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“DESARROLLO DE LOS FORMALISMOS DE GITMAN-LYAKHOVICH-TYUTIN Y HAMILTON-JACOBI APLICADOS A SISTEMAS GRAVITACIONALES DE ALTO ORDEN: GRAVEDAD DE WEYL”

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Development of the Gitman-Lyakhovich-Tyutin and Hamilton-Jacobi Frameworks Applied to Higher-Order Gravitational Systems: Weyl Gravity

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Abstract

This dissertation conducts an in-depth analysis of two higher order systems: the extended Maxwell-Chern-Simons theory and Weyl gravity. Higher-order theories, identifiable by the inclusion of time derivatives of third order or higher in the Lagrangian, exhibit interesting properties. On one hand, these theories demonstrate distinct dynamics, enabling them to fit experimental data and serve as viable models in situations where traditional theories deviate from empirical observations. On the other hand, challenges such as the Ostrogradsky instability and the loss of unitarity are prevalent in these theories. Due to their intricacy, the "Hamiltonization" procedure is highly non-trivial, necessitating the incorporation of sophisticated mechanisms. This study makes use of the Faddeev-Jackiw and Hamilton-Jacobi Hamiltonization methods, introducing additional degrees of freedom as an order reduction mechanism and obtaining a first-order Hamiltonian with equivalent dynamics. This process introduces constraints among variables, which must be properly treated for the dynamics to remain consistent. These frameworks are used to find the canonical structure of the actions, the symmetries of both theories and establish the algebra of the constraints.

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Conventions

Index notation is used throughout the thesis, employing the Greek alphabet $(\alpha, \beta, \gamma, ...)$ for space-time indices and the Latin alphabet $(a, b, c, ...)$ for space-only indices. Summation convention is assumed for repeated indices unless otherwise stated, with the number of terms being either explicitly mentioned or able to be deduced from the context.

Partial derivatives are indicated as $\partial_\mu A$, or as A_μ, with the time derivative $\partial_0 A$ also represented as an overhead dot $\dot{A}$. Covariant derivatives are denoted as $A_{;\mu}$, whereas the d'Alembert operator is represented as $\Box A = \partial_\mu \partial^\mu A$. While performing space-time decompositions in sections 3 and 4 some terms will present contractions of the spatial indices, to indicate this an overhead tilde will be used $\tilde{A} = A_i{}^i$.

There will be situations in which multiple objects are represented in a succinct manner, say A, B, C and D. We choose to express the set by means of a single index in parenthesis, $\gamma^{(\mu)} = \{A, B, C, D\}$.

The metric sign convention used in this work is the mostly plus metric convention $(-, +, +, +)$, where the minus sign is assigned to the dimension of time. Sections three and four deal with systems with constraints, where the symbol $\approx$ is used to represent equalities that are valid in the constrained hypersurface and not necessarily outside of it. Approximations of any other kind, such as Taylor series, are represented by $\simeq$.

Chapter 1

Introduction

Of the four fundamental interactions of nature, gravity is the only force that, to date, has not been represented by quantum field theory. In contrast, the electromagnetic field, as well as the strong and weak forces, have been incorporated into the standard model since the 1970s. But gravity, perhaps by its very nature, has refused to cooperate [1]. This stands as one of the pivotal challenges in modern physics, sparking numerous scientific advancements and an array of theories.

This work does not aim to obtain a quantum theory of gravity, but only the Hamiltonian description of two higher-order systems, the first step in the quantization procedure. This "Hamiltonization" is achieved by making use of the Faddeev-Jackiw and Hamilton-Jacobi methods, which have their roots in the Dirac quantization procedure and thus place the present paper within the framework of canonical quantization. Furthermore, due to the nature of the systems at hand, the work is placed within the field of extended theories of gravitation; specifically, within the higher-order gravitational theories.

In this chapter, we will discuss the current status of general relativity and the Lambda-CDM cosmological model. This is done with an emphasis on the physical properties of Einstein's general relativity (GR), including its phenomenology at cosmological scales, and its arduous relation with quantum mechanics; the quantization frameworks of GR, i.e. the canonical and covariant formulations; as well as alternatives to string theory and loop quantum gravity; including higher-order theories. Chapter 2 is devoted to quantization methods for higher-order theories, giving a brief review of the Dirac method and focusing on the Gitman-Lyakhovich-Tyutin (GLT) and Hamilton-Jacobi (HJ) frameworks. Chapter 3 analyzes the higher-order Maxwell-Chern-Simons action by means of the HJ and GLT methods, including an analysis of the gauge symmetries and a comparison of both methods. Finally, in Chapter 4 the Weyl gravity is analyzed by the HJ method and its gauge symmetries are obtained.

1.1 General Relativity and the Lambda-CDM model

1.1.1 Generalities

General relativity, formulated by Albert Einstein between 1907 and 1915, stands as the prevailing theory of gravitation, employed to explain gravitational phenomena. It integrates the principle of relativity and the invariance of the speed of light from his theory of special relativity with the principle of equivalence. Simultaneously, it significantly alters conceptions of space and time. Widely considered an elegant theory, derived from first principles, it generalized the concepts of time and space, incorporating them into a dynamic space-time continuum. At the same time, it describes gravity not as a force, but as the curvature of said space-time, caused by the unequal distribution of mass. Which in turn moves through space-time along geodesic lines.

But the popularity of the theory is due not only to the elegant description of nature and the simplicity of its equations, but also to the large number of tests and predictions that the theory has achieved [2]. In 1915, Einstein used his theory to calculate the bending of light by the gravity of the Sun; which was later measured by Arthur Eddington during the famous solar eclipse of May 29, 1919. That same year of 1915, Einstein successfully described the perihelion precession of Mercury. This without resorting to the inclusion of Vulcan, a hypothetical planet thought to orbit between Mercury and the Sun[1]. Additionally, the Pound-Rebka experiment in 1959 corroborated Einstein's prediction that gravitational effects could cause a frequency shift of light. Finally, the theory has also passed more recent tests, such as the detection of gravitational waves by the LIGO and Virgo collaborations [3] [4], which since 2016 have recorded multiple detection events.

The prevailing standard cosmological model, known as the ΛCDM or Lambda-CDM model, postulates that the universe originated from a big bang, underwent a transient phase of rapid expansion known as inflation, and subsequently expanded and cooled to its current state. Based on general relativity, this model proposes that the universe has three main components: a cosmological constant (λ) associated with dark energy, which has a negative pressure that counteracts gravity and generates the expansion of the universe [5]; followed by cold dark matter (CDM), which does not emit, absorb or reflect electromagnetic radiation; and finally ordinary matter.

The model, as demonstrated by its success [6], skillfully explains several phenomena, including the existence and structure of the cosmic microwave background, the large-scale structure of the universe and its accelerating expansion. It also provides an excellent fit to the large amount of observational data, based on a minimal number of cosmological parameters: the density of matter $\Omega_m h^2$, the density of atoms $\Omega_b h^2$, the expansion rate of the universe H_0, the amplitude of primordial fluctuations σ_8, their scale dependence n_s and the optical depth of the universe τ [7].

1.1.2 Issues

However, ΛCDM is not without its unsolved questions [8]. One such issue is the flatness problem, as the value of the matter density parameter Ω is very close to the critical value needed for the universe to be flat. The conundrum is that any deviation from the critical value would

[1] Although the notion of a new planet within our solar system may seem implausible today, it is crucial to have in mind that previous irregularities in the orbit of Uranus prompted the discovery of Neptune by Le Verrier.

cause Ω to increase rapidly, so the early universe must have had a density extremely closer to the critical one, not deviating by more than 1 part in 10^{55} [9]. There is also the cosmological horizon problem, in which multiple regions of the universe that are causally disconnected share similar properties. Due to their relative distance, these regions of space have not been able to physically interact with each other, not even light has been able to travel between them. However, this is in conflict with the observed homogeneity of the universe and the similarities in the observed values of the cosmic background radiation temperatures [10].

Further complications arise in relation to the densities of dark energy and dark matter, giving rise to the problem of cosmic coincidence. This phenomenon shows that, while the vacuum energy remains constant over time, dark matter energy strongly depends on the redshift. However, both densities currently exhibit comparable magnitudes [11]. In addition, there is tension in the scientific community over the value of the Hubble constant, which manifests itself as a difference between its predicted value using measurements, according to the early universe physics and its observed value in the late universe [12].

But the inconsistencies lie not only in the cosmological model, but are also found, and perhaps caused, by the peculiarities of general relativity itself, which is not exempt of shortcomings. Among the most notable ones are those regarding dark matter, dark energy, the cosmological constant and, of course, quantum mechanics.

