let $T : X \to Y$ a function with X and Y a linear spaces over the same field $\mathbb{F}$. We say that T is a linear transformation if T is homogeneous, i.e., $T(\alpha x) = \alpha(Tx)$ for all $x \in X$ and $\alpha \in \mathbb{F}$ and T is additive, i.e., $T(x + y) = Tx + Ty$ for all $x, y \in X$. For $T : X \to Y$, we will denote by $R(T)$ the range of T and by $N(T)$ the null space of T.

Definition 1.2. *A linear transformation T of a normed space X into a normed space Y is bounded if there exists a constant $\beta \geq 0$ such that*

$$\|Tx\| \leq \beta\|x\|$$

for every $x \in X$.

Suppose that we have that X and Y are normed spaces, then we denote $B(X, Y)$ to the set of all bounded linear transformations. We can define one norm for $B(X,Y)$, this norm are defined by:

$$\begin{aligned}\|T\| &= \sup_{\|x\|_X=1} \|Tx\|_Y \\ &= \min\{M \in \mathbb{R} : \|Tx\|_Y \leq M\|x\|_X \text{ for all } x \in X\}.\end{aligned}$$

One of the main results in Banach space theory is the inverse mapping theorem which says:

Theorem 1.3. *[26, Theorem 4.22](The inverse mapping theorem) If X and Y are Banach spaces and $T \in B(X,Y)$ is injective and surjective, then $T^{-1} \in B(Y,X)$.*

In particular $B(X) = B(X,X)$ is a $\mathbb{K}$-algebra, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Moreover, if X is a Banach space then $B(X)$ is a Banach algebra, since $B(X)$ be will Banach space.

The algebraic structure of a normed space allowed us to operate with vectors, and its topological structure gave us a notion of closeness, which interacts harmoniously with the algebraic operations. But, the algebra and topology are not enough to extend to abstract spaces the geometric concept of relative direction (or angle) between vectors, that is familiar in Euclidean geometry. However, the concept of orthogonality that emerges when we equip a linear space with an inner product will be the key for it. Therefore, we introduce the definition of inner product.

Definition 1.4. *Let X be a linear space over $\mathbb{F}$. A functional*

$$\langle\, , \,\rangle : X \times X \to \mathbb{F}$$

is an inner product on X if the following conditions are satisfied for all $x, y, z \in X$ and all $\alpha, \beta \in \mathbb{F}$:

- $\langle x, x\rangle > 0$ *for all* $x \neq 0$.
- $\langle x, x\rangle = 0$ *if and only if* $x = 0$.
- $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$.
- $\langle x, y\rangle = \overline{\langle y, x\rangle}$.

Proposition 1.5. *[26, Proposition 5.3] Let X a linear space endowed with a inner product $\langle\ ,\ \rangle : X \times X \to \mathbb{F}$. Then the function $\|\ \| : X \to \mathbb{R}$, defined by*

$$\|x\| = \langle x, x\rangle^{\frac{1}{2}}$$

for each $x \in X$, is a norm on X.

So, we can define a Hilbert space as an inner product space that is complete as a metric space with respect to the metric generated by the norm induced by the inner product. In fact, every Hilbert space is a Banach space since a Hilbert space is a Banach space whose norm is induced by an inner product.

Definition 1.6. *Two vectors x and y in any inner product space $(X, \langle\ ,\ \rangle)$ are said to be orthogonal if $\langle x, y\rangle = 0$ (notation: $x \perp y$).*

As a consequence of the above definition, if $M, N \subseteq X$ then we have; $x \perp M$ if $\langle x, y\rangle$ for all $y \in M$, and $M \perp N$ if $\langle x, y\rangle$ for all $x \in M$ and $y \in N$. Therefore, if M is a subset of an inner product space $(X, \langle\ ,\ \rangle)$, then we can define the orthogonal complement of M as the set

$$M^{\perp} := \{x \in X : x \perp M\}.$$

If N is another subset of X, then it is readily verify that

- $M \perp M^{\perp}$ and $M \cap M^{\perp} = \{0\}$ if $0 \in M$, otherwise $M \cap M^{\perp} \subseteq \{0\}$.
- $M \perp N$ if and only if $M \subseteq N^{\perp}$ if and only if $N \subseteq M^{\perp}$.
- $M \perp N$ if and only if $N \perp M$.
- $M \subset N$ implies $N^{\perp} \subset M^{\perp}$.

Proposition 1.7. *[26, Proposition 5.12] The orthogonal complement $M^{\perp}$ of every subset M of any inner product space X is a subspace closed of X.*

We will denote by $M \sqsubseteq X$ to say that $M \subseteq X$ is a closed subspace of X.

1.2 Some operators class in B(H)

Let $\mathcal{H}$ a Hilbert space, then $B(\mathcal{H})$ be a Banach algebra. We observe that if there exists T^{-1} for some $T \in B(\mathcal{H})$, then $T^{-1} \in B(\mathcal{H})$, [26, Corrollary 4.24]. We denote this operators in $B(\mathcal{H})$ by:

$$\begin{aligned} \mathcal{GL} &= \{T \in B(\mathcal{H}) : N(T) = \{0\},\, R(T) = \mathcal{H}\} \\ &= \{T \in B(\mathcal{H}) : \exists T^{-1} \in B(\mathcal{H})\}. \end{aligned}$$

Note that $\mathcal{GL}(\mathcal{H})$ is a group with the product in $B(\mathcal{H})$ (usual composition of functions). Moreover, $\mathcal{GL}(\mathcal{H})$ is open in $B(\mathcal{H})$.

Let $\mathcal{H}, \mathcal{K}$ a Hilbert spaces. We know that for $T \in B(\mathcal{H}, \mathcal{K})$ there exists a unique $T^* \in B(\mathcal{K}, \mathcal{H})$ such that $\langle Tx, y\rangle_{\mathcal{K}} = \langle x, T^*y\rangle_{\mathcal{H}}$ for all $(x, y) \in \mathcal{H} \times \mathcal{K}$ ([26, Proposition 5.65]). A useful consequence is $\|T\| = \sup_{\|x\|=1=\|y\|} |\langle Tx, y\rangle|$ ([26, Corollary 5.71]) and also, $T \in \mathcal{GL}(\mathcal{H}, \mathcal{K})$ if and only if $T^* \in \mathcal{GL}(\mathcal{K}, \mathcal{H})$ and $(T^*)^{-1} = (T^{-1})^*$ ([36, Lemma 6.14]).

Let $\mathcal{H}$ a Hilbert space and $T \in B(\mathcal{H})$. We say that T is a normal operator if $TT^* = T^*T$. If $T = T^*$ then T is a self-adjoint operator and we denote the set of all operators self-adjoint by:

$$\mathcal{A}(\mathcal{H}) = \{T \in B(\mathcal{H}) : T^* = T\},$$

this is a $\mathbb{K}$-subspace closed of $B(\mathcal{H})$. Some properties for this operators are the following:

Proposition 1.8. *[36, Lemma 6.8] Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces, let R, $S \in B(\mathcal{H}, \mathcal{K})$ and let $T \in B(\mathcal{K}, \mathcal{L})$. Let $\lambda\, \mu \in \mathbb{C}$. Then*

(a) $(\mu R + \lambda S)^* = \overline{\mu} R^* + \overline{\lambda} S^*$,

(b) $(TR)^* = R^*T^*$.

Theorem 1.9. *[36, Theorem 6.10] Let $\mathcal{H}$ and $\mathcal{K}$ Hilbert spaces and let $T \in B(\mathcal{H}, \mathcal{K})$.*

(a) $(T^*)^* = T$.

(b) $\|T^*\| = \|T\|$.

(c) $\|T^*T\| = \|T\|^2$.

Also, we say that T is positive if $T \in \mathcal{A}(\mathcal{H})$ and $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. Therefore we denote by

$$B(\mathcal{H})^+ = \{T \in B(\mathcal{H}) : T \geq 0\},$$

to set of all positive operators in $B(\mathcal{H})$, which is a cone convex closed in $\mathcal{A}(\mathcal{H})$.

Another operators class are the unitary operators ($T \in \mathcal{GL}(\mathcal{H})$ and $T^{-1} = T^*$). We denote by

$$\mathcal{U}(\mathcal{H}) := \{T \in B(\mathcal{H}) : T \text{ is unitary operator }\},$$

to set all unitary operators. Note that $\mathcal{U}(\mathcal{H})$ is a subgroup of $\mathcal{GL}(\mathcal{H})$.

If $\mathcal{H}$ is a Hilbert space, then $\mathcal{P}(\mathcal{H}) := \{T \in B(\mathcal{H}) : T^2 = T\}$ (oblique projector). We have the next result:

Proposition 1.10. *[26, Proposition 5.81] Let $P \in \mathcal{P}(\mathcal{H})$, and let $N = R(P) \sqsubseteq \mathcal{H}$. Then are equivalent:*

(a) $P = P_N$, *i.e.,* $N(P) = N^\perp$.

(b) $\|P\| = 1$.

(c) $P \in B(\mathcal{H})^+$.

(d) $P \in \mathcal{A}(\mathcal{H})$, *i.e.,* $P^* = P$.

(e) P *is normal operator.*

The operators that satisfy the proposition above are called orthogonal projector or projector, and are denoted by:

$$\mathbb{P}(\mathcal{H}) = \{P \in \mathcal{P}(\mathcal{H}) : P = P^*\}.$$

We can proof that the order $\leq$ of $\mathcal{A}(\mathcal{H})$, restricted to set $\mathbb{P}(\mathcal{H})$, there it does produce a lattice. In other words, if $P, Q \in \mathbb{P}(\mathcal{H})$ such that $R(P) = N$ and $R(Q) = M$, then they are equivalent:

- $P \leq Q$,
- $N \subseteq M$,
- $PQ = QP = P$,
- $Q - P \in \mathbb{P}(\mathcal{H})$ ($(Q-P)$ project onto $M \cap (N \cap M)^{\perp}$).

Another definition that we need is the invariant subspace of a operator $T \in B(\mathcal{H})$.

Definition 1.11. *Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})$. Let $M \subseteq \mathcal{H}$ a subspace, then we say that:*

1. *M is invariant by T (or T-invariant), if $T(M) \subseteq M$.*
2. *M reduces to T, if M and $M^{\perp}$ are T-invariant.*

So, we know that $N(T)$ and $\overline{R(T)}$ are invariant subspaces for $T \in B(\mathcal{H})$. The next result show another invariant subspaces of T.

Proposition 1.12. *[26, Proposition 5.76] If $T \in B(\mathcal{H}, \mathcal{K})$ with $\mathcal{H}, \mathcal{K}$ a Hilbert spaces, then*

(a) $N(T) = R(T^*)^{\perp} = N(T^*T)$.

(b) $\overline{R(T)} = N(T^*)^{\perp} = \overline{R(TT^*)}$.

(c) $N(T^*) = R(T)^{\perp} = N(TT^*)$.

(d) $\overline{R(T^*)} = N(T)^{\perp} = \overline{R(T^*T)}$.

The subspaces family mentioned in the Definition 1.11 we can denoted by:

$$Lat(T) = \{M \sqsubseteq \mathcal{H} : T(M) \subseteq M\}$$

and

$$Lat_r(T) = \{M \sqsubseteq \mathcal{H} : M \text{ reduces to } T\}.$$

Remark.

- *If $M \in Lat(T)$, then $\overline{M} \in Lat(T)$.*

- *If* $M \sqsubseteq \mathcal{H}$ *and let* $P_M \in \mathbb{P}(\mathcal{H})$*, then*

$$M \in Lat(T) \Longleftrightarrow TP_M = P_M T P_M$$

and

$$M \in Lat_r(T) \Longleftrightarrow TP_M = P_M T.$$

Identifying $M \longleftrightarrow P_M$ *we can say that*

$$Lat_r(T) \subseteq Lat(T) \subseteq \mathbb{P}(\mathcal{H}).$$

- *Let* $N, M \in Lat(T)$*:*

$$N \wedge M := N \cap M \in Lat(T)$$

and

$$N \vee M := \overline{N + M} \in Lat(T).$$

Similarly for $Lat_r(T)$*, since let* $N, M \sqsubseteq \mathcal{H}$ *arbitrary we have that*

$$(N \vee M)^{\perp} = (N + M)^{\perp} = N^{\perp} \wedge M^{\perp}$$

and

$$(N \wedge M)^{\perp} = N^{\perp} \vee M^{\perp}.$$

Proposition 1.13. *[26, Proposition 5.74, Corollary 5.75]Let* $T \in B(\mathcal{H})$ *and* $M \sqsubseteq \mathcal{H}$*. Then the following statements are equivalents:*

(a) $M \in Lat(T)$*.*

(b) $M^{\perp} \in Lat(T^*)$*.*

Remark. *Note that* $M \in Lat_r(T)$ *if and only if* $M \in Lat_r(T^*)$*.*

Corollary 1.14. *If* $T \in \mathcal{A}(\mathcal{H})$*, then* $Lat(T) = Lat_r(T)$*, i.e., for all* $M \sqsubseteq \mathcal{H}$ *subspace* T*-invariant it is satisfied that* $M^{\perp}$ *is* T*-invariant.*

1.3 Polar decomposition for an operator

First, we guarantee the existence of square root positive of an operator $T \in B(\mathcal{H})^+$.

Theorem 1.15. *[26, Theorem 5.85] Let $T \in B(\mathcal{H})^+$. Then there exists a unique $S \in B(\mathcal{H})^+$ such that $S^2 = T$. We denote $S = T^{1/2}$. Moreover, we have that*

$$C \in B(\mathcal{H}) \text{ such that } TC = CT \Longrightarrow T^{1/2}C = CT^{1/2}.$$

Proposition 1.16. *[26, Proposition 5.86] If $T \in B(\mathcal{H})^+$, then*

(a) $\|T^{1/2}\|^2 = \|T\| = \|T^2\|^{1/2}$.

(b) $N(T^{1/2}) = N(T) = N(T^2)$ *and* $\overline{R(T^{1/2})} = \overline{R(T)} = \overline{R(T^2)}$.