Dark matter, a hypothetical form of matter, derives its name from its characteristic of neither absorbing, reflecting, nor emitting electromagnetic radiation. This type of matter was originally proposed as a workaround to reconcile astronomical observations and theory. Specifically, its existence was postulated a posteriori by Zwicky in 1933 [13] to explain discrepancies in radial velocity measurements of several galaxies in the Coma cluster. Not only does its postulation raise questions, but also its implementation. For example, dark matter has been invoked to explain the absence of a Keplerian decay in rotational velocities with respect to distance from the center of many galaxies [14]. The caveat is that, in order to do this, a number of parameters have to be adjusted. This would not be a problem, if these values did not have to be adjusted for each galaxy [15]. This way, dark matter seems to know precisely where and exactly how much of it is needed to fit the measurements. Moreover, as we move to even larger scales, problems begin to appear for galaxy clusters, whose amount of gravitational lensing disagrees with theory.

In the discussion of the principles of general relativity, it is imperative to address the cosmological constant Λ. It was first proposed in 1917 by Albert Einstein out of a necessity to obtain a static solution for his equations [16]. It took the role of the intrinsic energy of space and acted as a repulsive force that counteracted the attractive force of gravity on large scales[2]. This was a compromise between his theory and cosmological considerations, as Einstein thought that the cosmological constant "detracted from the beauty of the theory". It has historically fallen in and out of favor over the years. After its conception, Hubble's discovery of the expansion of the universe [17] led Einstein to dismiss the constant from his equations, going so far as to regard it as his greatest failure [18]. Towards late 20th century, the seminal work of Perlmutter et al. demonstrated that the universe not only expands, but does so at an accelerating rate [19]. This led to a renewed interest in the cosmological constant as a repulsive force and prompted profound inquiries into its value and fundamental characteristics.

The first of these problems arises when quantum mechanics gets involved, since the estimates for the value of the vacuum energy density from quantum field theory (QFT) do not coincide with the observed values. That is, if one considers contributions from the zero-point energies of quantum fields to the cosmological constant, the cosmological constant has a value that is 60 orders of magnitude larger than the measured one; with initial estimates famously disagreeing by as much as 120 orders of magnitude. This is known today as the cosmological constant problem and has its roots with the historical work done by Zel'Dovich [20], and was later famously approached by Weinberg [21]. Even though modern research suggests that the degree of discrepancy is closer to 60 orders of magnitude [22], this still remains one of the biggest problems in physics today.

Alongside Λ lies the notion of "dark energy". When the accelerating expansion of the universe came to light, a plethora of models, collectively referred to as "dark energy models"

[2]At the time there was no knowledge that the universe was expanding, so a static solution was paramount

emerged[3]. The development of these models marked a turning point in cosmological research.

The most straightforward of which is the aforementioned cosmological constant; but dynamical models such as quintessence [24], K-essence [25], Chaplygin gas [26], holographic dark energy [27], chameleon fields [28], among others, have also been considered [29]. Dark energy models often run into the problem of cosmological coincidence. This refers to the remarkable similarity between the energy densities of dark energy and ordinary matter in the present cosmic epoch. This is despite their different evolutionary behaviors over time. It is important to stress that, although these models offer possible solutions to the cosmic coincidence problem, they often introduce additional complexities and are highly sensitive to initial conditions to match the observational data [30].

These problems have been addressed with varying degrees of success. Inflation helps with both the flatness and horizon problems and is an important part of the ΛCDM model [31]; despite its issues and questions, dark matter explains a wide range of astrophysical and cosmological phenomena; the anthropic principle has also been taken into consideration as to amend cosmological coincidences [32]. Dynamical models of dark energy (such as quintessence, k-essence, coupled dark energy and unified dark energy) also provide a framework for understanding how dark energy might change and interact with other components of the universe, and can also potentially address the problem of cosmic coincidences [33].

1.2 Quantum gravity

General relativity has one last major problem to discuss: its incompatibility with quantum mechanics. An issue that began with the failure of the covariant quantization program to obtain a quantum theory for gravity. Because of this, it is said that GR is a perturbatively non-renormalizable theory[4] [34].

[3] The name "dark energy" was coined by Michael Turner in 1998, in a parallel to the "dark matter" so named by Fritz Zwicky in the 1930s [23].

[4] At least in the strong sense, in which an infinite number of counter-terms do not have to be added.

The absence of a complete quantum theory of gravity is due to both technical challenges and the inherent difficulties in directly observing quantum gravitational effects [35]. However, the deeper challenges arise from fundamental conceptual disparities between quantum mechanics (QM) and Einstein's theory, as elegantly outlined by Rovelli [36]. Concepts such as time, space, causality, matter, and measurements undergo profound modifications in both realms. Notably, both theories rely on assumptions that directly contradict each other. While QM uses time as an external independent parameter, GR views it as a dynamical entity. Even quantum field theories, which tend to work in a fixed, non-dynamical space-time, often overlook the dynamical aspect of the metric tensor, one of the most important concepts of Einstein's theory.

It is natural to wonder whether a quantum theory of gravity (QG) is even necessary. The conventional perspective assumes the necessity of unification. Beginning with Maxwell's groundbreaking integration of electromagnetism, and extending through the developments of the standard model, these efforts have yielded significant advancements. However, the profound importance of a quantum theory of gravity becomes evident when considering environments where both gravitational and quantum effects play pivotal roles, such as near black holes or dense astrophysical bodies like neutron stars. These scenarios are anticipated to reveal the significance of quantum gravitational effects [37], particularly in phenomena which appear at the Planck length $l_{Planck} = \sqrt{G\hbar/c^3} \sim 10^{-35}m$. Such a theory would help us with complex issues including

- Addressing gravitational singularities. GR not only predicts the existence of singularities, points where space-time curvature becomes infinitely strong, but it also tells us that they are unavoidable in some real-world circumstances, such as at the center of black holes or the initial singularity of the Big Bang. These signal the breakdown of space-time and general relativity [38]. Quantum gravity could lead to a theory without divergences.

- Understanding the microscopic structure of space-time. To consider space-time as a dynamic entity that can undergo quantum fluctuations leads to quantization of its geometry [39].

- Advancing our understanding of the cosmos and black holes. To unravel the mysteries surrounding the early Universe, its primordial conditions after the big bang and the conclusive phases of black hole formation, a broader theoretical framework is essential [40]. The advent of a quantum theory of gravity promises to enrich our knowledge of the characteristics of black holes.

- The problem of time, which poses a major challenge. Quantum mechanics and Einstein's theory offer completely opposite notions of time, which makes them fundamentally mutually inconsistent. In quantum mechancis, time remains an external parameter, devoid of an inherent quantum operator. In contrast, in General Rrelativity, space-time emerges as a dynamical entity delineated by the metric tensor. It is clear that any attempt to bring quantum theory and GR together requires a revision of our understanding of time [36].

- Understanding early cosmology. According to the current standard cosmological model, in the early instants of the universe density, temperature and curvature were extreme, an analysis of such conditions requires a theory that incorporates both gravity and quantum mechanics.

Given the formidable challenge of unifying GR with QM and the profound scientific implications such a theory would entail, researchers have pursued various avenues in their quest for a quantum theory of gravity.

1.2.1 Frameworks

Initial efforts to quantize general relativity resulted in what is now known as the covariant quantization program. This method attempts to formulate quantum gravity as a quantum field theory that describes the gravitational field as the fluctuations of the metric tensor in a Minkowski space. The program, rooted in the early 1930s [1], traces its origins to the pioneering work of Rosenfeld [41]. However, it was not until 1952 when Gupta [42], building on the contributions of Fierz [43] and Pauli [44], consolidated the flat space quantization approach. The appearance of tools such as Feynman diagrams [45] and the introduction of Faddeev-Popov phantoms made it possible in 1967 to derive a consistent set of Feynman rules for general relativity [46]. This, together with the discovery of dimensional regularization [47], sparked renewed interest in perturbative analyses of the graviton propagator and facilitated investigations of various matter-gravity couplings, in particular at the one-loop level [34].

However, in 1974, the seminal work of t' Hooft and Veltman [48], along with that of Deser and Van Nieuwenhuizen [49], highlighted significant issues in the perturbative expansion of GR. Notably, they demonstrated that coupling even a single scalar to the pure (i.e., matter-free) theory compromised its finite behavior. By 1975, it became widely acknowledged that coupling GR to matter leads to a non-renormalizable theory. This corroborated echoed concerns raised as early as the 1930s by physicists such as Heisenberg, Bergman [50], and Bronstein [51].

However, by 1974, the work of t'Hooft & Veltman [48], as well as that of Deser & Van Nieuwenhuizen [49], revealed serious problems in the perturbative expansion of GR. Most notably, they showed that the finite behavior of the pure (i.e. matter-free) theory was destroyed by coupling it even to a single scalar. By 1975 it had become generally accepted that GR coupled to matter is not renormalizable. This confirmed assumptions and concerns of physicists like Heisenberg and Bergman, who pointed out the possible problems of the theory as early as the thirties [36], [51], [50].

Although a covariant quantum theory for gravity has not been achieved, work in this field has played an extremely important role in the development of QFT as a whole [34] and has led to advances such as the understanding of the aforementioned Faddeev-Popov ghosts, the background field method, the effective action, advances in gauge fixing, as well as progress in the field of constrained dynamics [52].

Moreover, this framework has paved the way for theories like supergravity and string theory, which directly evolved from the standard model and is firmly grounded in the techniques and conceptual framework of quantum field theory. It takes as basic entities of nature extended one-dimensional objects, called strings, rather than particles. This hypothesis gives rise to a comprehensive unified theory, encompassing a wealth of physical developments, such as fermions, Yang-Mills fields, as well as gravitons [36]. Among the many achievements of string theory are its elegant unification of different theories, its well-defined perturbative expansion, and its theoretical and mathematical richness and complexity. These theoretical advancements come with a significant amount of additional physics, including supersymmetry, extra dimensions, and an infinite array of fields with varying masses and spins, among other elements. Despite the absence of experimental observations confirming these theoretical constructs [53], string theory remains the most extensively investigated candidate theory of quantum gravity to date.

Another increasingly popular approach is the canonical framework. It began in the late 1960s with the work of DeWitt [37], and is based on Bergmann's phase space quantization program [54], as well as Dirac's analysis of constrained Hamiltonian systems [55]. This formulation attempts quantization based on the Hamiltonian formalism, treating the symplectic structure [5] as paramount. This implies choosing a specific parameterization of time and the loss of the manifest covariance of the theory. The goal is to obtain a quantum formulation in which operators are constructed from the metric tensor, whose algebra is consistent not only internally, but also with the constraints of the theory [56]. This removes the need to fix the background metric [57] and allows for a non-perturbative treatment.

The method had been properly established by 1961 with the ADM formulation of GR, which greatly simplified its Hamiltonian formulation and made its geometrical reading clearer [58]. Inspired by the ADM formalism, the work of John Wheeler and Bryce DeWitt resulted in the development of the Wheeler-DeWitt equation in 1967 [59]; a constraint equation that describes the quantum state of the gravitational field and any associated matter fields in a given space-time, meaning it imposes conditions on the possible quantum states of the universe. This formulation is not free of technical difficulties, as it presents challenges, such as the problem of time, difficulties in recovering the classical limit, as well as complications related to obtaining a

[5] i.e. the Poisson brackets' algebra

solution for a given and specific physical situation [60]. Although the Wheeler-DeWitt equation falls short of offering a comprehensive and self-contained theory for gravity in a quantum context, it has been instrumental as a foundational concept for the emergence and exploration of various theoretical frameworks, notably including loop quantum gravity (LQG).

Loop quantum gravity is based on the connection formulation of GR developed by Ashtekar [61]. Here, space and time are quantized, which means they are divided into discrete, fundamental units rather than being treated as continuous. This is done by using discrete structures called "loops", which are woven into what's called a spin network. The main advantage of LQG is its capacity to describe quantum space-time in a background-independent and nonperturbative fashion, while also providing a physical picture of the world at the Planck scale. The principal shortcoming of the theory relates to the formulation of the dynamics. In particular, recovering classical GR physics from loop gravity is not yet possible [1].

String Theory and Loop Quantum gravity are the two most prominent theories of quantum gravity to date, representing the covariant and canonical approaches, respectively. However, this does not mean that other approaches have not been considered [36]. Alternative research avenues include theories such as

- Causal dynamic triangulations, a path integral (or sum-over-histories) approach where space-time is discretized into three-dimensional analogs of triangles, connected to form a lattice [62].

- Non-commutative Geometry, in which a non-commutative algebra of functions over space-time is used [63].

- Regge calculus, an approximation scheme for general relativity and quantum gravity via space-time discretization [64].

- Penrose's twistor theory, which describes the physical information of space-time in the so-called twistor space; a vector space of four complex dimensions [65].

- Mini- and midisuperspaces: These concepts examine space-times featuring high degrees of symmetry and seek to represent quantum states of the gravitational field through functions of three-dimensional spatial geometry [66].

- Causal sets, in which the Lorentzian geometry is replaced by the set of locally finite posets, or causal sets; which encode the principles of causality and discreteness [67].
- Horava-Lifshitz Gravity, which introduces anisotropic scaling between space and time, breaking Lorentz invariance at high energies [68].
- Emergent gravity, in which gravity (and perhaps space-time itself) is a collective manifestation of very different underlying degrees of freedom [69].
- Modified theories of gravity, also called extended theories of gravity (ETG). Which seek to broaden the scope of GR in order to improve its behavior at different scales, leading to different equations of motion (EOM) as well as a different count of DOF.

Modified gravitational theories seek to amend the limitations of General Relativity (GR) at both infrared and ultraviolet scales. These modifications intend to preserve the foundational principles of Einstein's theory while addressing critical issues including inflation, dark energy/matter, and the large-scale structure of the universe, as well as incorporating aspects of quantum gravity. This framework makes use of higher-order curvature invariants as well as minimally (or non-minimally) coupled fields to the Einstein-Hilbert (EH) action [70] [71]. It's worth emphasizing that, within gravitational frameworks, there exists no inherent rationale for restricting the Lagrangian to a linear function of the Ricci scalar, minimally coupled to matter [72]

These theories have attracted attention for their inflationary features and their ability to address the limitations of the Standard Cosmological Model. In particular, cosmological models based on these theories align well with Microwave Cosmic Background Radiation data [73, 74]. Furthermore, via conformal transformations, it can be demonstrated that higher order terms and non-minimally coupled components invariantly reduce to GR minimally coupled with one or more scalar fields [75, 76]. Schemes such as superstrings and supergravity incorporate such terms, the inclusion of which is considered inevitable to derive the effective action of Quantum Gravity [72].

Within this field there is a diverse array of models, each with its own merit. These encompass metric and non-metric theories [77], as well as metric-affine [78] and purely affine theories [79]. Some proposals, in addition to the graviton, include scalar [80], vector [81] and tensor degrees of freedom [82]. Non-local models [83] and those challenging equivalence principles, both in the gravitational and matter sectors [68], have also been studied. Another approach is to extend the geometric structure of the manifold, with theories utilizing metric-independent affine structures, as in Palatini's variation method [78]. There are also theories that consider torsion [84] as well as non-metricity [85].

These theories have their specific strengths and weaknesses, and each of them has proven to be effective in dealing with the problems of general relativity. This paper focuses on the higher-order theories, as their properties have shown great potential in addressing the problems of GR.

1.3 Higher-Order Theories

Higher order theories, also called theories with higher order derivatives, incorporate time derivatives of order two or higher in their action formulations. This departure from conventional physics, established by Newtonian principles, introduces fourth-order derivatives in the equations of motion. Such formulations introduce a layer of complexity that departs from traditional paradigms, often resulting in improved system behavior by adding higher order terms.

Higher-order gravitational theories focus on the dynamics of the metric and/or a linear connection, but allow for much more complex structures, such as $R_{\alpha\beta\mu}R^{\alpha\beta\mu}$, $R_{\mu\nu}R^{\mu\nu}$, R^2, $\Box R$, $\nabla^\nu R \nabla_\nu R$. Therefore, it is not uncommon for such terms to result in higher-order derivatives acting on the metric tensor, with the appearance of field equations of order higher than two as a direct consequence[6]. These supplementary structures usually have a geometrical meaning akin to that of the metric in Einstein's theory. Some examples include the Starobinsky model, whose quadratic term can be associated with inflation [86]; the Lanczos-Lovelock Lagrangians, which correspond to the Euler density of a $2m$-dimensional manifold without boundary [87]; and the Chern-Simons term, which encodes information about the space-time's topology [88].

This field is very broad and has given rise to many different theories, one of the most notable examples are the $f(R)$ theories. These are based on general scalar curvature functions, as the name implies, and have remained relevant due to their explanation of large-scale phenomena [89]. While $f(R)$ theories do not have to include higher-order terms, there are models, such as the Starobinsky model, that take advantage of the quadratic scalar curvature term to deal with inflation [90]. Additionally, the quadratic or Ricci-based models $f(g_{\mu\nu}, R_{\mu\nu})$, where the Ricci tensor is incorporated to comply with the semi-classical corrections of quantum nature [91]. Other higher-order modifications can be given when making use of the full Riemann tensor $f(g_{\mu\nu}, R^{\alpha}_{\beta\mu\nu})$. Among these, Weyl's theory is notorious due to its conformal symmetry [92]. In addition to this, terms with covariant derivatives can be added, such as $\nabla_{mu} R \nabla^{\mu} R$ or even $\nabla^2 R$. The dynamics of these theories are even more complex, and are not the focus of this thesis.