Recall that if $T \in B(\mathcal{H})$, then $T^*T \in B(\mathcal{H})^+$. Moreover, $N(T^*T) = N(T)$ for all $T \in B(\mathcal{H})$, since $\langle T^*Tx, x\rangle = \|Tx\|^2$ for all $x \in \mathcal{H}$ and $T^*T \in B(\mathcal{H})^+$.

Definition 1.17. *Let $\mathcal{H}$ a Hilbert space. For each $T \in B(\mathcal{H})$ there are a positive operator $|T| = (T^*T)^{1/2} \in B(\mathcal{H})^+$, called module of T.*

This concept of the absolute value of an operator gives rise to a very important decomposition theorem for a bounded operator. This decomposition should be regarded as similar to the decomposition $z = |z|e^{i\theta}$ for complex numbers, with some limitations about the concept of absolute value of an operator.

One of the main similarities between T and $|T|$ is that $\|Tx\| = \||T|x\|$ for all $x \in \mathcal{H}$. These operators can be related using some kind of isometry between their images, that is the idea of the polar decomposition, which we will see after the following definitions.

Recall that an isometry between metric spaces is a function that preserves distance, and so every isometry is an injective contraction. Thus a linear isometry between normed spaces is an element of $B(X, Y)$ with X and Y a normed spaces, since an isometry is continuous [26, Proposition 4.37]. Now, for an isometry in an inner product space we have the next result.

Proposition 1.18. *[26, Proposition 5.21] Let X and Y be inner product spaces. A linear transformation $V \in L(X, Y)$ is an isometry if and only if*

$$\langle Vx_1, Vx_2 \rangle = \langle x_1, x_2 \rangle$$

for every $x_1, x_2 \in X$.

Another way to verify that a linear transformation between Hilbert spaces is an isometry is by means of its adjoint.

Proposition 1.19. *A transformation $V \in B(\mathcal{H}, \mathcal{K})$ of a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{K}$ is an isometry if and only if $V^*V = I$.*

Definition 1.20. *Let $\mathcal{H}$ a Hilbert space and let $U \in B(\mathcal{H})$ such that $S = R(U^*U) \sqsubseteq \mathcal{H}$. Then U defined as:*

$$U|_S : S \to U(S) \text{ is an isometry such that } U|_{S^\perp} \equiv 0. \tag{1.1}$$

The above implies that $N(U) = S^\perp$ and $R(U) = U(S) \sqsubseteq \mathcal{H}$.

Proposition 1.21. *[26, Proposition 5.88]Let $U \in B(\mathcal{H})$. Then U is a partial isometry if and only if there exists $S \sqsubseteq \mathcal{H}$ such that U satisfy 1.1 respect to S. In this case, if $M = R(U)$ and $S = R(U^*)$, then is satisfy:*

(a) U^* is a IP.

(b) $M = (N(U^*))^\perp \sqsubseteq \mathcal{H}$ and $S = (N(U))^\perp \sqsubseteq \mathcal{H}$.

(c) $UU^* = P_M$ and $U^*U = P_S$.

(d) $U = P_M U = U P_S = P_M U P_S$.

So, we can define a partial isometry how a bounded linear transformation that acts isometrically on the orthogonal complement of its null space. That is, $W \in B(\mathcal{H}, \mathcal{K})$ is a partial isometry if $W|_{N(W)^\perp} : N(W)^\perp \to \mathcal{K}$ is an lineal isometry. Since the range of a partial isometry is also a closed subspace, then the subspaces $N(W)^\perp$ and $R(W)$ are called initial and final spaces of the partial isometry W, respectively.

Theorem 1.22. *[26, Theorem 5.89] If $T \in B(\mathcal{H})$, then there exists a partial isometry $U \in B(\mathcal{H})$ with initial space $N(T)^{\perp}$ and final space $\overline{R(T)}$ such that $T = U|T|$ and $N(U) = N(|T|)$. Moreover, if $T = ZQ$, where $Q \in B(\mathcal{H})^{+}$ and Z is a partial isometry in $B(\mathcal{H})$ with $N(Z) = N(Q)$, then $Q = |T|$ and $Z = W$.*

The representation $U|T|$ is called the polar decomposition of T. Moreover, it is satisfy that $|T| = U^{*}T$.

1.4 Spectrum of bounded operators

An important set of complex numbers is the set of eigenvalues of a square matrix, since eigenvalues occur in a variety of applications in finite dimensional spaces. Then, it is natural to extend these notions to infinite dimensional spaces. So the notion of eigenvalue has to be replaced by a larger notion, the spectrum of the operator. This is what we aim to do in this section.

We recall that for $T \in B(\mathcal{H})$, the resolvent of T is denoted by $\rho(T)$ and is defined as:

$$\begin{aligned}\rho(T) &= \{\lambda \in \mathbb{C} : (T - \lambda I) \in \mathcal{GL}(\mathcal{H})\} \\ &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ has an inverse in } B(\mathcal{H})\} \\ &= \{\lambda \in \mathbb{C} : N(T - \lambda I) = \{0\} \text{ and } R(T - \lambda I) = \mathcal{H}\}.\end{aligned}$$

So we can define the spectrum of T, denoted by $\sigma(T)$, as the complement of $\rho(T)$ by:

$$\begin{aligned}\sigma(T) &= \mathbb{C} \setminus \rho(T) \\ &= \{\lambda \in \mathbb{C} : N(T - \lambda I) \neq \{0\} \text{ or } R(T - \lambda I) \neq \mathcal{H}\} \\ &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible }\}.\end{aligned}$$

We note that, $\sigma(T)$ can be split into many disjoint parts. A classical partition of spectrum is given by the point spectrum, the continuous spectrum and the residual spectrum of an operator that are defined as follows:

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not one to one}\},$$
$$\sigma_c(T) = \{\lambda \in \mathbb{C} : N(T - \lambda I) = \{0\}, \overline{R(T - \lambda I)} = \mathcal{H} \text{ and } R(T - \lambda I) \neq \mathcal{H}\},$$
$$\sigma_r(T) = \{\lambda \in \mathbb{C} : N(T - \lambda I) = \{0\}, \text{ and } R(T - \lambda I) \neq \mathcal{H}\}.$$

They are pairwise disjoint and

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T).$$

Now, we begin with some basic results over spectrum and resolvent set of an operator in $B(\mathcal{H})$.

Proposition 1.23. *[26, Corollary 6.12] The resolvent set $\rho(T)$ is nonempty and open, and the spectrum $\sigma(T)$ is compact.*

Proposition 1.24. *[26, Proposition 6.13] The spectrum $\sigma(T)$ is nonempty.*

There are some overlapping parts of the spectrum which are commonly used too, as is the approximate point spectrum, $\sigma_{ap}(T)$, defined by:

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bounded below}\}.$$

An alternative definition of approximate point spectrum is given in the next Proposition.

Proposition 1.25. *[26, Proposition 6.15] The following assertions are pairwise equivalent.*

(a) $\lambda \in \sigma_{ap}(T)$.

(b) *There exists an $\mathcal{H}$-valued sequence $\{h_n\}_{n\in\mathbb{N}}$ of unit vectors such that $\|(\lambda I - T)h_n\| \to 0$.*

(c) *For every $\epsilon > 0$ there exists a unit vector $h_\epsilon \in \mathcal{H}$ such that $\|(\lambda I - T)h_\epsilon\| < \epsilon$.*

Let $\mathbb{D}$ be the open unit disk about the origin in $\mathbb{C}$, and let $\mathbb{T} = \partial\mathbb{D}$ denote the unit circle about the origin in $\mathbb{C}$, then with these definitions we have the next result for different class of operators in $B(\mathcal{H})$.

Proposition 1.26. *[26, Corollary 6.18] Let $T \in B(\mathcal{H})$, then the following is satisfied:*

(a) If T is a normal operator, then $\sigma_p(T^*) = \sigma_p(T)^*$.

(b) If $T \in \mathcal{U}(\mathcal{H})$, then $\sigma(T) \subseteq \mathbb{T}$.

(c) If $T \in \mathcal{A}(\mathcal{H})$, then $\sigma(T) \subset \mathbb{R}$.

(d) If $T \in B(\mathcal{H})^+$, then $\sigma(T) \subset [0, \infty)$.

(e) If $T \in \mathcal{P}(\mathcal{H})$ is a nontrivial, then $\sigma(T) = \sigma_p(T) = \{0, 1\}$.

Proposition 1.27. *[27, Proposition 2.R] If $\mathcal{A}$ is a unital complex Banach algebra, and if $S, T \in \mathcal{A}$ commute, then*

$$\sigma(S+T) \subseteq \sigma(S) + \sigma(T) \quad \text{and} \quad \sigma(ST) \subseteq \sigma(S) \cdot \sigma(T).$$

Remark. *If $\mathcal{M} \in Lat(T)$, then it may happen that $\sigma(T|_{\mathcal{M}}) \not\subseteq \sigma(T)$. However, if $\mathcal{M} \in Lat_r(T)$ then $\sigma(T|_{\mathcal{M}}) \subseteq \sigma(T)$.*

1.5 Spectrum characterization for compact operators

We know that a bounded operator sends bounded sets in bounded sets, but if that operator also reaches finite range, then it sends bounded sets in relatively compact. This feature is what brings us to the next class of operators.

Definition 1.28. *A linear transformation between normed spaces $T \in \mathcal{L}(X, Y)$ is compact (or completely continuous) if $\overline{T(A)}$ is compact in Y whenever A is bounded in X.*

The term completely continuous operator is suggested by the following result, where it is also guaranteed that not every operator bounded is compact.

Theorem 1.29. *[26, Theorem 4.49] If $T \in \mathcal{L}(X, Y)$ is compact, then $T \in B(X, Y)$.*

Recall that a transformation lineal between normed spaces is of finite rank if it has a finite dimensional range.

Theorem 1.30. *[26, Proposition 4.50, Corollary 4.51] Let $T \in \mathcal{L}(X, Y)$. Then*

(a) If T is a bounded operator and of finite rank, then T is a compact operator.

(b) If $\dim(X) < \infty$, then T is a compact operator.

Recall that a two-sided ideal $\mathcal{I}$ of an algebra $\mathcal{A}$ is a subalgebra of $\mathcal{A}$ such that the product of every element of $\mathcal{I}$ with any element of $\mathcal{A}$ is again an element of $\mathcal{I}$.

Proposition 1.31. *[26, Proposition 4.54] Let X be a normed space. Then the set of compact operators is a two-sided ideal of the normed algebra $B(X)$.*

A fundamental fact for solving equations involving a compact operator is that the adjoint operator to a compact operator are also compact.

Theorem 1.32. *[24, Theorem(Schauder) 4.10] If $T : X \to Y$ be a compact operator then T^* is a compact operator.*

Now we consider $\mathcal{H}$ a Hilbert space. Let Γ any subset of $\mathbb{C}$, then

$$\Gamma^* := \{\overline{\lambda} \in \mathbb{C} : \lambda \in \Gamma\}.$$

It is clear, $\Gamma^{**} = \Gamma$, $(\mathbb{C} \setminus \Gamma)^* = \mathbb{C} \setminus \Gamma^*$ and $(\Gamma_1 \cup \Gamma_2)^* = \Gamma_1^* \cup \Gamma_2^*$.

Theorem 1.33. *[27, Theorem 2.6] Let $T^* \in B(\mathcal{H})$ the adjoint of $T \in B(\mathcal{H})$, then*

$$\rho(T) = \rho(T^*)^*, \;\; \sigma(T) = \sigma(T^*)^*, \;\; \sigma_c(T) = \sigma_c(T^*)^*$$

and

$$\sigma_r(T) = \sigma_p(T^*)^* \setminus \sigma_p(T).$$

The spectral mapping theorem is an important result in spectral theory. There are several versions of the spectral theorem that are equivalent in some way.

Theorem 1.34. *[27, Theorem 2.7, Spectral mapping theorem for polynomials] Let $T \in B(X)$ on a complex Banach space X. If p is an arbitrary polynomial with complex coefficients, then*

$$\sigma(p(T)) = p(\sigma(T)).$$

Remark. *As a direct consequence we have the following results:*

- $\sigma(T^n) = \sigma(T)^n$ *for every* $n \geq 0$.
- $\sigma(\alpha T) = \alpha\sigma(T)$ *for every* $\alpha \in \mathbb{C}$.
- *If* $T \in \mathcal{GL}(X)$, *then* $\sigma(T^{-1}) = \sigma(T)^{-1}$.
- *If* $T \in B(\mathcal{H})$, *then* $\sigma(T^*) = \sigma(T)^*$.

An extension of the Spectral Mapping Theorem for polynomials which holds for normal operators acting on a Hilbert space is the next result.

Theorem 1.35. *[27, Theorem 2.8] Let $T \in B(\mathcal{H})$ and $p(\cdot,\cdot) : \Gamma \times \Gamma \to \mathbb{C}$ is a polynomial in λ and $\overline{\lambda}$, then*

$$\sigma(p(T,T^*)) = p(\sigma(T),\sigma(T^*)) = \{p(\lambda,\overline{\lambda}) \in \mathbb{C} : \lambda \in \sigma(T)\}.$$

The spectral theory of compact operators have a big role in the Spectral Theorem for compact normal operators. Although the spectrum theory of compact operators can also be developed on complex non-zero Banach spaces we assume that operators act on complex non-zero Hilbert spaces $\mathcal{H}$. The fundamental result for characterizing the spectrum of a compact operator is the Fredholm alternative.

Theorem 1.36. *[27, Theorem 2.18, Fredholm Alternative] Let $T \in B(\mathcal{H})$ be a compact operator. If $\lambda \in \mathbb{C} \setminus \{0\}$, then $\lambda \in \rho(T) \cup \sigma_p(T)$ or, equivalently,*

$$\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}.$$

Moreover, for all $\lambda \in \mathbb{C} \setminus \{0\}$, it holds; $\dim(N(T - \lambda I)) = \dim(N(T^* - \overline{\lambda} I)) < \infty$.

The above theorem also guarantees that a compact operator only has a number of linearly independent eigenvectors associated with the same nonzero eigenvalue, that is, the corresponding space associated with any nonzero eigenvalue has finite dimension.

In the next Proposition, we will see that if a compact operator has infinite eigenvalues, these can be arranged in a sequence that converges to zero, among other very important properties of the spectrum of a compact operator.