Higher-order gravitational theories serve as a framework designed to address the limitations of General Relativity (GR) across both infrared and ultraviolet scales. This approach seeks to maintain the foundational insights of Einstein's theory while tackling conceptual and experimental challenges that have arisen in recent years within Astrophysics, Cosmology, and High Energy Physics. These models usually attempt to incorporate problems such as inflation, dark energy and dark matter, the large-scale structure of the universe, and a quantum description of gravity [72].

In the low energy limit of string/M-theory, higher-order terms are anticipated, appearing as one-loop corrections in the process of field quantization on curved spacetimes. These terms are nearly inevitable in any perturbative framework aimed at constructing a self-consistent

[6] There are exceptions to this, e.g. the Lanczos-Lovelock Lagrangians.

theory of quantum gravity [93]. It has been shown that employing higher-order terms as counter-terms eliminates singularities when quantized matter fields are taken into account. This, in turn, improves the behavior of General Relativity in the ultraviolet regime [94]. They have worked as tools for characterizing various properties of strongly coupled conformal field theories within the framework of holography [95]. Certain higher-order gravity models exhibit asymptotic freedom [96], while others inherently generate an accelerating expansion [97] [98]. Furthermore, these theories have been proven to be renormalizable [99] [100], a characteristic that has kept them relevant in recent years, notwithstanding associated issues [96].

The intricate structure provided by the higher-order terms not only enhances the dynamics, but can also lead to several undesired behaviors. Higher order theories often bring in additional degrees of freedom or extra fields, often leading to stability problems. It is crucial to take into account the presence of ghost fields, characterized by negative kinetic energy and often associated with higher order terms. The negative sign of the ghosts suggests that the energy linked to these fields lacks a lower bound, resulting in a linear dependence of the corresponding Hamiltonian on one or more of the conjugate momenta [101, 102]. Similar instabilities, as well as nonphysical behavior, can also be produced by the presence of non-minimal [103] or strong couplings [104]. Further pathologies include the so-called gradient instabilities [105], violation of multiple energy conditions [106], and even modifications of the gravitational wave speed [107].

Despite these complications, active research persists in the field. Studies have explored systems in which the additional degrees of freedom, including ghost degrees of freedom, can be eliminated [108, 109, 110, 111]. In addition to this, in supersymmetric systems, the singular behavior of the ghosts is benign, thus preserving the unitarity of the associated quantum theory [112].

Moreover, in recent years, additional studies have supported the idea that the addition of these terms improves the behavior of galaxy rotation curves [113] removing the need for dark mater.

This consistent behavior is also evident in gravitating structures such as stars, spiral and elliptical galaxies, as well as clusters of galaxies [114]. Moreover, researchers have examined the behavior of these modified theories in the high-curvature limit, leading to the discovery of exact cosmological solutions [115]. Regarding black holes, these theories have found success describing their energetic properties [116] as well as the structure of their solutions [117]. Results such as these show that, while there are challenges to overcome, these theories have the potential to achieve a realistic description of gravitational systems. In this work, two specific higher-order theories will be addressed: the Extended Maxwell-Chern-Simons theory and Weyl gravity. These two systems will be described in more detail in the following chapters, but a brief overview is provided below.

The Maxwell-Chern-Simons (MCS) theory is an extension of Maxwell's electromagnetism that incorporates a Chern-Simons term into the standard electromagnetic Lagrangian. This new term introduces a topological mass generation mechanism and leads to the emergence of massive vector bosons. Its higher-order extension introduces additional powers of both the electromagnetic field tensor and its derivatives in order to account for more complex physical phenomena and to describe the behavior of electromagnetic fields in nontrivial backgrounds [118]. Three-dimensional Chern-Simons (CS) gravitational theories are regarded as intriguing toy models, providing avenues to explore different facets of gravitational interaction and the fundamental principles of quantum gravity. The Extended Maxwell-Chern-Simons (EMCS) theory exhibits numerous properties akin to higher-dimensional gravity models, which are typically more challenging to investigate. This theory holds intriguing applications in both gravitational and cosmological contexts [119, 120].

The other focus of this thesis is Weyl Gravity, also known as Conformal Gravity. In addition to diffeomorphism invariance, the theory exhibits invariance under conformal transformations of the metric. These transformations maintain the causal structure of space-time while enabling the rescaling of distances but also maintaining angles. There are multiple ways to incorporate this symmetry into a theory of gravity, if one follows Weyl's original work one can embed the symmetry at a more fundamental level within the manifold, arriving at Weyl [121] geometry.

We, on the other hand, work on a Riemannian manifold and embed the symmetry in the structure of the Lagrangian, which is composed only of the squared Weyl tensor. This choice calls for a higher-order theory, Weyl gravity, whose galactic and cosmological applications have been well studied [122].

The Hamiltonian description of both systems will be shown in subsequent sections. Due to the presence of higher-order (time) derivatives, special techniques are needed. More precisely, we will use both the Hamilton-Jacobi and Gitman-Lyakhovich-Tyutin methods to obtain a Hamiltonian description for Maxwell-Chern-Simons theory, while Weyl gravity will be worked out only with the Hamilton-Jacobi method.

Chapter 2

Method

This section it will be shown how the Hamiltonization of high-order Lagrangian systems can be performed by making use of the Gitman-Lyakhovich-Tyutin (GLT) and Hamilton-Jacobi (HJ) procedures. The chapter is intended to be instructive, so that the calculations in subsequent sections can be carried out in a straightforward manner.

The description of a physical system can be done by either the Lagrangian or the Hamiltonian formalisms. The former has the advantage of fulfilling the requirements of special relativity in a very simple way, but in order to obtain the quantum description of a system we must begin with the latter. To go from one framework to another it is necessary to identify the canonical conjugate momenta. For non-singular systems, the momenta will be independent functions of the velocities, which allows to solve the velocities as functions of the coordinates and momenta; which, in turn, allows us to obtain the Hamiltonian. For singular systems this will not be possible, at least for some momenta, a fact that can also be seen by the non-invertibility of the system's Hessian matrix. Which means that we are dealing with a singular system and that the Hamiltonian cannot be constructed using the usual methods. Consequently, multiple methodologies have been developed to deal with the complex structure of these systems. Below we start by showing a brief overview of the Dirac-Bergmann algorithm, upon which the GLT framework is developed.

2.1 The Dirac-Bergmann Algorithm

The treatment of singular Lagrangian systems was initially outlined by Dirac in 1950 [123], followed by Bergmann in 1951 [124]. Their primary motivation was to elucidate the structure of field theories such as electromagnetism and GR, both of which are gauge theories containing degrees of freedom that do not affect the physical state of the system. Dirac discovered that the symmetries of a singular system are intertwined with its constraints, along with a method to convert them into constrained Hamiltonian systems, where dynamics occur within a subspace of the phase space. Furthermore, Dirac noted that a spacetime foliation simplifies the constraint structure of gravity, albeit at the expense of relinquishing the explicit symmetry of the Lagrangian picture [125]. Dirac's original formalism is not used in this work, but it might be instructive to give a brief overview. For a more detailed description see reference [126], as well as Dirac's own "Lectures on quantum mechanics" [55]. I hope this brief summary provides a basis on which to understand both procedures used in the following sections.

- Obtain the conjugate momenta $p^a = \partial\mathcal{L}/\partial\dot{q}_a$, where a goes from one to n, the number of generalized coordinates. The non-invertibility of s of these relations will give the so-called primary constraints $\phi_i(q,p) = 0$, where $i = 1, \ldots, s$.
- Second, one has to define the canonical Hamiltonian $\mathcal{H}_c(q,p)$ via a Legendre transformation. Even though some velocities cannot be solved for in terms of positions and momenta the system can always be described using the Hamiltonian formulation. Since the Hamiltonian will always be a function of the coordinates q and momenta p, exclusively.
- The next step is to define the primary Hamiltonian $\mathcal{H}_P = \mathcal{H}_c + \lambda^i\phi_i$. This is done via the addition of the primary constraints as well as their corresponding Lagrange multipliers λ^i.
- Afterwards, one must impose the so-called consistency conditions upon the constraints. This means that the equations of motion (EOM) for each constraint mush vanish, at least weakly $\dot{\phi}_i \approx \{\phi_i, \mathcal{H}_c\} + \{\phi_i, \phi_j\} \approx 0$. Two functions, F_1 and F_2, defined on the phase space, are considered to be weakly equal if they are identical only when the constraints

are satisfied (i.e., within the constrained subspace), but not across the entire phase space. This equivalence is denoted by $fF_1 \approx F_2$.