Proposition 1.37. *[27, Corollary 2.20] Let $T \in B(\mathcal{H})$ be a compact operator. Then*

(a) An infinite sequence of distinct points of $\sigma(T)$ converges to zero.

(b) 0 is the only possible accumulation point of $\sigma(T)$).

(c) If $\lambda \in \mathbb{C} \setminus \{0\}$, then λ is an isolated point of $\sigma(T)$.

(d) $\sigma(T) \setminus \{0\}$ is a discrete subset of $\mathbb{C}$.

(e) $\sigma(T)$ is countable.

Chapter 2

The Image-Kernel (P,Q)-inverse

In this chapter, we introduce a generalized inverse for bounded linear operators which we can characterize as an inverse with prescribed range and null space. The idea of “prescribing” was considered by Djordjević and Wei in the set of rings where generalized inverses with certain idempotents were studied ([12]). However, some of these ideas can be traced back at least to Bott-Duffin in 1953 ([6]). M.P. Drazin himself considered that Bott-Duffin’s ideas have to be recognized as a direct precursor of the (b, c)-inverse with b, c elements in a semigroup and, consenquetly, of most other known types of uniquely-defined outer generalized inverses ([13]).

Much of the recent work on generalized inverses has taken place in algebras and, more generally, rings and semigroups. However, the richer structure found in the algebra of bounded linear operators can give better insight of the phenomena.

2.1 Introduction

Let X and Y be Banach spaces and $B(X, Y)$ the space of bounded linear operators from X to Y. An operator $S \in B(Y, X)$ is called a (bounded) inverse of $T \in B(X,Y)$ if $TS = I_Y$ and $ST = I_X$, where I_Y, I_X are the identity operators in the respective spaces. If there is no risk of confusion, we use I to denote I_X or I_Y.

An operator $S \in B(Y,X)$ is an inner inverse for $T \in B(X,Y)$ if

$$TST = T$$

and, in this case, we say that T is inner regular. S is an outer inverse for T if

$$STS = S$$

and, we say that T is outer regular. Neither inner inverse nor outer inverse are unique when they exist. Moreover, inner inverse is not unique even if we prescribe its range and null space. On the other hand, if we prescribe the range and null space for an outer inverse we have uniqueness (see [11]). Several classes of generalized inverses found in the literature are special cases of outer inverses with prescribed range and null space.

We know that a projection is an idempotent linear transformation of a linear space into itself. For example, the null transformation and the identity in $B(X)$ are projections. If S is an outer or inner inverse of T, then ST and TS are projections.

If X_1 and X_2 are subspaces of X such that $X_1 \cap X_2 = \{0\}$ and $X_1 + X_2 = X$, we say that X_1 is complemented with X_2. Additionally, if X_1 and X_2 are closed, then we write $X = X_1 \oplus X_2$.

It is well-know that $T \in B(X,Y)$ is inner invertible if and only if $N(T)$ and $R(T)$ are closed and complemented subspaces of X and Y respectively [11, Corollary 1.1.5]. Moreover, if M and N are such that $Y = R(T) \oplus M$ and $X = N \oplus N(T)$, then T has the following matrix form:

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} N \\ N(T) \end{bmatrix} \to \begin{bmatrix} R(T) \\ M \end{bmatrix},$$

with $T_1 \in B(N, R(T))$ invertible. If S is any inner inverse of T such that $R(ST) = N$ and $N(TS) = M$, then S has the following matrix form:

$$S = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & S_2 \end{bmatrix} : \begin{bmatrix} R(T) \\ M \end{bmatrix} \to \begin{bmatrix} N \\ N(T) \end{bmatrix},$$

with $S_2 \in B(M, N(T))$ arbitrary ([11, Theorem 1.1.6]).

2.2 Definition and characterizations of (P,Q)-inverse

The image-kernel (p,q)-inverse was introduced by Kantún-Montiel in the context of rings ([22]). This inverse was later studied by Mosić et al. ([32]), where an algebraic characterization was given without the explicit use of ideals of the ring. This allows us to give the following definition in the space of bounded linear operators:

Definition 2.1. *Let $P \in B(X)$ and $Q \in B(Y)$ be projections and $T \in B(X,Y)$. If there exists $S \in B(Y,X)$ such that*

$$S = PS, \qquad STP = P, \qquad SQ = O, \qquad I - Q = (I-Q)TS,$$

then S is called the image-kernel (P,Q)-inverse of T.

Remark.

(i) The image-kernel (P,Q)-inverse is unique when it exists. Indeed, suppose that $S_1, S_2 \in B(Y,X)$ are two image-kernel (P,Q)-inverses of $T \in B(X,Y)$, $P \in B(X)$, $Q \in B(Y)$. Then

$$\begin{aligned} S_1 = PS_1 = S_2TPS_1 = S_2TS_1 \\ = S_2TS_1 - 0 = S_2TS_1 - S_2QTS_1 \\ = (S_2 - S_2Q)TS_1 = S_2(I-Q)TS_1 \\ = S_2(I-Q) = S_2. \end{aligned}$$

(ii) The image-kernel (P,Q)-inverse S of T is an outer inverse:

$$STS = STPS = PS = S.$$

(iii) Since the image-kernel (P,Q)-inverse S of T is an outer inverse, then TS and ST are projections. They are related to P and Q as follows:

(a) $R(ST) = R(P)$. Indeed, $STP = P$ implies $R(P) \subseteq R(ST)$, and $S = PS$ implies $ST = PST$, hence $R(ST) \subseteq R(P)$. Thus $R(ST) = R(P)$.

(b) $N(TS) = R(Q)$. Indeed, $SQ = 0$ implies $TSQ = 0$, hence $R(Q) \subseteq N(TS)$. Moreover, from $I-Q = (I-Q)TS$ it follows that $N(TS) \subseteq N(I-Q) = R(Q)$. Hence, $N(TS) = R(Q)$.

Example 1. Let $T \in B(X,Y)$ be Fredhom operator, i.e. $\dim(N(T)) < \infty$ and $\dim(Y/R(T)) < \infty$. Since $\dim(N(T)) < \infty$ and $\dim(Y/R(T)) < \infty$ then there are closed subspaces $X_1 \subseteq X$ and $Y_2 \subseteq Y$ such that $X = X_1 \oplus N(T)$ and $Y = R(T) \oplus Y_2$. We have the following block matrix representation for T

$$T = \begin{bmatrix} T_1 & T_3 \\ T_4 & T_2 \end{bmatrix} : \begin{bmatrix} X_1 \\ N(T) \end{bmatrix} \to \begin{bmatrix} R(T) \\ Y_2 \end{bmatrix},$$

where $T_1 \in B(X_1, R(T))$ is invertible.

Let $P \in B(X)$ and $Q \in B(Y)$ be the projections defined by

$$P := \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ N(T) \end{bmatrix} \to \begin{bmatrix} X_1 \\ N(T) \end{bmatrix},$$

$$Q := \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} : \begin{bmatrix} R(T) \\ Y_2 \end{bmatrix} \to \begin{bmatrix} R(T) \\ Y_2 \end{bmatrix}.$$

Then $S \in B(Y,X)$, defined by

$$S := \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(T) \\ Y_2 \end{bmatrix} \to \begin{bmatrix} X_1 \\ N(T) \end{bmatrix},$$

is the image-kernel (P,Q)-inverse of T.

Example 2. Let $X = l^2(\mathbb{N})$, the space of square-summable sequences, and let $T, P, Q \in B(X)$ defined by

$$Tx := (\tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \tfrac{1}{4}x_4, \dots),$$

$$Px := (0, x_2, x_3, \dots, x_{n-1}, x_n, 0, 0, 0, \dots); \; n \geq 3,$$

$$Qx := (0, \dots, 0, x_n, x_{n+1}, x_{n+2}, \dots),$$

for $x = (x_1, x_2, x_3, \dots) \in X$. We remark that $P \neq I - Q$. We have that P, Q are projections and

$$Sx := (0, 2x_1, 3x_2, \dots, (n-1)x_{n-2}, nx_{n-1}, 0, 0, 0, \dots).$$

is the image-kernel (P,Q)-inverse for T. A simple calculation shows:

$$\begin{aligned}
PSx &= (0, 2x_1, 3x_2, \dots, (n-1)x_{n-2}, nx_{n-1}, 0, 0, 0, \dots) \\
&= Sx, \\
STPx &= ST(0, x_2, x_3, \dots, x_{n-1}, x_n, 0, 0, 0, \dots) \\
&= S(\tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \dots, \tfrac{1}{n-1}x_{n-1}, \tfrac{1}{n}x_n, 0, \dots) \\
&= (0, x_2, x_3, \dots, x_{n-1}, x_n, 0, 0, 0, \dots) \\
&= Px, \\
SQx &= S(0, \dots, 0, x_n, x_{n+1}, x_{n+2}, \dots) \\
&= \mathbf{0}, \\
(I-Q)TSx &= (I-Q)(x_1, x_2, \dots, x_{n-2}, x_{n-1}, 0, 0, 0, \dots) \\
&= (x_1, x_2, \dots, x_{n-2}, x_{n-1}, 0, 0, 0, \dots) \\
&= (I-Q)x.
\end{aligned}$$

Example 3. Let $X = l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$, and let $T = U \oplus U^* \in B(X)$ where U, U^* are defined by

$$Ux := (x_2, x_3, x_4, \dots),$$

$$U^*y := (0, y_1, y_2, y_3, \dots),$$

where $x = (x_1, x_2, x_3, \dots)$, $y = (y_1, y_2, y_3, \dots) \in l^2(\mathbb{N})$. Let $P, Q \in B(X)$ defined by

$$P(x \oplus y) := (0, x_2, x_3, x_4, \dots) \oplus (y_1, y_2, y_3, \dots),$$

$$Q(x \oplus y) := (0, 0, 0, \dots) \oplus (y_1, 0, 0, 0, \dots).$$

Then T is image-kernel (P,Q)-invertible with inverse S, as can be verified immediately, defined by

$$S(x \oplus y) := (0, x_1, x_2, x_3, \dots) \oplus (y_2, y_3, y_4, \dots).$$

Let $T \in B(X, Y)$ and M, N be given subspaces of X, Y respectively. If there exist $0 \neq S \in B(Y, X)$ outer inverse of T such that $R(S) = M$ and $N(S) = N$, then S is the unique outer inverse of T with prescribed range M and nullspace N, and M and N are complemented in X and Y respectively (see [11, Theorem 1.1.10]). We denote this S by $T^{(2)}_{M,N}$. The image-kernel (P, Q)-inverse can be characterized as an outer generalized inverse with prescribed range $R(P)$ and nullspace $R(Q)$ as follows:

Theorem 2.2. Let $T \in B(X, Y)$ be a nonzero operator and let $P \in B(X)$, $Q \in B(Y)$ be projections. The following statements are equivalent:

(a) S is the image-kernel (P, Q)-inverse of T,

(b) $S = T^{(2)}_{R(P),R(Q)}$.

Proof. $a) \Rightarrow b)$. Suppose that S is the image-kernel (P, Q)-inverse of T. Since $STS = STPS = PS = S$, we obtain that S is an outer inverse of T. Now, since $R(S) = R(PS) \subseteq R(P) = R(STP) \subseteq R(S)$, then $R(P) = R(S)$. On the other hand, $SQ = 0$ implies that $R(Q) \subseteq N(S)$. Also, since $N(S) \subseteq N((I - Q)TS) = N(I - Q) = R(Q)$, we have that $R(Q) = N(S)$.

$b) \Rightarrow a)$. Suppose that $S = T^{(2)}_{R(P),R(Q)}$, then $R(S)$ and $N(S)$ are closed and complemented subspaces and we have the following matrix form for S (see [11]):

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} N(Q) \\ N(S) \end{bmatrix} \rightarrow \begin{bmatrix} R(S) \\ N(P) \end{bmatrix},$$

with $S_1 \in B(N(Q), R(S))$ invertible.

Now, since T is an inner inverse of S then T have the following matrix form:

$$T = \begin{bmatrix} S_1^{-1} & T_3 \\ T_4 & T_2 \end{bmatrix} : \begin{bmatrix} R(S) \\ M \end{bmatrix} \rightarrow \begin{bmatrix} N \\ N(S) \end{bmatrix},$$

with $S_1^{-1} \in B(R(S), N)$ the inverse of S_1, $M = N(P)$ and $N = N(Q)$. A direct calculation shows that $S = PS$, $STP = P$, $SQ = 0$ and $I - Q = (I - Q)TS$. □

Remark. From the previous theorem, it is natural to denote the unique inverse S from Definition 2.1 by $T^{(2)}_{P,Q}$.

As for several other classes of generalized inverses, the image-kernel (P, Q)-inverse is related to certain decompositions of spaces X and Y such that the restriction of T is invertible:

Theorem 2.3. Let $T \in B(X, Y)$ and let $P \in B(X)$, $Q \in B(Y)$ be projections. The following statements are equivalent:

(a) T is image-kernel (P, Q)-invertible,

(b) $R(TP)$ is closed, $R(TP) \oplus R(Q) = Y$ and the restriction $T|_{R(P)} : R(P) \to R(TP)$ is invertible.

Proof.

a) $\Rightarrow$ *b*). Suppose that $S = T^{(2)}_{P,Q}$. Since $I - TS$ is a projection from Y on $N(S) = R(Q)$, then $Y = R(TS) \oplus R(Q)$. Further, since $R(TS) = T(R(S)) = T(R(P)) = R(TP)$ we have that $R(TP)$ is closed and $Y = R(TP) \oplus R(Q)$. Now, for the invertibility of $T|_{R(P)} : R(P) \to R(TP)$ it is clear that it is onto. To see that $T|_{R(P)}$ is also $1 - 1$ on $R(P)$, suppose that there exists $z \in R(P)$ such that $Tz = 0$. Since $z \in R(P) = R(S)$, there exists $y \in Y$ such that $Sy = z$. Then $Tz = 0$ implies $0 = STz = STSy = Sy$ and thus $z = 0$. Therefore $T|_{R(P)}$ is $1 - 1$ and onto, and hence invertible.

b) $\Rightarrow$ *a*). Suppose that $R(TP)$ is closed, $R(TP) \oplus R(Q) = Y$ and the reduction $T|_{R(P)} = R(P) \to R(TP)$ is invertible. Since P is a projection, $X = R(P) \oplus N(P)$. Then T has the following matrix form with respect to these decomposition of spaces:

$$T = \begin{bmatrix} T_1 & T_3 \\ T_4 & T_2 \end{bmatrix} : \begin{bmatrix} R(P) \\ N(P) \end{bmatrix} \to \begin{bmatrix} R(TP) \\ R(Q) \end{bmatrix}.$$

Since T maps $R(P)$ onto $R(TP)$ (with $T_1 = T|_{R(P)}$ invertible), it follows that $T_4 = 0$. Now, let S be the operator defined by

$$S = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(TP) \\ R(Q) \end{bmatrix} \to \begin{bmatrix} R(P) \\ N(P) \end{bmatrix}.$$

A direct calculation shows that $STS = S$, $R(S) = R(P)$ and $N(S) = R(Q)$. Thus, by Theorem 2.2, $S = T^{(2)}_{P,Q}$. □

Note that, even if $X = Y$, the decomposition of X like as domain of T and codomain of T can be different, i.e. $P \neq I - Q$. Moreover, even that $R(Q)$ being complemented by $R(TP)$, it does not imply $R(TP) = R(I - Q)(= N(Q))$.

Now, we can give a matrix form of the operator T from decompositions mentioned in the proof of previous Theorem.

Proposition 2.4. *Let $T \in B(X, Y)$. T is image-kernel (P, Q)-invertible if only if T has the following matrix form:*

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} : \begin{bmatrix} R(P) \\ N(P) \end{bmatrix} \to \begin{bmatrix} R(TP) \\ R(Q) \end{bmatrix},$$

with T_1 invertible. Moreover its image-kernel (P, Q)-inverse have the form:

$$T^{(2)}_{P,Q} = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(TP) \\ R(Q) \end{bmatrix} \to \begin{bmatrix} R(P) \\ N(P) \end{bmatrix}.$$

Proof. Suppose that T is image-kernel (P, Q)-invertible, from the proof of the Theorem 2.3 we know that T has the following matrix form

$$T = \begin{bmatrix} T_1 & T_3 \\ 0 & T_2 \end{bmatrix} : \begin{bmatrix} R(P) \\ N(P) \end{bmatrix} \to \begin{bmatrix} R(TP) \\ R(Q) \end{bmatrix}.$$

We claim that also $T_3 = 0$. Indeed, let $T^{(2)}_{P,Q}$ is the image-kernel (P, Q)-inverse of T then $T^{(2)}_{P,Q}$ has the following matrix form (again from the proof of Theorem 2.3)

$$T^{(2)}_{P,Q} = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(TP) \\ R(Q) \end{bmatrix} \to \begin{bmatrix} R(P) \\ N(P) \end{bmatrix}.$$

Since $T^{(2)}_{P,Q}T$ is the projection from X on $R(T^{(2)}_{P,Q}) = R(P)$, from the matrix form

$$T^{(2)}_{P,Q}T = \begin{bmatrix} I & T_1^{-1}T_3 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(P) \\ N(P) \end{bmatrix} \to \begin{bmatrix} R(P) \\ N(P) \end{bmatrix},$$

we observe that $T_1^{-1}T_3 = 0$ and this implies that $T_3 = 0$. □

2.3 Relation with other generalized inverses

Recall $T \in B(X)$ is group invertible if there exist $S \in B(X)$ such that $TST = T$, $STS = S$ and $ST = TS$. If this inverse exists, then it is unique and we write $S = T^{\#}$ the group inverse of T. We have that T is group invertible if and only if $X = R(T) \oplus N(T)$ [39, Theorem 2.1.1 and Theorem 2.2.1]. In the special case where $X = Y$ we can relate the image-kernel (P, Q)-inverse of an operator T with the group inverse of operator TP as follows.

Theorem 2.5. Let $T \in B(X)$ and let $P, Q \in B(X)$ be projections such that $R(Q) = N(P)$. The following statements are equivalent:

(a) T is image-kernel (P, Q)-invertible,

(b) $N(P) = N(TP)$ and TP is group invertible.

Moreover, $P(TP)^{\#}$ is the image-kernel (P, Q)-inverse of T.

Proof.

a) $\Rightarrow$ *b)*. Suppose that $S = T^{(2)}_{P,Q}$. Then we have $R(TP) = T(R(P)) = T(R(S)) = R(TS)$. Since $N(P) = R(Q) = N(S) = N(TS)$ and $N(P) = N(STP) \supseteq N(TP) \supseteq N(P)$, we get $N(TP) = N(TS)$. Finally, since TS is projection in $B(X)$ it is implies that $X = R(TS) \oplus N(TS)$, then we get $X = R(TP) \oplus N(TP)$. Therefore TP is group invertible.

b) $\Rightarrow$ *a)*. For simplify notation denote $P(TP)^{\#} = S$. A simple calculation shows that $S = STS$. We will prove that $R(S) = R(P)$ and $N(S) = R(Q)$. Indeed,

$$\begin{aligned} N(S) &= N(TS) = N(TP(TP)^{\#}) = N(TP) \\ &= R(Q). \end{aligned}$$

For $R(S) = R(P)$, first we prove that $R(TP) = R((TP)^{\#}TP)$. In fact, since

$$\begin{aligned} R((TP)^{\#}(TP)) &= R((TP)(TP)^{\#}) \subseteq R(TP) \\ &= R((TP)(TP)^{\#}(TP)) \subseteq R((TP)(TP)^{\#}) \\ &= R((TP)^{\#}(TP)). \end{aligned}$$

Now, TP is group invertible implies $X = R(TP) \oplus N(TP)$. Moreover $R(TP) = R((TP)^{\#}TP)$ and $N(TP) = N(P)$, then $X = R((TP)^{\#}TP) \oplus N(P)$.

Let $x \in X$, $x = x_1 + x_2$ where $x_1 \in R((TP)^{\#}TP)$ and $x_2 \in N(P)$. Then $Px = Px_1$ and this implies that $R(P) \subseteq P(R((TP)^{\#}TP)) = R(P(TP)^{\#}TP)$. Finally, from

$$\begin{aligned} R(S) &= R(P(TP)^{\#}) \subseteq R(P) \\ &\subseteq R(P(TP)^{\#}TP) \subseteq R(P(TP)^{\#}) \\ &= R(S) \end{aligned}$$

it follows that $R(S) = R(P)$. Hence, by Theorem 2.2, we have $S = T^{(2)}_{P,Q}$. □

Example 4. Let $X = l^2(\mathbb{N})$ and $T, P, Q \in B(X)$ defined by

$$Tx := (\tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \tfrac{1}{4}x_4, \dots),$$

$$Px := (x_1, x_2, x_1, 0, 0, 0, \dots),$$

$$Qx := (0, 0, x_3, x_4, x_5, \dots),$$

for $x = (x_1, x_2, x_3, \dots) \in X$. Then TP is group invertible with

$$(TP)^{\#}x = (3x_2, 2x_1, 0, 0, 0, ...).$$

Since $N(TP) = N(P) = R(Q)$, by Theorem 2.5, T is image-kernel (P, Q)-invertible. Moreover, we have that

$$S := P(TP)^{\#} = (3x_2, 2x_1, 3x_2, 0, 0, 0, ...)$$

is the image-kernel (P, Q)-inverse of T.

A result analogous to the Theorem 2.5, but only in one sense, can be obtained if we consider the operator $(I - Q)T$ and other conditions as shown below.

Proposition 2.6. *Let $T \in B(X)$ and let $P, Q \in B(X)$ be projections such that $R(P) = N(Q)$. If $TP = (I - Q)TP$ with $N(Q) \subseteq R(TP)$ and $(I - Q)T$ is group invertible with $N((I - Q)T) = R(Q)$, then T is image-kernel (P, Q)-invertible and $T^{(2)}_{P,Q} = ((I - Q)T)^{\#}(I - Q)$.*

Proof. To simplify the notation let us denote $((I-Q)T)^{\#}(I-Q) = S$. A direct calculation show that $S = STS$. We will prove that $R(P) = R(S)$ and $R(Q) = N(S)$.
First,

$$\begin{aligned} R(S) &= R(ST) = R(((I-Q)T)^{\#}(I-Q)T) \\ &= R((I-Q)T(((I-Q)T)^{\#}) = R((I-Q)T) \\ &\subseteq R(I-Q) = N(Q) = R(P) \subseteq R(TP) = R((I-Q)TP) \\ &\subseteq R((I-Q)T) = R((I-Q)T(((I-Q)T)^{\#}) \\ &= R(((I-Q)T)^{\#}(I-Q)T) = R(ST) = R(S). \end{aligned}$$

Then, $R(P) = R(S)$.

Now, since $R(Q) = N(I-Q)$, it is enough to prove that $N(I-Q) = N(S)$. Indeed, since $(I-Q)T$ is group invertible then $X = R((I-Q)T) \oplus N((I-Q)T) = R(TP) \oplus N((I-Q)T) = R(P) \oplus N((I-Q)T)$. Also, we know that $N((I-Q)T) = N((I-Q)T(((I-Q)T)^{\#})$, then $X = R(P) \oplus N((I-Q)T(((I-Q)T)^{\#})$.

Now, we observe the following:

$$\begin{aligned} N(S) &= N(TS) = N(T(((I-Q)T)^{\#}(I-Q)) \\ &\subseteq N((I-Q)T(((I-Q)T)^{\#}(I-Q)) \\ &\subseteq N((((I-Q)T)^{\#}(I-Q)T(((I-Q)T)^{\#}(I-Q)) \\ &= N((((I-Q)T)^{\#}(I-Q)) = N(S) \end{aligned}$$

then $N(S) = N((I-Q)T(((I-Q)T)^{\#}(I-Q))$.

Finally, from the definition of S, we have $N(I-Q) \subseteq N(S)$. To prove equality, suppose that exists $x \in N(S)$ such that $x \notin N(I-Q)$. Then $x \in N(S) = N((I-Q)T(((I-Q)T)^{\#}(I-Q))$ implies that $(I-Q)T(((I-Q)T)^{\#}(I-Q)x = 0$. Therefore $(I-Q)x \in N((I-Q)T(((I-Q)T)^{\#}) = N((I-Q)T) = R(Q) = N(I-Q)$, then $(I-Q)x = (I-Q)(I-Q)x = 0$ and this implies that $x \in N(I-Q)$, which is a contradiction. Hence, by Theorem 2.2 we have $S = T^{(2)}_{P,Q}$. □

Recall, for a Banach space X and $T \in B(X)$, T is Drazin invertible if exists $S \in B(X)$ such that:

$$TS = ST, \quad STS = S, \quad \text{and} \quad T(I - ST) \text{ is nilpotent.}$$

The Drazin inverse will be denoted by T^d.

Generalizing the inverse Drazin we have: T is generlized Drazin invertible if exists $S \in B(X)$ such that:

$$TS = ST, \quad STS = S, \quad \text{and} \quad T(I - ST) \text{ is quasi-nilpotent.}$$

It will be denoted by T^D.

If X is a Hilbert space, then $S \in B(X)$ is called the Moore-Penrose inverse of T, in notation $S = T^{\dagger}$, if the following is satisfied:

$$TST = T, \quad STS = S, \quad (TS)^* = TS \quad \text{and} \quad (ST)^* = ST,$$

where A^* denotes the adjoint operator of A.

It should be noted that the image-kernel (P,Q)-inverse, developed in this work, can be related with the group inverse, the Drazin inverse, the generalized Drazin inverse and the Moore-Penrose inverse in the following way.

Let $T \in B(X)$. Suppose that for a projection $P \in B(X)$ exists $S \in B(X)$ such that $S = T^{(2)}_{P,Q}$ with $Q = I - P$. Then, we have the following:

1. If $P = I$, then $S = T^{-1}$.
2. If $R(P) = R(T)$ and $R(Q) = N(T)$, then $S = T^{\#}$.
3. If $R(P) = R(T^n)$ and $R(Q) = N(T^n)$ for some $n \in \mathbb{N}$, then $S = T^d$.
4. If $R(P) = R(T^{\infty})$ and $R(Q) = N(T^{\infty})$, then $S = T^D$.
5. If $T, P \in B(X)$ with X a Hilbert space, $R(P) = R(T^*)$ and $R(Q) = N(T^*)$, then $S = T^{\dagger}$.

2.4 The Image-kernel (P,Q)-inverse on B(H)

In this section we will see a type of operators for which the image-kernel (P,Q)-inverse exists, for this we consider a Hilbert space $\mathcal{H}$ and let $T \in B(\mathcal{H})$, then, by [15, Lemma 1.1], for each $\epsilon > 0$ there exists a closed subspace X_ϵ of $\mathcal{H}$ such that

(1) $N(T) \subset X_\epsilon$, $\|Tx\| < \epsilon\|x\|$ for every $x \in X_\epsilon \setminus \{0\}$,

(2) $\|Tx\| \geq \epsilon\|x\|$ for every $x \in X_\epsilon^\perp$.

We remark that $X_\epsilon^\perp$ and $T(X_\epsilon^\perp)$ are closed subspaces, indeed, since X_ϵ is closed and T is bounded below on $X_\epsilon^\perp$ ([26, Corollary 4.24]). These subspaces, $X_\epsilon^\perp$ and $T(X_\epsilon^\perp)$, will allow us to define adequate projections for our (P,Q)-inverse as we will see next.

Using some of the results obtained in 2.2 and the spaces mentioned above, we show a construction of a (P,Q)-inverse for any operator $T \in B(\mathcal{H})$.

Let $\epsilon > 0$ and let X_ϵ a closed subspace of $\mathcal{H}$ that satisfy (1) and (2), then by projection Theorem second version [26, Theorem 5.25], we have that $\mathcal{H} = X_\epsilon \oplus X_\epsilon^\perp$ and by projection Theorem third version [26, Theorem 5.52] there exist a unique projection $P : \mathcal{H} \to \mathcal{H}$ such that $R(P) = X_\epsilon^\perp$ and $N(P) = X_\epsilon$. Then, since $T(X_\epsilon^\perp)$ is a closed subspace of $\mathcal{H}$, once again we can do the same, so we have that $\mathcal{H} = T(X_\epsilon^\perp) \oplus \big(T(X_\epsilon^\perp)\big)^\perp$ and there exists a unique projection $Q : \mathcal{H} \to \mathcal{H}$ such that $R(Q) = \big(T(X_\epsilon^\perp)\big)^\perp$ and $N(Q) = T(X_\epsilon^\perp)$.