- The consistency conditions will reduce to either
  - Trivial identities, when the constraints are directly zero $\phi_a(q,p) \equiv 0$.
  - Constraints that will be imposed on the Lagrange multipliers λ, such that some multipliers will be expressed in terms of positions, momenta, and the remaining λ's.
  - Additional constraints will be imposed on the coordinates as well as the momenta, known as secondary constraints $\psi_m(q,p) = 0$.
- If any secondary constraint exist, it must also satisfy the consistency conditions. This may result in additional identities, more constraints on the λ's, as well as more constraints.
- Once all the constraints have been identified and their consistency conditions have been met, this constraint gathering process is completed. All non-primary constraints are then grouped under ψ_m, with the index m naturally being extended to account for this.
- The total Hamiltonian $\mathcal{H}_T$ is then constructed by with help of the primary Hamiltonian $\mathcal{H}_P$ by means of constraints on the Lagrange multipliers.
- Both types of constraints (ϕ_a, ψ_m) are further classified into first and second class; denoted by γ and ξ, respectively. First class γ constraints are characterized in that their Poisson bracket vanishes with all other constraints, while second class ξ constraints will have at least one Poisson bracket that does not vanish.

At this point, the equations of motion derived from the so-called Extended Hamiltonian $\mathcal{H}E = \mathcal{H}C + \lambda^i \gamma_i$ correspond to Lagrange's equations for the original Lagrangian. Given that the Lagrange multipliers λ^i are entirely arbitrary, the phase space transformations induced by the primary first-class constraints γ_i do not alter the physical state of the system. In other words, the first-class constraints serve as generators of gauge transformations [126]. Moreover, despite the equations of motion defined by $\mathcal{H}E$ not being strictly identical to the original Lagrangian equations of motion, both theories exhibit concordance in the evolution of physical variables. The temporal evolution of the phase-space function F is governed by a Poisson Bracket with either the extended $F, \mathcal{H}E$ or the total $F, \mathcal{H}_T$ Hamiltonian, with physical trajectories confined to the subspace where the constraints are satisfied.

We now have two options: we can either eliminate the second-class constraints without affecting the gauge freedom generated by the first-class constraints, or we can impose a gauge. The former involves replacing the Poisson bracket with Dirac's bracket. $\{A, B\}_D$, which allows the weak inequalities of the second-class constraints to become strong inequalities; this new bracket will be shown in more detail in the following section. See [127] and [128] for more detailed descriptions as well as examples.

2.2 Gitman-Lyakhovich-Tyutin Framework

Dirac's work pioneered the Hamiltonian analysis of singular systems, but the generalization to higher order systems would not come until much later. In 1850, Ostrogradsky managed to analyze non-singular higher order systems [129], where he showed the equivalence between a Lagrangian of fourth order and a canonical system of $4n$ equations of first order. The treatment of higher order singular systems would not take place until the works of Gitman, Lyakhovich and Tyutin in 1983 [130].

This approach is based on the introduction of new DOFs as an order-reduction mechanism, and maintains Lagrangian-consistent dynamics through the introduction of constraints. This allows the system to be of first order, and for the Hamiltonian picture to be introduced via Dirac's procedure, with the classification of the constraints into first and second-class being paramount to the identification of the gauge symmetries.

Let us start with Lagrangian system with N degrees of freedom

$$\mathcal{L}(q_1, \dot{q}_1, ..., q_1^{(K_1)}, q_2, \dot{q}_2, ..., q_2^{(K_2)}, ..., q_N, \dot{q}_N, ..., q_N^{(K_N)}) \quad , \quad q_i^{(K_i)} = \frac{d^{K_i}}{dt^{K_i}}(q_i) \tag{2.1}$$

Where K_i is the highest-order time derivative of the variable q_i. The procedure is easily generalized for field theories. The equations of motion, as obtained through the principle of stationary action, are shown below

$$\frac{\delta \mathcal{L}}{\delta q_i} \equiv \sum_{s_i=0}^{K} (-1)^{s_i} \frac{d^{s_i}}{dt^{s_i}} \left(\frac{\partial \mathcal{L}}{\partial q_i^{(s_i)}} \right) = 0 \quad , \quad i = 1, ..., N. \tag{2.2}$$

To switch to a Hamiltonian description, we need to perform a Legendre transformation, which in turn requires defining the canonical momenta of the system. For a higher order system, this can become a convoluted process, but here, the derivatives are defined as new coordinates. This change of variables, to be imposed via Lagrange multipliers, is shown below

$$\begin{aligned} q_i^{(0)} &= q_i = q_{(1)i}, \\ q_i^{(s_i)} &= q_{(s_i+1)i} \quad , \quad s_i = 1, \dots, K_i - 1, \\ q_i^{(K)} &= v_i. \end{aligned} \tag{2.3}$$

Hereafter, indices in parentheses do not represent time derivatives, but the parentheses are kept for consistency. To maintain the dynamics of (2.1), this change of variable must be present in the Lagrangian. Iterating over (2.3b) one can find relationships between the velocities and the coordinates, producing the following constraints

$$\begin{aligned} &\dot{q}_{(s)i} - q_{(s+1)i} = 0 \quad , \quad s_i = 1, ..., K_i - 1, \\ &\dot{q}_{(K)i} - v_i = 0. \end{aligned} \tag{2.4}$$

These equations are introduced into the Lagrangian through the Lagrange multipliers

$$I' = \int \mathcal{L}' dt = \int \mathcal{L}(q, v) + \sum_{s_i=1}^{K-1} p^{(s_i)i} \left(\dot{q}_{(s_i)i} - q_{(s_i+1)i} \right) + \pi^i \left(\dot{q}_{(K)i} - v_i \right) dt. \tag{2.5}$$

The higher order Lagrangian $\mathcal{L}$ has been converted to a first order extended Lagrangian $\mathcal{L}'$. Here, the Lagrange multipliers have been labeled $p^{(s_i)i}$ and π^i since, as will be show below, these correspond to the conjugate momenta of $q_{(s_i)i}$, and $q_{(K_i)i}$, respectively.

$$
\begin{aligned}
p^{(s_i)i} &= \frac{\partial \mathcal{L}'}{\partial \dot{q}_{(s_i)i}} = p^{(s_i)i} \quad , \quad s_i = 1, ..., K_i - 1, \\
p^{(K_i)i} &= \frac{\partial \mathcal{L}'}{\partial \dot{q}_{(K_i)i}} = \pi^i.
\end{aligned}
\tag{2.6}
$$

Next, we calculate the EOM for $\mathcal{L}'$

$$
\begin{aligned}
\frac{\delta \mathcal{L}'}{\delta q_{(1)i}} &\equiv \frac{\partial \mathcal{L}}{\partial q_{(1)i}} - \frac{d}{dt}\left(p^{(1)i}\right) = 0, \\
\frac{\delta \mathcal{L}'}{\delta q_{(s)i}} &\equiv \frac{\partial \mathcal{L}}{\partial q_{(s)i}} - p^{(s-1)i} - \frac{d}{dt}\left(p^{(s)i}\right) = 0, \qquad s = 2, ..., K, \\
\frac{\delta \mathcal{L}'}{\delta v_i} &\equiv \frac{\partial \mathcal{L}}{\partial v_i} - \pi^i = 0, \\
\frac{\delta \mathcal{L}'}{\delta p^{(s)i}} &\equiv \dot{q}_{(s)i} - q_{(s+1)i} = 0, \qquad s = 1, ..., K - 1, \\
\frac{\delta \mathcal{L}'}{\delta \pi^i} &\equiv \dot{q}_{(K)i} - v_i = 0.
\end{aligned}
\tag{2.7}
$$

These can always be recombined to recover (2.2), thus demonstrating that the dynamics are equivalent. See reference [131] for an example. For non-singular systems, the velocities can be solved using (2.7) in terms of the variables (i.e. coordinates and momenta) $\dot{q}_{(s)i} = f(q, p)$, which can then be used to perform a Legendre transformation to obtain the Hamiltonian picture. But for singular systems, not all velocities will be solvable. Instead, some constraints will arise, a situation that must be dealt with.

The singularity of the system can be identified by means or the Hessian matrix W_{ij}.The system is said to be singular if the following condition is met

$$
\det W^{ij} = \det \frac{\partial^2 \mathcal{L}}{\partial \dot{q}_{(K_i)i} \partial \dot{q}_{(K_j)j}} = 0 \tag{2.8}
$$

The number of solvable velocities is determined by the rank and nullity of the Hessian

$$
\operatorname{rank} W^{ij} = r_W < N, \tag{2.9}
$$

with the nullity of the Hessian $n_W = N - r_W > 0$ determining the number of Lagrangian constraints the system has

$$\phi_l = \frac{\partial \mathcal{L}'}{\partial \dot{q}_{(l)i}} - p^{(l)i} = 0, \quad l = 1, ..., n_W. \tag{2.10}$$

By analogy with the Dirac procedure for singular Lagrangians, this relations are called "primary constraints", as they are a consequence of the singularity condition without using the EOM.