Now, we take $X_\epsilon^* = T(X_\epsilon^\perp)$ and $(X_\epsilon^*)^\perp := \big(T(X_\epsilon^\perp)\big)^\perp$. So, we have that $X_\epsilon^* = R(TP)$ is closed, $\mathcal{H} = X_\epsilon^* \oplus (X_\epsilon^*)^\perp = R(TP) \oplus R(Q)$ and $T|_{X_\epsilon^\perp} : X_\epsilon^\perp \to X_\epsilon^*$ is invertible. Then, by Theorem 2.3, T is image-kernel (P,Q)-invertible. Moreover, by Proposition 2.4, T has the following matrix form:

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} : \begin{bmatrix} X_\epsilon^\perp \\ X_\epsilon \end{bmatrix} \to \begin{bmatrix} X_\epsilon^* \\ (X_\epsilon^*)^\perp \end{bmatrix},$$

with T_1 invertible. And its image-kernel (P,Q)-inverse have the form:

$$T^{(2)}_{P,Q} = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X^*_\epsilon \\ (X^*_\epsilon)^\perp \end{bmatrix} \to \begin{bmatrix} X^\perp_\epsilon \\ X_\epsilon \end{bmatrix}.$$

From all the above arises the following question posed for the inverse type image-kernel (P,Q) of a given operator T in $B(X,Y)$ with X and Y a different Banach spaces, that is:

Let $T^{(2)}_{P,Q}$ be a (P,Q)-inverse for an operator $T \in B(X,Y)$, does there exist another projections P' and Q' such that $T^{(2)}_{P',Q'} = T^{(2)}_{P,Q}$?

From this, we have the following questions:

1. Can we describe the class of all pairs (P,Q) of projectors such that this pairs given us the same (P,Q)-inverse for T?

2. Does exist a way to choose "best" pair of projections (P,Q)?

Chapter 3

The ϵ-{2,3,4} inverse

The generalized inverses of matrices and linear operators have been defined in different ways throughout history, allowing a development in solving problems with different approaches, for example; analysis, algebra or geometry. So in this chapter we defined as a new generalized inverse, called ϵ-{2,3,4}, that exists in the case when a Hilbert space operator does not necessarily has closed range. It is shown that, in the case when the Moore-Penrose exists and ϵ is small enough, these inverses are equal. Also, it gives a matrix form of such inverse. Additionally, we studied the ϵ-{2,3,4} inverse for the adjoint, positive and unitary operators, all this to express a form of the ϵ-{2,3,4} inverse using the polar decomposition of an operator.

3.1 Introduction

In this chapter we will always consider that $\mathcal{H}$ and $\mathcal{K}$ are complex Hilbert spaces of infinite dimension unless otherwise specified. Let $B(\mathcal{H}, \mathcal{K})$ denotes the set of bounded linear transformations from $\mathcal{H}$ to $\mathcal{K}$. In the case when $\mathcal{H} = \mathcal{K}$, we denoted $B(\mathcal{H}, \mathcal{H})$ by $B(\mathcal{H})$ which is a Banach algebra. E.H.Moore was the first to give an explicit definition of the generalized inverse of an arbitrary matrix. This definition was given in an abstract published in the Bulletin of the American Mathematical Society in 1920. In modern notation Moore's definition can be written as follows [17]: If $T \in B(\mathcal{H}, \mathcal{K})$

with $R(T)$ closed in $\mathcal{K}$, then $T^\dagger$ is the unique operator in $B(\mathcal{K},\mathcal{H})$ that satisfies:

$$TT^\dagger = P_{R(T)} \quad \text{and} \quad T^\dagger T = P_{R(T^\dagger)}.$$

Independently of Moore's work, in 1955 Penrose defined the generalized inverses of a matrix, that was extended to an arbitrary $T \in B(\mathcal{H},\mathcal{K})$ with closed range, like the unique matrix (operator) $T^\dagger \in B(\mathcal{K},\mathcal{H})$ such that:

$$(1) \;\; TT^\dagger T = T, \qquad (2) \;\; T^\dagger TT^\dagger = T^\dagger,$$
$$(3) \;\; TT^\dagger = (TT^\dagger)^*, \qquad (4) \;\; T^\dagger T = (T^\dagger T)^*.$$

The equivalence between of Penrose's definition and Moore's definition was pointed out by Rado in 1956. In our days we refer to $T^\dagger$ as the Moore-Penrose inverse (for short MP-inverse), and it is well known that $T \in B(\mathcal{H},\mathcal{K})$ has closed range if and only if there exists the unique operator $T^\dagger \in B(\mathcal{K},\mathcal{H})$ which satisfies the equations $(1)-(4)$ (for more details see [18]). However, the range of an operator T is not necessarily closed, for example, any integral operator whose kernel doesn't have a finite rank provides an example of this type of operators. A fundamental distinction between the case of an operator with closed range and the case of an operator with not closed range is that, the generalized inverse of an operator with not closed range turns out to be an unbounded operator, and therefore approximations to such a generalized inverse by bounded operators must necessarily converge only in the pointwise sense at best [17].

3.2 Reduced minimum modulus

Another definition that we consider is the reduced minimum modulus operators on a Hilbert space, that we can say measures the closeness for the range of this operator.

Definition 3.1. *[26, Defintion 4.19] A linear transformation T of a normed space X into a normed space Y is bounded below if there exists a constant $\alpha > 0$ such that*

$$\alpha\|x\| \leq \|Tx\|$$

for every $x \in X$,

The minimum modulus of an operator $T \in B(\mathcal{H})$ is

$$m(T) := \inf_{h \in \mathcal{H}} \frac{\|Th\|}{\|h\|}.$$

Remark. *Let $T \in B(\mathcal{H})$. If T is bounded below then $m(T) > 0$.*

Proposition 3.2. *[30] Let $T \in B(\mathcal{H})$. Then $m(T) > 0$ if and only if $R(T)$ is closed and $N(T) = \{0\}$.*

Definition 3.3. *Let $T \in B(\mathcal{H})$. We define*

$$\gamma(T) := \inf \left\{ \frac{\|Th\|}{\|h\|} \middle| h \in N(T)^{\perp}, h \neq 0 \right\}$$

as the reduced minimum modulus of T.

If $T = 0$ then $\gamma(T) = 0$. Also, we know that if T is an invertible operator, then $\gamma(T) = \|T^{-1}\|^{-1}$.

Theorem 3.4. *[30, Theorem 1.4.1] Let $T \in B(\mathcal{H})$, then $\gamma(T) > 0$ if and only if $R(T)$ is closed.*

Remark. *As T is defined on a Hilbert space $\mathcal{H}$ it is easy to see that*

$$\begin{aligned}\gamma(T) =& \inf\{\|Th\| : h \in (N(T))^{\perp}, \|h\| = 1\}\\ =& \inf(\sigma(T^*T)^{1/2} \setminus \{0\})\\ =& \inf(\sigma(TT^*)^{1/2} \setminus \{0\})\\ =& \gamma(T^*).\end{aligned}$$

3.3 Definition and basic results

Now, following Campbell's idea [9] we can give the following definition of generalized inverse for an operator $T \in B(\mathcal{H}, \mathcal{K})$.

Definition 3.5. *Let $T \in B(\mathcal{H}, \mathcal{K})$ and $\epsilon \geq 0$, then a ϵ-{2,3,4} inverse of T is the operator $S \in B(\mathcal{K}, \mathcal{H})$ that satisfy the following equations:*

$$(\epsilon)\ \|TST - T\| \leq \epsilon, \qquad (2) \quad STS = S,$$

$$(3) \quad TS = (TS)^*, \qquad (4)\ ST = (ST)^*.$$

We could denote the $\epsilon - \{2,3,4\}$ inverse of T as $T^{\epsilon-\{2,3,4\}}$, but by simplicity, we denote it by T^{ϵ}. We have the next assertions are satisfied from the definition.

Remark.

(a) If $\epsilon = 0$, then $0 - \{2,3,4\}$ inverse is the MP-inverse.

(b) It is clear that, if $\epsilon < \epsilon_1$, then the ϵ-$\{2,3,4\}$ inverse is an ϵ_1-$\{2,3,4\}$ inverse.

(c) Through this paper we will denote $TST - T = E$ and assume that $\epsilon > 0$. From definition we have that:

1. $R(E) \subseteq R(I - TT^{\epsilon})$ and $N(I - T^{\epsilon}T) \subseteq N(E)$.
2. $TT^{\epsilon} \in B(\mathcal{K})$ is a orthogonal projection with $N(TT^{\epsilon}) = N(T^{\epsilon})$.
3. $T^{\epsilon}T \in B(\mathcal{H})$ is a orthogonal projection with $R(T^{\epsilon}T) = R(T^{\epsilon})$.
4. $R(T^{\epsilon})$ is closed in $\mathcal{H}$. Therefore, there exists the MP-inverse for any ϵ-inverse of T.

(d) If T has an ϵ-{2,3,4} inverse, then T^* have an ϵ-{2,3,4} inverse and $(T^*)^{\epsilon} = (T^{\epsilon})^*$, where $T^*(T^*)^{\epsilon}T^* - T^* = E^*$.

(e) If T is invertible (respectively, T has closed range) then for any $0 < \epsilon < \gamma(T)$, T has a ϵ-{2, 3, 4} inverse and $T^{\epsilon} = T^{-1}$ (respectively, $T^{\epsilon} = T^{\dagger}$).

In the above definition we can think to S as the computed estimate and the E as error term. By Remark 3.3, it is clear that, when the ϵ-{2,3,4} inverse exists for a non-closed range operator, it is not necessary unique. As shown in the following example.

Example 5. *Let $\mathcal{H} = l^2(\mathbb{N})$ the space square summable sequences, $\epsilon > 0$ and $T \in B(\mathcal{H})$ defined by*

$$Th := (\tfrac{1}{2}h_2, \tfrac{1}{3}h_3, \tfrac{1}{4}h_4, \ldots), \;\; h = (h_1, h_2, h_3, \ldots) \in \mathcal{H}.$$

We remark that $R(T) \neq \overline{R(T)}$ (and T don't have MP-inverse).

Set $n_\epsilon = \min\{n \in \mathbb{N} \big| \frac{1}{\epsilon} < n\}$, *and* $S_{n_\epsilon} \in B(\mathcal{H})$ *defined by*

$$S_{n_\epsilon} h := (0, 2h_1, 3h_2, 4h_3, \ldots, n_\epsilon h_{n_\epsilon - 1}, 0, 0, 0, \ldots).$$

Then, we have $TS_{n_\epsilon}T - T = E$ *where*

$$Eh := (\underbrace{0, \ldots, 0}_{n_\epsilon - 1}, \tfrac{1}{n_\epsilon+1} h_{n_\epsilon+1}, \tfrac{1}{n_\epsilon+2} h_{n_\epsilon+2}, \tfrac{1}{n_\epsilon+3} h_{n_\epsilon+3}, \ldots)$$

with $\|E\| \leq \epsilon$, *and*

$$\begin{aligned} S_{n_\epsilon} T S_{n_\epsilon} =& S_{n_\epsilon} T(0, 2h_1, 3h_2, 4h_3, \ldots, n_\epsilon h_{n_\epsilon-1}, 0, 0, 0, \ldots) \\ =& S_{n_\epsilon}(h_1, h_2, h_3, \ldots, h_{n_{\epsilon-1},0,0,0,\ldots}) \\ =& (0, 2h_1, 3h_2, 4h_3, \ldots, n_\epsilon h_{n_\epsilon-1}, 0, 0, 0, \ldots) \\ =& S_{n_\epsilon}. \end{aligned}$$

On the other hand, since

$$T^* u := (0, \tfrac{1}{2} u_1, \tfrac{1}{3} u_2, \tfrac{1}{4} u_3, \ldots)$$

and

$$S^* u := (2u_2, 3u_3, 4u_4, \ldots, n_\epsilon u_{n_\epsilon}, 0, 0, 0, \ldots)$$

for all $u \in \mathcal{H}$. *Then we have*

$$TS_{n_\epsilon} = (TS_{n_\epsilon})^* \quad \text{and} \quad S_{n_\epsilon} T = (S_{n_\epsilon} T)^*.$$

So, S_{n_ϵ} *is a* ϵ-*{2,3,4} inverse of* T.

Therefore, for any integer $n > n_\epsilon$,

$$S_n h := (0, 2h_1, 3h_2, 4h_3, \ldots, nh_{n-1}, 0, 0, 0, \ldots),$$

is an ϵ-*{2,3,4} inverse for* T *different to* S_{n_ϵ}. *So,* T *has not unique* ϵ-*{2,3,4} inverse.*

The next proposition show a relation between the MP-inverse of T^ϵ and some projections with range and nullspace prescribed.

Proposition 3.6. *Let* $T \in B(\mathcal{H}, \mathcal{K})$ *and suppose there exists* T^ϵ *for some* $\epsilon > 0$. *Then for each pair of projections* P *and* Q, *in* $\mathcal{H}, \mathcal{K}$ *respectively, such that* $R(P) = R(T^\epsilon)$ *and* $N(Q) = N(T^\epsilon)$, *we have that* $(T^\epsilon)^\dagger$ *satisfy the next equalities:* $(T^\epsilon)^\dagger T^\epsilon = Q$ *and* $T^\epsilon (T^\epsilon)^\dagger = P$.

Proof. Let $P \in B(\mathcal{H})$ and $Q \in B(\mathcal{K})$ such that $R(P) = R(T^\epsilon)$ and $N(Q) = N(T^\epsilon)$. We define $M = N(P)$ and $N = R(Q)$ closed subspaces in $\mathcal{H}$ and $\mathcal{K}$, respectively. Then, we have that $\mathcal{K} = N(T^\epsilon) \oplus N$, so we can define $T^\epsilon|_N \in B(N, R(T^\epsilon))$ invertible operator. We define the operator;

$$S : M \oplus R(T^\epsilon) \to N(T^\epsilon) \oplus N \ \text{ defined by } \ S(h) = (T^\epsilon|_N)^{-1}(h_2)$$

where $h = h_1 + h_2 \in M \oplus R(T^\epsilon)$. It's clear that $S \in B(\mathcal{H}, \mathcal{K})$. Elementary operation show that $S = (T^\epsilon)^\dagger$ with $(T^\epsilon)^\dagger T^\epsilon = Q$ and $T^\epsilon(T^\epsilon)^\dagger = P$ □

3.4 Matrix representation and norm for the ϵ-{2,3,4} inverse

A block matrix representation for a generalized inverse through adequate decomposition of the spaces $\mathcal{H}$ and $\mathcal{K}$ is always a good tool to deal with it. For example, it is known that MP-inverse (when it exists) has nice operator matrix form in the respect of decomposition $\mathcal{H} = N(T) \oplus N(T)^\perp$ and $\mathcal{K} = R(T)^\perp \oplus R(T)$. The next theorem gives us a similar operator matrix representation of an ϵ-{2,3,4} inverse.

Theorem 3.7. *Let $T \in B(\mathcal{H}, \mathcal{K})$ and let $\epsilon > 0$. Then there exists a ϵ-{2,3,4} inverse of T if and only if there exists a closed subspace M of $\mathcal{H}$ such that $T(M^\perp)$ is a closed subspace of $\mathcal{K}$ and $N(T) \subseteq M$ with $\|Tx\| \leq \epsilon\|x\|$ for all $x \in M$.*

Proof.

($\Leftarrow$) Let $\epsilon > 0$ and let M a closed subspace of $\mathcal{H}$ with the assumptions of theorem. We consider decompositions for $\mathcal{H}$ and $\mathcal{K}$ as follows: $\mathcal{H} = M^\perp \oplus M$ and $\mathcal{K} = T(M^\perp) \oplus T(M^\perp)^\perp$. We define; $N := T(M^\perp)$. Then we have the following matrix form for T:

$$T = \begin{bmatrix} T_1 & T_3 \\ T_4 & T_2 \end{bmatrix} : \begin{bmatrix} M^\perp \\ M \end{bmatrix} \to \begin{bmatrix} N \\ N^\perp \end{bmatrix},$$

where $T_1 \in B(M^\perp, N)$ is an invertible operator (like as a bounded below and surjective). Moreover, $T_3 = 0$. Really, if we suppose that exists $x_M \in M$ such that $0 \neq T_3 x_M \in M^{\perp}$, then there exists $x_{M^{\perp}} \in M^{\perp}$ such that $T_1(x_{M^{\perp}}) = T_3(x_M)$. In this case $T(x_{M^{\perp}} + x_M) = 0$, that implies $x_{M^{\perp}} + x_M \in N(T) \subseteq M$ and, consequently, $x_{M^{\perp}} \in M$. Hence, $x_{M^{\perp}} = 0$ and $0 = T_3 x_M$.

In the respect to the same decomposition for $\mathcal{H}$ and $\mathcal{K}$, we consider $S \in B(\mathcal{K}, \mathcal{H})$ defined as follows:

$$S = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} N \\ N^{\perp} \end{bmatrix} \to \begin{bmatrix} M^{\perp} \\ M \end{bmatrix}.$$

The operator S is a ϵ-{2,3,4} inverse of T. In fact,

$$\begin{aligned} E = TST - T &= \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} - \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \\ &= \begin{bmatrix} I|_N & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} - \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \\ &= \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & -T_2 \end{bmatrix}. \end{aligned}$$

This, $\|E\| = \|T_2\|$ and $T_2 = T|_M$ then $\|E\| \leq \epsilon$.

Now,

$$\begin{aligned} STS &= \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I|_N & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= S \end{aligned}$$

On the other hand, by the shape of T and S operators, we have that T^* and S^* has the following matrix form:

$$T^* = \begin{bmatrix} T_1^* & 0 \\ 0 & T_2^* \end{bmatrix} \quad \text{and} \quad S^* = \begin{bmatrix} (T_1^{-1})^* & 0 \\ 0 & 0 \end{bmatrix}.$$

Then,

$$S^*T^* = \begin{bmatrix} I|_N & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad T^*S^* = \begin{bmatrix} I|_{M^\perp} & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, $TS = (TS)^*$ and $ST = (ST)^*$. So, we conclude that $S = T^\epsilon$.

($\Rightarrow$) Let $\epsilon > 0$ and let $S = T^\epsilon$. By (2) of Definition 3.5 we have that T is a inner inverse of S. Then $N(S)$ and $R(S)$ are closed and complemented subspaces of $\mathcal{K}$ and $\mathcal{H}$, respectively (see [11, Corollary 1.1.5]). Moreover, $R(S) = R(ST)$ and, by (4) of Definition 3.5, $R(ST)^\perp = R(I - ST)$. Similarly, $N(TS) = N(S)$ and $N(TS)^\perp = N(I - TS)$. Then, we have the following matrix form for S and T:

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} N(I-TS) \\ N(TS) \end{bmatrix} \to \begin{bmatrix} R(ST) \\ R(I-ST) \end{bmatrix},$$

with $S_1 \in B(N(TS)^\perp, R(S))$ invertible, and, since T is a inner inverse of S with TS and ST orthogonal projections ([11, Theorem 1.1.6]),

$$T = \begin{bmatrix} S_1^{-1} & 0 \\ 0 & T_2 \end{bmatrix} : \begin{bmatrix} R(ST) \\ R(I-ST) \end{bmatrix} \to \begin{bmatrix} N(I-TS) \\ N(TS) \end{bmatrix}.$$

Set $M = R(I - ST)$. Then, $N(T) \subseteq M$ and, for $x \in M$,

$$\|Tx\| = \|T(I - ST)x\| = \|Ex\| < \epsilon\|x\|.$$

Moreover, $T(M^\perp) = N(I - TS)$, that is a closed subspace of $\mathcal{K}$. □

By the proof of the Theorem 3.7, for the operator $U \in B(\mathcal{H}, \mathcal{K})$ define as

$$U = \begin{bmatrix} 0 & 0 \\ 0 & -T_2 \end{bmatrix} : \begin{bmatrix} R(ST) \\ R(I-ST) \end{bmatrix} \to \begin{bmatrix} N(I-TS) \\ N(TS) \end{bmatrix},$$

we have that $\|U\| < \epsilon$ and $T + U$ has the closed range. Hence we have next:

Corollary 3.8. *Let $T \in B(\mathcal{H}, \mathcal{K})$ and let $\epsilon \geq 0$ be such that there exists an ϵ-{2,3,4} inverse of T. Then there exists an operator $U \in B(\mathcal{H}, \mathcal{K})$ such that $\|U\| \leq \epsilon$ and $T + U$ has closed range.*

Remark. Let $T \in B(\mathcal{H},\mathcal{K})$ has an ϵ-{2,3,4} inverse T^ϵ. Then, by Theorem 3.7 we have the next matrix form of T and T^ϵ:

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} : \begin{bmatrix} M^\perp \\ M \end{bmatrix} \to \begin{bmatrix} N \\ N^\perp \end{bmatrix}$$

and

$$T^\epsilon = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} N \\ N^\perp \end{bmatrix} \to \begin{bmatrix} M^\perp \\ M \end{bmatrix},$$

for some closed subspace M of $\mathcal{H}$ and $N = T(M^\perp)$. So, for $0 \neq k \in N$, there exists the unique $0 \neq h \in M^\perp$ such that $k = T_1 h$. Moreover, since $T^\epsilon k = T^\epsilon k_1 + T^\epsilon k_2 = T_1^{-1} k_1$ with $k = k_1 + k_2 \in N \oplus N^\perp = \mathcal{K}$ then we have:

$$\begin{aligned} \|T^\epsilon\| &= \sup\left\{ \frac{\|T^\epsilon k\|}{\|k\|} \middle| k \in N, k \neq 0 \right\} \\ &= \sup\left\{ \frac{\|T_1^{-1} T_1 h\|}{\|T_1 h\|} \middle| h \in M^\perp, h \neq 0 \right\} \\ &= \left(\inf\left\{ \frac{\|T_1 h\|}{\|h\|} \middle| h \in M^\perp, h \neq 0 \right\} \right)^{-1} \leq \frac{1}{\gamma(T_1)}. \end{aligned}$$

Hence, $\|T^\epsilon\| \leq \gamma(T_1)^{-1} = \|T_1^{-1}\|$.

3.5 "Error bound" for the ϵ-{2,3,4} inverse

A linear operator $T \in B(\mathcal{H},\mathcal{K})$ with the closed range has the MP-inverse and, in this case, for ϵ small enough, $T^\epsilon = T^\dagger$ (see Corollary 3.3). The next theorem give us norm error estimate between the MP-inverse and an arbitrary ϵ-{2,3,4} inverse.

Theorem 3.9. *Let $T \in B(\mathcal{H},\mathcal{K})$ has closed range. Then, for all $\epsilon \geq 0$ and every ϵ-{2,3,4} inverse of T, holds $\|T^\epsilon - T^\dagger\| \leq \dfrac{\epsilon}{(\gamma(T))^2}$.*

Proof. Let $\gamma(T) > \epsilon \geq 0$. In this case $T^\epsilon = T^\dagger$ and the proof follows immediately. Now, let $\epsilon \geq \gamma(T) > 0$ and $A = T^\epsilon - T^\dagger \in B(\mathcal{K},\mathcal{H})$. By the definitions of the MP and ϵ-{2,3,4} inverses we have:

(1.1) $TAT = E$,

(1.2) $ATT^\dagger + T^\dagger TA + ATA - A = 0$,

(1.3) $TA - A^*T^* = 0$,

(1.4) $AT - T^*A^* = 0$.

Where shall use this equations to solve for A in terms of T, $T^\dagger$ and E. From (1.1) we get

$$T^\dagger TATT^\dagger = T^\dagger ET^\dagger. \qquad (1.5)$$

From (1.2) we get

$$(I - T^\dagger T)ATA(I - TT^\dagger) - (I - T^\dagger T)A(I - TT^\dagger) = 0. \qquad (1.6)$$

Now, from (1.3) we get

$$T^\dagger TA(I - TT^\dagger) = 0. \qquad (1.7)$$

Analogously, from (1.4) we get

$$(I - T^\dagger T)ATT^\dagger = 0. \qquad (1.8)$$

Finally, using (1.7) and (1.8) we have

$$\begin{aligned}(I - T^\dagger T)A(I - TT^\dagger) &= (I - T^\dagger T)ATT^\dagger TT^\dagger TA(I - TT^\dagger)\\ &= (I - T^\dagger T)ATT^\dagger T(0)\\ &= (I - T^\dagger T)ATT^\dagger(0)\\ &= 0\end{aligned}$$

and

$$\begin{aligned}A \;=\;& T^\dagger TATT^\dagger + (I - T^\dagger T)ATT^\dagger + T^\dagger TA(I - TT^\dagger) +\\ & (I - T^\dagger T)A(I - TT^\dagger).\end{aligned}$$

Hence,

$$\begin{aligned}\|A\| \;\leq\;& \|T^\dagger TATT^\dagger\| + \|(I - TT^\dagger)ATT^\dagger\| +\\ & \|T^\dagger TA(I - TT^\dagger)\| + \|(I - T^\dagger T)A(I - TT^\dagger)\|\\ \leq\;& \|T^\dagger ET^\dagger\|\\ <\;& \epsilon\|T^\dagger\|^2\\ =\;& \frac{\epsilon}{(\gamma(T))^2}.\end{aligned}$$

□

Remark. By Theorem 3.9, it is obvious that $T^{\epsilon_n} \to T^{\dagger}$, when $\epsilon \to 0$. Matter in fact, we only need that $\epsilon_n \to \gamma(T)_+$.

Corollary 3.10. *Let $T \in B(\mathcal{H}, \mathcal{K})$. If $\{T_n\}_{n\in\mathbb{N}}$ is a sequence of ϵ-{2,3,4}inverses of T in $B(\mathcal{K}, \mathcal{H})$ such that the sequence $\{TT_nT - T\}_{n\in\mathbb{N}}$ converge to zero, then $\{T_n\}_{n\in\mathbb{N}}$ converge to $T^{\dagger}$.*

3.6 The ϵ-{2,3,4} inverse of the module of T

In this section we show the shape of an ϵ-{2,3,4} inverse for $|T| \in B(\mathcal{H})^+$, where $|T|$ is an operator that belongs to polar decomposition of T, $T = U|T|$. We begin with some results about the ϵ-{2,3,4} inverse of an adjoint, positive, unitary operator that will help us for the objective of this section.

Proposition 3.11. *If $T \in \mathcal{A}(\mathcal{H})$ has an ϵ-{2,3,4} inverse T^{ϵ}, then $T^{\epsilon} \in \mathcal{A}(\mathcal{H})$.*

Proof. Let $T \in \mathcal{A}(\mathcal{H})$ and $\epsilon > 0$. Suppose that there exist a ϵ-{2,3,4} inverse of T, T^{ϵ}. By 3.3 we have that $(T^*)^{\epsilon} = (T^{\epsilon})^*$, for the adjoint operator T^*. On the other hand, we know that $T^{\epsilon} = (T^*)^{\epsilon}$ since $T \in \mathcal{A}(\mathcal{H})$. So, we have $T^{\epsilon} = (T^{\epsilon})^*$, i.e., $T^{\epsilon} \in \mathcal{A}(\mathcal{H})$. □

Lemma 3.12. *Let $U \in \mathcal{U}(\mathcal{H})$ Then, for any $\epsilon > 0$, we can choose $U^{\epsilon} = U^* \in \mathcal{U}(\mathcal{H})$.*

Proof. Since $U \in \mathcal{U}(\mathcal{H})$, then U is invertible. So, we know that $U^{\epsilon} = U^{-1}$, but $U^{-1} = U^*$, therefore $U^{\epsilon} = U^*$. Finally, $(U^{\epsilon})^* = (U^*)^* = U$ and, on the other hand, $(U^{\epsilon})^{-1} = (U^{-1})^{-1} = U$. Therefore, $U^{\epsilon} \in \mathcal{U}(\mathcal{H})$. □

Proposition 3.13. *Let $T \in B(\mathcal{H})$ and $U, V \in \mathcal{U}(\mathcal{H})$. If there exists ϵ-{2,3,4} inverse of T, then there exists an ϵ-{2,3,4} inverse of UTV and $(UTV)^{\epsilon} = V^*T^{\epsilon}U^*$.*

Proof. We show that $V^*T^{\epsilon}U^*$ satisfy the equations (ϵ), (2), (3) and (4) in the Definition 3.5 for UTV.