With the system now being first-order in the time derivatives, the canonical Hamiltonian can be constructed

$$\mathcal{H}_c = \sum_{s_i=1}^{K_i-1} p^{(s_i)i} \dot{q}_{(s_i)i} - \mathcal{L}'. \tag{2.11}$$

It is worth pointing out that, while we cannot solve for all the velocities, the Hamiltonian is (still) only a function of the variables and momenta; as can be seen by taking its total differential [132]. By explicitly writing $\mathcal{L}'$, the Hamiltonian can be written using the initial Lagrangian $\mathcal{L}$ as

$$\mathcal{H}_c = \sum_{s_i=1}^{K-1} p^{(s_i)i} q_{(s_i+1)i} - \mathcal{L}(q, v). \tag{2.12}$$

The dynamics of this Hamiltonian are that of a canonical system with $2N$ variables. The fundamental Poisson brackets are

$$\{q_{(s_i)i}, p^{(s_j)j}\} = \delta_{s_i}^{s_j} \delta_i^j \quad , \quad s_i = 0, ..., K_i - 1 \tag{2.13}$$

with $i, j = 1, ..., N$ for both equations. The bracket between two phase-space functions A, B includes all variables and momenta

$$\{A, B\} = \sum_{s_i=1}^{K-1} \left(\frac{\partial A}{\partial q_{(s_i)i}} \frac{\partial B}{\partial p^{(s_i)i}} - \frac{\partial A}{\partial p^{(s_i)i}} \frac{\partial B}{\partial q_{(s_i)i}} \right). \tag{2.14}$$

Finally, the Hamilton equations of motion are

$$\begin{aligned}
\dot{q}_{(s_i)i} &= \left\{q_{(s_i)i}, \mathcal{H}_c\right\} = \frac{\partial \mathcal{H}_c}{\partial p_{(s_i)i}} = q_{(s_i+1)i}, \\
\dot{p}^{(s_i)i} &= \left\{p^{(s_i)i}, \mathcal{H}_c\right\} = -\frac{\partial \mathcal{H}_c}{\partial q_{(s_i)i}} = -p^{(s_i+1)i} + \frac{\partial \mathcal{L}}{\partial q_{(s_i)i}}.
\end{aligned} \tag{2.15}$$

where $s_i = 1, ..., K_i - 1$. For the dynamics to remain unchanged, the constraints arising from the definition of momenta (2.10) must be added to this system of equations. This is done via Lagrange multipliers, bringing us to the primary Hamiltonian

$$\mathcal{H}' = \mathcal{H}_c + \lambda^l \phi_l, \quad l = 1, ..., n_H. \tag{2.16}$$

Following Dirac's steps, the evolution of a function $A(p, q)$ of the phase space variables is

$$\{A, \mathcal{H}'\} = \{A, \mathcal{H}_c\} + \lambda^l \{A, \phi_l\}. \tag{2.17}$$

This was done by exploiting the fact that $\phi_l = 0$. However, this is only taken into account after applying the Poisson bracket, so the second term is not zero directly. This is known as as a weak equality and is indicated as $\phi_l \approx 0$. Weak equalities are only enforced, or valid, within the constrained surface of motion; outside which, the condition might not necessarily hold.

Additionally, the constraints must remain unchanged with the passage of time. A sensible requirement stemming, again, from Dirac's procedure must be fulfilled for the dynamics to be consistent. To this end, we must analyze the evolution of the constraints by means of the Poisson bracket

$$\dot{\phi}_i = \{\phi_i, \mathcal{H}'\} = \{\phi_i, \mathcal{H}_c\} + \lambda^l \{\phi_i, \phi_l\} \approx 0. \tag{2.18}$$

There are four situations that can arise from this; but first, it is necessary to define $\Phi_{ij} = \phi_i, \phi_j$ as the matrix representing the Poisson brackets between the constraints. Depending on whether Φ_{ij} is singular, and whether or not $\{\phi_i, \mathcal{H}_c\}$ vanishes, the following scenarios can be identified

1. The bracket involving the Hamiltonian vanishes weakly $\{\phi_l, \mathcal{H}_c\} \approx 0$, but $\det \Phi_{ij} \not\approx 0$.

This means that all multipliers are weakly zero: $\lambda^l \approx 0$.

2. If $\{\phi_l, \mathcal{H}_c\} \not\approx 0$ and $\det \Phi_{ij} \not\approx 0$ all Lagrange multipliers can be fully determined via $\lambda^l = -\Phi^{kl}\{\phi_k, \mathcal{H}_c\}$. Therefore, the EOM will take the ensuing form $\dot{A} = \{A, \mathcal{H}_c\} - \{A, \phi_i\}\Phi^{ij}\{\phi_j, \mathcal{H}_c\}$.
3. If $\{\phi_l, \mathcal{H}_c\} \approx 0$ and likewise $\det \Phi_{ij} \approx 0$, then some of the multipliers are fixed, depending on the rank of Φ_{ij}.
4. Finally, having $\{\phi_l, \mathcal{H}_c\} \not\approx 0$ while $\det \Phi_{ij} \approx 0$ will lead us to an iterative process; where we could obtain more constraints, which must be added to the system and whose consistency conditions must be analyzed. This scenario will be discussed down below.

If Φ_{ij} is singular, say of rank r_Φ, there must be $n_\Phi = n_H - r_\Phi$ null vectors ζ^r; which, when multiplied by the matrix, result in the null vector

$$\Phi_{rl}\zeta^r \approx 0, \quad r = 1, ..., n_\Phi. \tag{2.19}$$

By inserting the null vectors into (2.18) we are able to find further conditions

$$\begin{aligned} \zeta^r\{\phi_r, \mathcal{H}'\} &= \zeta^r\{\phi_r, \mathcal{H}_c\} + \lambda^l\zeta^r\{\phi_r, \phi_l\} \\ &= \zeta^r\{\phi_r, \mathcal{H}_c\} \approx 0. \end{aligned} \tag{2.20}$$

Which may or may not produce more constraints, further restricting the motion in phase space. These new constraints, ψ_l, are called secondary constraints, given that they appear after the equations of motion have been used. Although they are labeled as secondary, they are no less important, and their consistency must also be verified.

$$\dot{\psi}_k = \{\psi_k, \mathcal{H}_c\} + \lambda^l_{fc}\{\psi_k, \phi_l\} + \lambda^l_{sc}\{\psi_k, \psi_l\} \approx 0. \tag{2.21}$$

Where λ_{fc} and λ_{sc} are the Lagrange multipliers of the first and second-class constraints, respectively, and l runs up to the number of primary constraints and k up to the number of secondary constraints. Notice that the secondary constraints are now involved in the dynamics. This might, again, result in one of the four aforementioned scenarios, with the possibility of finding even more constraints. This iterative process ends once no more constraints are found, but all constraints obtained after using the equations of motion are called secondary.

Once we have verified that the set of constraints is independent, for instance, by contracting with null vectors, we proceed to perform the classification of the constraints into first and second class, denoted as γ_a and ξ_b, respectively. This classification is determined through the Poisson bracket. A given function $A(q, p)$ is classified as first class if its Poisson brackets with all constraints evaluate to zero

$$\begin{aligned} \{A, \gamma_a\} &\approx 0, \\ \{A, \xi_b\} &\approx 0, \end{aligned} \tag{2.22}$$

Otherwise, if at least one bracket doesn't vanish, it is said to be second class.

Physically permissible paths in the reduced phase space must comply with all constraints, no matter it's classification. The second-class constraints ξ_b, besides restricting the dynamics, do not play any relevant role in the theory. Therefore, it would be desirable for the dynamics to automatically satisfy these constraints. Dirac devised a way to do this by modifying the Poisson brackets, which gave rise to what is now known as Dirac's bracket. The consistency conditions for the second-class constraints

$$\dot{\xi}_b \approx \{\xi_b, \mathcal{H}_c\} + \lambda^a_{fc}\{\xi_b, \gamma_a\} - \lambda^c_{sc}\{\xi_b, \xi_c\} \approx 0, \tag{2.23}$$

leads us to the following equation

$$\{\xi_b, \mathcal{H}_c\} - \lambda^c_{sc}\{\xi_b, \xi_c\} \approx 0. \tag{2.24}$$

which suggests the introduction the matrix consisting of the Poisson brackets of all second class constraints

$$\Delta^{ab} = \left(\{\xi^a, \xi^b\}\right). \tag{2.25}$$

Next, we express the equations of motion as follows

$$\dot{A} \approx \{A, \mathcal{H}_c\} + \lambda^a_{fc}\{A, \gamma_a\} - \{A, \xi_b\}\Delta^{bc}\{\xi_c, \mathcal{H}\}. \tag{2.26}$$

By introducing Dirac's bracket

$$\{A, B\}_D = \{A, B\} - \{A, \xi_a\}\Delta^{ab}\{\xi_b, B\}, \tag{2.27}$$

the second-class constraints can be eliminated from the set, leaving us with the extended Hamiltonian, which contains the Hamiltonian $\mathcal{H}_c$ as well as the first-class constraints γ_a

$$\mathcal{H}_E = \mathcal{H}_c + \lambda^a \gamma_a \tag{2.28}$$

This is the final Hamiltonian. The EOM, derived using the extended Hamiltonian and the Dirac bracket, remain consistent with the original Lagrangian.

$$\dot{A} = \{A, \mathcal{H}_E\}_D \approx \{A, \mathcal{H}_c\}_D + \lambda^a \{A, \gamma_a\}_D. \tag{2.29}$$

Notice that the Lagrange multipliers λ^a above are arbitrary. An important feature of the γ_a is that they are the generators of gauge transformations. This can be seen by first writing the infinitesimal change of a phase-space function, A, with the help of the Hamiltonian equation of motion

$$\begin{aligned} A(t) &\simeq A(0) + \left.\frac{dA}{dt}\right|_{t=0} t + \frac{1}{2}\left.\frac{d^2A}{dt^2}\right|_{t=0} t^2 + \dots \\ &\simeq A_0 + \left(\{A, \mathcal{H}_c\} + \lambda^a\{A_0, \gamma_a\}\right) t + \dots \end{aligned} \tag{2.30}$$

The arbitrariness of λ^a means that this change won't be affected by using a different multiplier $\tilde{\lambda}^a$

$$A(t) \simeq A_0 + \left(\{A, \mathcal{H}_c\} + \tilde{\lambda}^a\{A_0, \gamma_a\}\right) t + \dots \tag{2.31}$$

Their difference is given by

$$\delta A = \epsilon^a \{A_0, \gamma_a\} \tag{2.32}$$

Where $\epsilon^a = (\lambda^a - \tilde{\lambda}^a)t$. This are the infinitesimal form of the gauge transformations. With the first-class constraints, the gauge generator can be constructed

$$G = \sum_a \epsilon_a \gamma^a \tag{2.33}$$

This generator is a crucial component in understanding the symmetries and gauge transformations within the system. Additionally, one can fix the gauge in order to remove all first class constraints, turning them into second class and absorbing them with a new Dirac bracket. This procedure removes the non-physical degrees of freedom of the system. With this in hand, it is possible to proceed with a first quantization of the system to proceed to the quantum realm, but at this point the GLT method has been fully executed. We have successfully transformed a higher order Lagrangian into a Hamiltonian whose dynamics are fully equivalent. The higher order derivatives were removed by introducing additional variables, the constraints that arose as a consequence of this have been correctly dealt with and the gauge symmetries of the system have been properly identified.

2.3 Hamilton-Jacobi Framework

In addition to the Dirac approach and the methods that build upon it, there are other techniques that can be used to obtain the Hamiltonian formulation for higher order systems. These usually share the idea of adding degrees of freedom and imposing conditions on the system to preserve the system's dynamics over time. Among these, the Hamilton-Jacobi framework (HJ) is of particular relevance [133].

The HJ framework[1] is based on Carathéodory's so-called "complete figure of the variational calculus" [134]. It begins by writing down the Hamilton-Jacobi equation for the system and makes use of the system's singularity to write the equations of motion as total differential equations in many variables. This formalism provides a natural way of dealing with both symmetries and the singularity of the system. In general, a system can have two different sets of constraints, which are identified with the system's Hamiltonians and labeled as involutive and non-involutive. The integrability of the system is ensured by Frobenius integrability Conditions, dealing with the non-involutive Hamiltonians by introducing the generalized brackets. At the end of the procedure, one can eliminate the additional degrees of freedom, retaining only the

[1]perhaps better known as the Hamilton-Jacobi method for higher order systems

original ones. The gauge symmetries of the system can be obtained by means of the fundamental differential, which describes fully the dynamics of the system.

Before continuing, perhaps a brief review of the standard HJ method is in order. The HJ theory provides the most powerful method for obtaining Hamilton's equations and is based on canonical transformations.

A canonical transformation is that transformation between two Hamiltonians, $H(q,p,t)$ and $K(Q,P,t)$, which preserves the equations of motion in both systems, i.e. the Hamilton equations are satisfied for both (q,p) and (Q,P). The HJ method aims at obtaining a canonical transformation

$$H + \frac{\partial S}{\partial t} = K, \tag{2.34}$$

where $S(q,P,t)$ is a generating function of type two, such that the New Hamiltonian is trivial

$$K(Q,P,t) = 0.$$

Naturally, the new variables (Q,P) have trivial equations of motion

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = 0 \quad \rightarrow \quad Q_i = \beta_i,$$
$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0 \quad \rightarrow \quad P_i = \alpha_i,$$

with α, β being constants. Here, (2.34) becomes, by means of the transformation equations

$$p_i = \frac{\partial S}{\partial q_i}, \quad Q_i = \frac{\partial S}{\partial P_i}, \tag{2.35}$$

a first order partial differential equation for the generating function S, called the Hamilton-Jacobi equation

$$H\left(q_1, ..., q_n, \frac{\partial S}{\partial q_1}, ..., \frac{\partial S}{\partial q_n}, t\right) + \frac{\partial S}{\partial t} = 0. \tag{2.36}$$

The approach for working with higher-order systems is based on the HJ method shown above. This formalism was extended to treat singular systems by Güler [135], followed by generalizations for higher order derivative Lagrangians by Pimentel and Teixeira [136], who also worked with applications in the gravitational field [137] and topologically massive theories [138].

In the previous section we showed how a higher-order Lagrangian can be transformed into an equivalent first-order one. A similar order-reduction mechanism will be done here as well. Take, for simplicity, an N-dimensional space with the following action

$$I = \int \mathcal{L}(q, \dot{q}, ..., q^{(K)}, t) dt. \tag{2.37}$$

Where q the generalized coordinate, $\dot{q}$ the generalized velocity, and $q^{(K)}$ the time derivative of order k, and t is the time. By introducing additional variables, either based on geometrical considerations (such as with the extrinsic curve) or simply as an order reduction mechanism (as in the GLT method), the system can be transformed into a first-order one

$$I = \int_{t_0}^{t_1} \mathcal{L}(q^1, \dot{q}^1, q^2, \dot{q}^2, ..., q^N, \dot{q}^N, t) dt. \tag{2.38}$$

According to Carathéodory's equivalent Lagrangian method, an equivalent system can be constructed with the addition of the total derivative of a generic function, say $S(q^i, t)$

$$\mathcal{L}^*(q^i, \dot{q}^i, t) = \mathcal{L}(q^i, \dot{q}^i, t) + \frac{dS(q^i, t)}{dt}. \tag{2.39}$$

The equivalence between these two systems is established by means of the variational problem. Two Lagrangians are equivalent to each other if there exists a function $S(q^i, t)$ such that the Lagrangians $\mathcal{L}$ and $\mathcal{L}^*$ have equal extrema. It should be noted that Carathéodory's definition of extrema for the variational problem is different from the modern definition to which the reader may be accustomed. Carathéodory defines an extremum as the curve that minimizes the value of the action I, rather than its variation δI.

To guarantee this equivalence, it is necessary to find two functions $S(q^i, t)$ and $\alpha^i(q^j, t)$ such that for neighborhoods of the curve $\dot{q}^i = \alpha^i(q^j, t)$ the following two conditions are satisfied

$$\begin{aligned} &\mathcal{L}^*(q^i, \dot{q}^i = \alpha^i(q^j, t), t) = 0, \\ &\frac{\partial \mathcal{L}^*}{\partial \dot{q}^i} = 0. \end{aligned} \tag{2.40}$$

Making use of (2.39) these conditions can be transformed into a more useful form

$$\begin{aligned} &\frac{\partial S}{\partial t} + \frac{\partial S}{\partial q^i}\dot{q}^i - \mathcal{L} = 0, \\ &\frac{\partial \mathcal{L}}{\partial \dot{q}^i} = \frac{\partial S}{\partial q^i} = p_i. \end{aligned} \tag{2.41}$$

The HJ formalism identifies S as the generating function and makes (2.41a) the HJ partial differential equation for S.

The transition from the Lagrangian to the Hamiltonian formulation requires that the generalized velocities can be uniquely expressed in terms of the canonical momenta. We consider the case in which the Hessian matrix is singular, i.e.

$$\det W_{ij} = \det\left(\frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j}\right) = 0, \tag{2.42}$$

In such case, the transformation between the Lagrangian and Hamiltonian formulations is not invertible, and will lead us into a phase space that does not correspond to the original configuration space. In other words, the Hamiltonian will not correspond to the original Lagrangian. Assuming that the Hessian is of rank $P < N$. We can split the N-dimensional space into a regular space of dimension P and a null space of dimension $R = N - P$. We shall name the coordinates of the regular space q^a, $a = 1, ..., P$ and those of the null space in a foreshadowing manner as t^z, $z = 1, ..., R$.