(ϵ)

$$\begin{aligned}\|UTV(V^*T^\epsilon U^*)UTV - UTV\| &= \|UTT^\epsilon TV - UTV\| \\ &= \|UTV + UEV - UTV\| \\ &= \|UEV\| = \|E\| \leq \epsilon.\end{aligned}$$

(2)

$$V^*T^\epsilon U^*(UTV)V^*T^\epsilon U^* = V^*T^\epsilon TT^\epsilon U^* = V^*T^\epsilon U^*.$$

(3)

$$\begin{aligned}\Big(UTV(V^*T^\epsilon U^*)\Big)^* &= \big(V^*T^\epsilon U^*\big)^* V^*T^\epsilon U^* \\ &= U(T^\epsilon)^*T^*U^* \\ &= UTT^\epsilon U^* \\ &= UTV(V^*T^\epsilon U^*).\end{aligned}$$

(4) Analogous to (3).

□

Lemma 3.14. *Let $T \in B(\mathcal{H})$ has an ϵ-{2,3,4} inverse T^ϵ. If $T \in B(\mathcal{H})^+$ then $T^\epsilon \in B(\mathcal{H})^+$.*

Proof. Since $T \in \mathcal{A}(\mathcal{H})$, by 3.11, we have that $T^\epsilon \in \mathcal{A}(\mathcal{H})$. Now, for all $x \in \mathcal{H}$ we have:

$$\begin{aligned}\langle T^\epsilon x, x\rangle &= \langle T^\epsilon TT^\epsilon x, x\rangle \\ &= \langle T(T^\epsilon x), T^\epsilon x\rangle \\ &\geq 0.\end{aligned}$$

□

Theorem 3.15. *Let $T = W|T|$ be the polar decomposition of $T \in B(\mathcal{H})$. Then T has an ϵ-{2,3,4} inverse if and only if $|T|$ has one. In this case $|T|^\epsilon = T^\epsilon W$ and $T^\epsilon = |T|^\epsilon W^*$.*

Proof. ($\Rightarrow$) Let $T = U|T|$ the polar decomposition of T. We show that $T^\epsilon U$ satisfy the equations (ϵ), (2), (3) and (4) in the Definition 3.5 for $|T|$.

(ϵ)

$$\begin{aligned}\||T|(T^{\epsilon}U)|T| - |T|\| &= \|U^{*}TT^{\epsilon}T - U^{*}T\| \\ &\leq \|U^{*}\|\|TT^{\epsilon}T - T\| \\ &\leq 1 \cdot \epsilon = \epsilon.\end{aligned}$$

(2)

$$\begin{aligned}T^{\epsilon}U|T|T^{\epsilon}U &= T^{\epsilon}UU^{*}TT^{\epsilon}U \\ &= T^{\epsilon}TT^{\epsilon}U \\ &= T^{\epsilon}U.\end{aligned}$$

(3)

$$\begin{aligned}\Big(|T|(T^{\epsilon}U)\Big)^{*} &= U^{*}(TT^{\epsilon})^{*}U \\ &= U^{*}TT^{\epsilon}U \\ &= |T|(T^{\epsilon}U).\end{aligned}$$

(4) Analogous to (3).

($\Leftarrow$) If $|T|$ has an ϵ-{2,3,4} inverse $|T|^{\epsilon}$, then, similarly to previous case, we can proof that $|T|^{\epsilon}W^{*}$ is an ϵ-{2,3,4} inverse of T. □

Example 6. *Let $\mathcal{H} = l^2(\mathbb{N})$ the space square summable sequences, $\epsilon > 0$ and $T \in B(\mathcal{H})$ defined in 5. Then, we know from 5 that*

$$T^{\epsilon}h := (0, 2h_1, 3h_2, 4h_3, \ldots, n_{\epsilon}h_{n_{\epsilon}-1}, 0, 0, 0, \ldots),$$

is a ϵ-{2,3,4} inverse of T, where $n_{\epsilon} = \min\{n \in \mathbb{N} | \frac{1}{\epsilon} < n\}$ and $h \in \mathcal{H}$. Now, we know by 1.22 that the polar decomposition of T is $W|T|$ with W a partial isometry and $|T|$ the module of T, defined by:

$$Wh := (h_2, h_3, h_4, \ldots) \quad \text{and} \quad |T|h := (0, \tfrac{h_2}{2}, \tfrac{h_3}{3}, \tfrac{h_4}{4}, \ldots),$$

for all $h = (h_1, h_2, h_3, \ldots,) \in \mathcal{H}$. Therefore, by Theorem 3.15, an ϵ-{2,3,4} inverse of $|T|$ is defined by:

$$|T|^{\epsilon}h := (0, 2h_2, 3h_3, 4h_4, \ldots, n_{\epsilon}h_{n_{\epsilon}}, 0, 0, 0, \ldots,).$$

A direct calculation show that $|T|^{\epsilon}$ satisfy the conditions (ϵ), (2), (3) and (4) of Definition 3.5.

3.7 Matrix relation between the ϵ-{2,3,4} inverse and the MP inverse

Let $T \in B(\mathcal{H}, \mathcal{K})$ has an ϵ-{2,3,4} inverse, T^ϵ. Then, we have the next matrix form of T and T^ϵ:

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} : \begin{bmatrix} M^\perp \\ M \end{bmatrix} \to \begin{bmatrix} N \\ N^\perp \end{bmatrix}$$

and

$$T^\epsilon = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} N \\ N^\perp \end{bmatrix} \to \begin{bmatrix} M^\perp \\ M \end{bmatrix},$$

for some closed subspace M of $\mathcal{H}$ with $N(T) \subseteq M$ and $N = T(M^\perp)$ a subspace closed of $\mathcal{K}$. On the other hand, we know that $\mathcal{H} = N(T) \oplus N(T)^\perp$ and if $R(T) = \overline{R(T)}$ then $\mathcal{K} = R(T)^\perp \oplus R(T)$. So we have the next matrix form of T and $T^\dagger$:

$$T = \begin{bmatrix} T_0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} N(T)^\perp \\ N(T) \end{bmatrix} \to \begin{bmatrix} R(T) \\ R(T)^\perp \end{bmatrix}$$

and

$$T^\dagger = \begin{bmatrix} T_0^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(T) \\ R(T)^\perp \end{bmatrix} \to \begin{bmatrix} N(T)^\perp \\ N(T) \end{bmatrix}.$$

So we have,

$$N(T) \subset M \Longrightarrow M^\perp \subset N(T)^\perp,$$

$$N \subset R(T) \Longrightarrow R(T)^\perp \subset N^\perp.$$

Therefore, we have the following

$$\mathcal{H} = N(T) \oplus M^\perp \oplus M_1 \quad \text{and} \quad \mathcal{K} = N \oplus N_1 \oplus R(T)^\perp$$

with $M^\perp \oplus M_1 = N(T)^\perp$ [by ProjTheo 2°v $M_1 = N(T)^\perp \ominus M^\perp$] and $N \oplus N_1 = R(T)$ [By ProjTheo 2°v $N_1 = R(T) \ominus N$].

We set

$$T_{0,1}^{-1} : \underset{n}{N} \underset{\longmapsto}{\longrightarrow} \underset{T_1^{-1}(n)}{M^\perp} \quad \text{and} \quad T_{0,2}^{-1} : \underset{n_1}{N_1} \underset{\longmapsto}{\longrightarrow} \underset{T_0^{-1}(n_1)}{M_1},$$

then

$$T_0^{-1} = T_{0,1}^{-1} \oplus T_{0,2}^{-1}.$$

Therefore, $T_0^{-1}|_N = T_1^{-1}$. That is, $T^\dagger|_N = T^\epsilon$.

3.8 Some results over the spectrum of positive operators that have an ϵ-{2,3,4} inverse

Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})$. Let $S \sqsubseteq \mathcal{H}$ a subspace, then we say that:

- If we define $T_S : S \to S$ as $T_S x = Tx$ for all $x \in S$, then it is clear that $T_S \in B(\mathcal{H})$.
- We know that S reduces to T, if S and $S^{\perp}$ are T-invariant. Therefore, we say that T is completly reduced by the pair $(S, S^{\perp})$ if $\mathcal{H} = S \oplus S^{\perp}$. In this case we write $T = T_S \oplus T_{S^{\perp}}$. Also, we have $N(T) = N(T_S) \oplus N(T_{S^{\perp}})$, $R(T) = R(T_S) \oplus R(T_{S^{\perp}})$ and $T^n = T_S^n \oplus T_{S^{\perp}}^n$,

For an operator $T \in B(\mathcal{H})$ we present a series of results that are well known:

- $T^*T \in B(\mathcal{H})^+$.
- $R(T)$ is closed if and only if $R(T^*T)$ is closed.
- $N(T)$ and $N(T)^{\perp}$ are invariant under T^*T and by [37, Theorem 5.4] it has

$$\sigma(T^*T) = \sigma(T^*T|_{N(T)}) \bigcup \sigma\left(T^*T|_{N(T)^{\perp}}\right) \subseteq \{0\} \bigcup [0, \|T\|^2].$$

Now some results that will help us with respect to the spectrum of an ϵ-{2,3,4} inverse operator.

Proposition 3.16. *[29, Proposition 3.1] Let $T \in L(\mathcal{H})$ be a positive operator. Then we have the following results.*

1. $\sigma(T) \setminus \{0\} = \sigma(T|_{N(T)^{\perp}}) \setminus \{0\}$.
2. $\sigma(T) = \sigma_a(T)$.
3. $0 \notin \sigma(I + T)$, *that is* $(I + T)^{-1} \in B(\mathcal{H})$.
4. *If* $0 \notin \sigma(T)$, *then* $0 \neq \lambda \in \sigma(T) \Leftrightarrow \frac{1}{\lambda} \in \sigma(T^{-1})$.

Theorem 3.17. *[28, Theorem 2.1] Let $T \in B(H,K)$ nonzero. The $R(T)$ is closed if and only if there exists $\gamma > 0$ such that $\sigma(T^*T|_{N(T)^\perp}) \subseteq [\gamma, \|T\|^2]$. And this case, $\sigma(T^*T) \subseteq \{0\} \cup [\gamma, \|T\|^2]$.*

Proposition 3.18. *[28, Proposition 2] Let $\mathcal{H}$ a complex Hilbert space and $T \in B(\mathcal{H})$ a self adjoint operator. Then every isolated spectral value of T is an eigenvalue of T.*

Then we have the following results for the spectrum of an ϵ-{2,3,4} inverse operator if we consider $T \in B(\mathcal{H})$:

$$T^\epsilon = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} N \\ N^\perp \end{bmatrix} \to \begin{bmatrix} M^\perp \\ M \end{bmatrix},$$

for some closed subspace M of $\mathcal{H}$ with $N(T) \subseteq M$ and $N = T(M^\perp)$ a subspace closed of $\mathcal{H}$.

Lemma 3.19. *If $T \in B(\mathcal{H})^+$ has an ϵ-{2,3,4} inverse T^ϵ, then $\sigma(T^\epsilon) \setminus \{0\} = \sigma(T_1^{-1})$.*

Proof. We known that $T^\epsilon \in \mathcal{A}(\mathcal{H})$ and that $T(M^\perp)$ reduce to T^ϵ. Now, since $T^\epsilon|_{T(M^\perp)} = T_1^{-1}$, we have $\sigma(T^\epsilon) = \sigma(T^\epsilon|_{T(M^\perp)^\perp}) \oplus \sigma(T^\epsilon|_{T(M^\perp)}) = \{0\} \cup \sigma(T_1^{-1})$. Therefore, $\sigma(T^\epsilon) \setminus \{0\} = \sigma(T_1^{-1})$. □

Lemma 3.20. *If $T \in B(\mathcal{H})^+$ has an ϵ-{2,3,4} inverse T^ϵ, then $I + T^\epsilon \in \mathcal{GL}(\mathcal{H})$.*

Proof. Since T positive, then T^ϵ is positive. So, we have $I + T^\epsilon$ is positive. Now, suppose that $0 \in \sigma(I + T^\epsilon)$. Then, there exists a sequence $\{h_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ of unit vectors such that $\|(I+T^\epsilon)h_n\| \to 0$ as $n \to \infty$. Since $|\langle (I + T^\epsilon)h_n, h_n\rangle| \leq \|(I + T^\epsilon)h_n\|\|h_n\| = \|(I + T^\epsilon)h_n\|$ (Cauchy-Schwartz inequality) then $\langle (I + T^\epsilon)h_n, h_n\rangle \to 0$ as $n \to \infty$. That is equal to $\langle T^\epsilon h_n, h_n\rangle \to -1$. But, on the other hand, $\langle T^\epsilon h, h\rangle \geq 0$ for all $h \in \mathcal{H}$. Hence, this is a contradiction. Therefore, $0 \notin \sigma(I + T^\epsilon)$. □

Chapter 4

The ϵ-{2,3,4} inverse for compact operators

In this chapter we study the existence and the construction of the ϵ-{2,3,4} inverse of a compact operator T in $B(\mathcal{H})$. The Compact operators are those whose properties most closely resemble those of linear operators in finite-dimensional spaces therefore their spectral theory will be a direct extension of those and a simple way to get into this study. Many of the operators that appear in the study of integral equations are also compact operators. Thus, this application of spectral theory leads to Fredholm's theory of integral equations.

4.1 Existence of the ϵ-{2,3,4} inverse

First, we observe a relationship between the Drazin inverse and the Moore-Penrose inverse as follows. We know that if $T \in B(\mathcal{H})$ then

$$T^*(TT^*)^\dagger = T^\dagger = (T^*T)^\dagger T^*.$$

Furthermore, since T^*T and TT^* are self adjoint operators we obtain

$$(T^*T)^\dagger = (T^*T)^d \text{ and } (TT^*)^\dagger = (TT^*)^d.$$

In fact, as $T^*T(T^*T)^\dagger = (T^*T(T^*T)^\dagger)^* = ((T^*T)^\dagger)^*(T^*T)^* = ((T^*T)^*)^\dagger(T^*T) = (T^*T)^\dagger T^*T$. Therefore,

$$T^*(TT^*)^d = T^\dagger = (T^*T)^d T^*. \tag{4.1}$$

Now, an auxiliary result that will help us for what we are looking for.

Proposition 4.1. *Let $T \in B(\mathcal{H}, \mathcal{K})$. There exists $T^\dagger$ if and only if $0 \notin acc\ \sigma(T^*T)$.*

Proof.

$\Rightarrow$] There exist $T^\dagger$ if and only if $R(T)$ is closed. Then, by Theorem 3.17, we have $\sigma(T^*T) \subseteq \{0\} \cup [\gamma, \|T\|^2]$ for some $\gamma > 0$. Therefore $0 \in iso(\sigma(T^*T))$ or $0 \notin \sigma(T^*T)$. So, $0 \notin acc(\sigma(T^*T))$.

$\Leftarrow$] We know that $T^*T \in B^+(\mathcal{H})$, then by [29, Theorem 4.4] we have $R(T^*T)$ is closed if and only if $0 \notin acc\ (\sigma(T^*T))$. Then, by 3.8, we have that $R(T)$ is closed. Therefore, there exists $T^\dagger$.

□

To star we take $T \in B(\mathcal{H})$. If we set $A := T^*T$, then $A \in B^+(\mathcal{H})$ is a positive operator. We have the following cases:

1. If $0 \notin \sigma(A)$ then there exists $A^{-1} \in B(\mathcal{H})$. As A is Drazin invertible with $A^d = A^{-1}$, we have that there exists $T^\dagger$, by Ec.4.1. ($0 \notin \sigma(A) \Rightarrow \exists T^\dagger$).
2. If $0 \in iso(\sigma(A))$, then we have, by Proposition 4.1, there exists $T^\dagger$ ($0 \in iso(\sigma(A)) \Rightarrow \exists T^\dagger$).

From the above we can conclude that there exists T^ϵ for all $\epsilon \in [0, \gamma(T))$.

4.2 Construction of an ϵ-{2,3,4} inverse for compact operators in B($\mathcal{H}$)

Let $T \in B(\mathcal{H}, \mathcal{K})$ be a compact operator. Then $T^* \in B(\mathcal{K}, \mathcal{H})$ is a compact operator and, $A' = TT^* \in B(\mathcal{K})$ is a compact positive self-adjoint operator with

$$\sigma(A') \setminus \{0\} = \sigma_p(A') \setminus \{0\}$$

and

$$\sigma(A') = \{0\} \cup \{\lambda_n \in \mathbb{R}^+ \, : \, n \in \mathbb{N}\},$$

where λ_n are eigenvalues of A' such that $\dim N(A' - \lambda_n I) = m_n < \infty$. Moreover, $A = T^*T \in B(\mathcal{H})$ is a compact positive self-adjoint operator too with $\sigma(A) = \sigma(A')$, $\sigma_p(A') = \sigma_p(A)$ and $\dim N(A - \lambda_n I) = m_n < \infty$.

We can suppose that the eigenvalues of A' are ordered in the way that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n > \cdots \to 0.$$

For arbitrary $\epsilon > 0$, let $N_\epsilon \in \mathbb{N}$ be such that $\lambda_{N_\epsilon+1} < \epsilon^2 < \lambda_{N_\epsilon}$.

Now, set

$$N_\epsilon^* := span \cup_{n=1}^{N_\epsilon} N(A' - \lambda_n I)$$

Note that N_ϵ^* is (closed) finite-dimensional subspace of $\mathcal{K}$.

Let $(N_\epsilon^*)^\perp$ the complement orthogonal for N_ϵ^*. Then $T^*\left((N_\epsilon^*)^\perp\right)$ is closed in $\mathcal{H}$. If $M_\epsilon = T^*\left((N_\epsilon^*)^\perp\right)$ then we going to prove $M_\epsilon^\perp = T^*(N_\epsilon^*)$ or, equivalently, $M_\epsilon = (T^*(N_\epsilon^*))^\perp$. In fact, let $T^*z \in M_\epsilon = T^*\left((N_\epsilon^*)^\perp\right) \subseteq \mathcal{H}$. Then

$$\begin{aligned}
T^*z \in M_\epsilon &\Leftrightarrow \langle z, k\rangle = 0 \text{ for all } k \in N_\epsilon^*,\\
&\Leftrightarrow \lambda_i^{m_i}\langle z, \sum_{i=1}^{N_\epsilon}\sum_{j=1}^{m_i}\alpha_i^j k_i^j\rangle = \lambda_i^{m_i}\cdot 0 = 0;\\
&\qquad \text{where } k = \left(\sum_{i=1}^{N_\epsilon}\sum_{j=1}^{m_i}\alpha_i^j k_i^j\right) \text{ for all } k \in N_\epsilon^*,\\
&\Leftrightarrow \langle z, \sum_{i=1}^{N_\epsilon}\sum_{j=1}^{m_i}\lambda_i^{m_i}\alpha_i^j k_i^j\rangle = 0 \text{ for all } k \in N_\epsilon^*,\\
&\Leftrightarrow \langle z, \sum_{i=1}^{N_\epsilon}\sum_{j=1}^{m_i}\alpha_i^j A' k_i^j\rangle = 0 \text{ for all } k \in N_\epsilon^*,\\
&\Leftrightarrow \langle T^*z, T^*\left(\sum_{i=1}^{N_\epsilon}\sum_{j=1}^{m_i}\alpha_i^j k_i^j\right)\rangle = 0 \text{ for all } k \in N_\epsilon^*,\\
&\Leftrightarrow \langle T^*z, T^*k\rangle = 0 \text{ for all } k \in N_\epsilon^*,\\
&\Leftrightarrow T^*z \perp T^*(N_\epsilon^*)\\
&\Leftrightarrow T^*z \in \left(T^*(N_\epsilon^*)\right)^\perp.
\end{aligned}$$

Therefore,

$$M_\epsilon^\perp = T^*(N_\epsilon^*).$$

Now, we can proof the following statements:

- If $x \in M_\epsilon^\perp$ then $\|Tx\| \geq \epsilon\|x\|$. In fact, let $x \in M_\epsilon^\perp$. Then

$x = T^*\left(\sum_{i=1}^{N_\epsilon}\sum_{j=1}^{m_i}\alpha_i^j k_i^j\right)$ with $\alpha_i^j \in \mathbb{F}$ and $k_i^j \in \mathcal{K}$. So,

$$\begin{aligned}
\|Tx\|^2 &= \langle Tx, Tx\rangle \\
&= \langle x, T^*\left(T\left(T^*\left(\sum_{i=1}^{N_\epsilon}\sum_{j=1}^{m_i}\alpha_i^j k_i^j\right)\right)\right)\rangle \\
&= \langle x, T^*\left(\sum_{i=1}^{N_\epsilon}\sum_{j=1}^{m_i}\alpha_i^j A' k_i^j\right)\rangle \\
&= \langle x, T^*\left(\left(\sum_{i=1}^{N_\epsilon}\lambda_i^{m_i}\right)\left(\sum_{i=1}^{N_\epsilon}\sum_{j=1}^{m_i}\alpha_i^j k_i^j\right)\right)\rangle \\
&= \left(\sum_{i=1}^{N_\epsilon}\lambda_i^{m_i}\right)\langle x, T^*\left(\sum_{i=1}^{N_\epsilon}\sum_{j=1}^{m_i}\alpha_i^j k_i^j\right)\rangle \\
&\geq \epsilon^2\langle x, x\rangle \\
&= \epsilon^2\|x\|^2.
\end{aligned}$$

Therefore, $\|Tx\| \geq \epsilon\|x\|$ for all $x \in M_\epsilon^\perp$.

- If $x \in M_\epsilon$ then $\|Tx\| \leq \epsilon\|x\|$. In fact, let $x \in M_\epsilon$. Then $x = T^*\left(\sum_{i=N_\epsilon+1}^{\infty}\sum_{j=1}^{m_i}\alpha_i^j k_i^j\right)$ with $\alpha_i^j \in \mathbb{F}$ and $k_i^j \in \mathcal{K}$. So, we have

$$\begin{aligned}
\|Tx\|^2 &= \langle Tx, Tx\rangle \\
&= \langle x, T^*\left(T\left(T^*\left(\sum_{i=N_\epsilon+1}^{\infty}\sum_{j=1}^{m_i}\alpha_i^j k_i^j\right)\right)\right)\rangle \\
&= \langle x, T^*\left(\sum_{i=N_\epsilon+1}^{\infty}\sum_{j=1}^{m_i}\alpha_i^j A' k_i^j\right)\rangle \\
&= \langle x, T^*\left(\left(\sum_{i=N_\epsilon+1}^{\infty}\lambda_i^{m_i}\right)\left(\sum_{i=N_\epsilon+1}^{\infty}\sum_{j=1}^{m_i}\alpha_i^j k_i^j\right)\right)\rangle \\
&= \left(\sum_{i=N_\epsilon+1}^{\infty}\lambda_i^{m_i}\right)\langle x, T^*\left(\sum_{i=N_\epsilon+1}^{\infty}\sum_{j=1}^{m_i}\alpha_i^j k_i^j\right)\rangle \\
&\leq \epsilon^2\langle x, x\rangle \\
&= \epsilon^2\|x\|^2.
\end{aligned}$$

Therefore, $\|Tx\| \leq \epsilon\|x\|$ for all $x \in M_\epsilon$.

Now we define the operator S as follows,

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} N \\ N^\perp \end{bmatrix} \to \begin{bmatrix} M_\epsilon^\perp \\ M_\epsilon \end{bmatrix}$$

where $N = T(M_\epsilon^\perp)$. If T has the following matrix form,

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} : \begin{bmatrix} M_\epsilon^\perp \\ M_\epsilon \end{bmatrix} \to \begin{bmatrix} N \\ N^\perp \end{bmatrix},$$

and $S_1 := T_1^{-1}$, then a simple calculation shows that S satisfy the conditions of Definition 3.5, and therefore $S = T^\epsilon$.

Remark. *We note that* $M_\epsilon^\perp = T^*(N_\epsilon^*) \subseteq R(T^*) \subseteq \overline{R(T^*)} = N(T)^\perp \Leftrightarrow N(T) \subseteq M_\epsilon$. *Also,* M_ϵ *is a closed subspace of* $\mathcal{H}$. *Therefore, by Theorem 3.7, there exists an* ϵ-*{2,3,4} inverse of* T *if* T *be a compact operator.*

Theorem 4.2. *Let* $T \in B(\mathcal{H}, \mathcal{K})$ *be a compact operator, then for all* $\epsilon > 0$ *there exists an* ϵ-*{2,3,4} inverse for* T.

Conclusions

In this thesis work 2 types of generalized inverses were defined: the image-kernel (P, Q) inverse on Banach spaces and the ϵ-{2,3,4} inverse on infinite-dimensional complex Hilbert spaces.

For the image-kernel (P, Q)-inverse we can highlight the following: the Definition of the (P, Q)-inverse on the space of bounded linear operators between Banach spaces was given (see Definition 2.1). It was proven that the (P, Q)-inverse is unique when it exist, and also is an outer inverse (see Remark 2.2). One of the main results that was formulated and demonstrated is Theorem 2.2, in which the (P, Q)-inverse was characterized as an outer generalized inverse with range and nullspace prescribed. A block matrix representation for operator T, as well as its (P, Q)-inverse was given (see Proposition 2.4). It was determined that under certain conditions the (P, Q) inverse of T may be related to the group inverse of operator TP (see Theorem 2.5).

For the ϵ-{2,3,4} inverse we can highlight the following: the Definition of the ϵ-{2,3,4} inverse on the space of bounded linear operators between infinite-dimensional complex Hilbert spaces was given (see Definition 3.5). Unfortunately, as noted in Example 5, the ϵ-{2,3,4} inverse for an operator T is not "unique". Also, a block matrix representation for an ϵ-{2,3,4} inverse was given through adequate decomposition of his domain and codomain. As observed in the Theorem 3.7 the existence of the ϵ-{2,3,4} inverse was characterized by the existence of a closed subspace such that it satisfies some properties. Another very important result for the work was characterize the existence of an ϵ-{2,3,4} inverse through its module, where the module is an positive operator that belong to the polar decomposition of operator (see Theorem 3.15). Finally, we can mention that the construction of the ϵ-{2,3,4} inverse (see Chapter 4), which was given for compact operators, will be very important in the work that is being developed.

Future research

From all the work developed in this thesis for (P,Q)-inverse, the following question is posed for image-kernel (P,Q) inverse of a given operator T in $B(X,Y)$ with X and Y a Banach spaces:

Let $T^{(2)}_{P,Q}$ be a (P,Q)-inverse for an operator $T \in B(X,Y)$, are there any another projections P' and Q' such that $T^{(2)}_{P',Q'} = T^{(2)}_{P,Q}$?

From this, we have the following questions:

1. Can we describe the class of all pairs (P,Q) of projectors such that these pairs have give us the same (P,Q)-inverse of T?
2. Does exist a way to choose the "best" pair of projections (P,Q)?

Also, we will use the definition of approximate nullity considered by G.Edgar et al in *Weighing operator spectra* to distinguish between the different infinite cardinals for which they have a dimension smaller than or equal to the dimension of Hilbert space $\mathcal{H}$ but larger than or equal to $\aleph_0$. So we have the following question:

1. We know that if T has a closed range then we can choose $M_\epsilon = N(T)$ for every $0 < \epsilon \leq \gamma(T)$, but what happen if T doesn't has closed range? A partial answer is given if we consider the space $R(E_T([0,\epsilon)))$ where E_T denote the spectral measure for $|T|$.

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