In a regular space, we are able to, via the definition of momentum, obtain the velocities as functions of the coordinates and momenta

$$p_a = \frac{\partial \mathcal{L}}{\partial \dot{q}^a} = \frac{\partial S}{\partial q^a} \longrightarrow \dot{q}^a = \dot{q}^a\left(t, t^z, q^b, \frac{\partial S}{\partial q^b}\right). \tag{2.43}$$

Notice that there is no dependence on the null-space momenta.

Naturally, null-space velocities cannot be solved for; but they are still relevant, as they are equivalent to the primary constraints in Dirac's terminology and will constrain the motion to a phase space of lower dimensionality

$$p_z = \frac{\partial \mathcal{L}}{\partial \dot{t}^z} = \frac{\partial S}{\partial t^z}. \tag{2.44}$$

We may write (2.44) as

$$\frac{\partial S}{\partial t^z} + \mathcal{H}_z\left(t, t^z, q^a, \frac{\partial S}{\partial q^a}\right) = 0, \tag{2.45}$$

Where

$$H_z \equiv -\frac{\partial \mathcal{L}}{\partial \dot{t}^z}. \tag{2.46}$$

Notice that H_z does not depend on t^z, otherwise, the momenta p_z would be invertible[2]. We continue by explicitly separating the null and regular terms in (2.41a)

$$\frac{\partial S}{\partial t} + \frac{\partial S}{\partial t^z}\dot{t}^z + \frac{\partial S}{\partial q^a}\dot{q}^a - \mathcal{L} = 0. \tag{2.47}$$

By recognizing the partial derivatives of S as momenta, via (2.43) and (2.44), we can identify the Hamiltonian function H_0

$$H_0 \equiv \frac{\partial S}{\partial t^z}\dot{t}^z + \frac{\partial S}{\partial q^a}\dot{q}^a - \mathcal{L}\left(t, t^z, q^a, \dot{t}^z, \dot{q}^a\right). \tag{2.48}$$

Which reveals (2.41a) the desired HJ partial differential equation.

$$\frac{\partial S}{\partial t} + H_0\left(t, t^z, q^a, \dot{q}^a\right) = 0. \tag{2.49}$$

[2]And therefore part of the regular space, and not the null one.

Notably, H_0 does not depend on $\dot{t}^z$, as can be seen by taking the total differential

$$\begin{aligned}
dH_0 &= \frac{\partial S}{\partial t^z} d\dot{t}^z + \dot{t}^z d\left(\frac{\partial S}{\partial t^z}\right) + \frac{\partial S}{\partial q^a} d\dot{q}^a + \dot{q}^a d\left(\frac{\partial S}{\partial q^a}\right) - d\mathcal{L} \\
&= \left(\frac{\partial S}{\partial t^z} - \frac{\partial \mathcal{L}}{\partial \dot{t}^z}\right) d\dot{t}^z + \dot{t}^z d\left(\frac{\partial S}{\partial t^z}\right) + \left(\frac{\partial S}{\partial q^a} - \frac{\partial \mathcal{L}}{\partial \dot{q}^a}\right) d\dot{q}^a + \dot{q}^a d\left(\frac{\partial S}{\partial q^a}\right) - \frac{\partial \mathcal{L}}{\partial t} dt - \frac{\partial \mathcal{L}}{\partial t^z} dt^z \\
&- \frac{\partial \mathcal{L}}{\partial q^a} dq^a \\
&= \dot{t}^z dp_z + \dot{q}^a dp_a - \frac{\partial \mathcal{L}}{\partial t} dt - \frac{\partial \mathcal{L}}{\partial t^z} dt^z - \frac{\partial \mathcal{L}}{\partial q^a} dq^a
\end{aligned}$$

Where (2.43) and (2.44) have been used.

Furthermore, (2.45) can be identified as additional HJ equations for the null-space variables. By grouping the time variable together with the null-space variables $t^\alpha = (t, t^z)$, as well as identifying $p_0 = \partial S/\partial t$, we may write these equations in a succinct manner

$$\frac{\partial S}{\partial t^\alpha} + H_\alpha \left(t^\beta, q^a, \frac{\partial S}{\partial q^a}\right) = 0 \quad , \alpha, \beta = 0, 1, ..., R. \tag{2.50}$$

These are the HJ partial differential equations for the system. The structure of the equation suggests that each H_α is a Hamiltonian that generates dynamical evolution through its respective parameter t^α. Therefore, starting from a higher-derivative Lagrangian, we have obtained a set of HJ equations for several independent variables t^α. Singular systems in the HJ formalism are systems of several independent variables, with the null-space variables treated on equal footing with t.

From the HJ theory, the conjugate momenta are defined to be in the direction of the gradient of the generating function S, as can be seen from (2.43) and (2.44). Renaming the null-space momenta as π_z, we can rewrite (2.50) as

$$\mathcal{H}'_\alpha(t^\beta, q^a, \pi_\beta, p_a) \equiv \pi_\alpha + H_\alpha(t^\beta, q^a, p_a) \tag{2.51}$$

as well as identify them as a set of canonical constraints $\mathcal{H}'_\alpha = 0$.

The system is now completely described by the set of HJ partial differential equations

$$\begin{aligned} &\mathcal{H}'_{\alpha}(t^{\beta}, q^{a}, \pi_{\beta}, p_{a}) = 0 \\ &\pi_{\alpha} = \frac{\partial S}{\partial t^{\alpha}}, \quad p_{a} = \frac{\partial S}{\partial q^{a}} \end{aligned} \tag{2.52}$$

These HJ equations form a set of $R+1$ first-order partial differential equations. Typically, when we have a time-dependent Hamiltonian we can write Hamilton's equations of motion as follows

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \quad \frac{dq}{dt} = +\frac{\partial H}{\partial p}.$$

If the parameters t^{α} form an independent set, as we have so far assumed, the system can be described by the following equations

$$\begin{aligned} &dq^{a} = \frac{\partial \mathcal{H}'_{\alpha}}{\partial p_{a}} dt^{\alpha}, \quad dp_{a} = -\frac{\partial \mathcal{H}'_{\alpha}}{\partial q^{a}} dt^{\alpha} \\ &dS = p_{a} dq^{a} + \pi_{\alpha} dt^{\alpha} - \mathcal{H}'_{\alpha} dt^{\alpha} \end{aligned} \tag{2.53}$$

These are called the characteristic equations. The first two are simply Hamilton's equations of motion with several independent variables. The latter is obtained from the total differential of the generating function S

$$\begin{aligned} dS &= \frac{\partial S}{\partial t^{\alpha}} dt^{\alpha} + \frac{\partial S}{\partial q^{a}} dq^{a} \\ &= -H_{\alpha} dt^{\alpha} + p_{a} dq^{a} \\ &= -\left(\mathcal{H}'_{\alpha} - \pi_{\alpha}\right) dt^{\alpha} + p_{a} dq^{a} \\ &= p_{a} dq^{a} + \pi_{\alpha} dt^{\alpha} - \mathcal{H}'_{\alpha} dt^{\alpha}. \end{aligned}$$

The independence of the parameters t^{α} is crucial for this procedure and is ensured by an integrability theorem, which will be discussed later. For now, notice that we now have solutions of the form $q^{a} = q^{a}(t^{\alpha}), \quad p_{a} = p_{a}(t^{\alpha})$, in which the variables t^{α} are evolution parameters in a phase space[3] spanned by the variables (q, p).

[3] Usually called reduced phase space.

At this point it is possible to describe the evolution of a function $F(t^\alpha, \pi_\alpha, q^a, p_a)$ in an extended phase space spanned by $(t^\alpha, \pi_\alpha, q^a, p_a)$ via the fundamental differential

$$dF = \{F, \mathcal{H}'_\alpha\} dt^\alpha \tag{2.54}$$

with the help of the extended Poisson brackets

$$\{A, B\} \equiv \frac{\partial A}{\partial t^\alpha}\frac{\partial B}{\partial \pi_\alpha} - \frac{\partial B}{\partial t^\alpha}\frac{\partial A}{\partial \pi_\alpha} + \frac{\partial A}{\partial q^a}\frac{\partial B}{\partial p_a} - \frac{\partial B}{\partial q^a}\frac{\partial A}{\partial p_a}. \tag{2.55}$$

The fundamental differential contains all the dynamics of the system and shows explicitly how each $\mathcal{H}'_\alpha$ is associated with the evolution parameters t^α.

The HJ equations are a necessary condition for the existence of an extremum to the action, but they are not sufficient. While obtaining the characteristic equations, we assumed that the new parameters t^α were mutually independent. However, this is not ensured by the Hamilton-Jacobi equations alone. The independence of t^α implies that the evolution of the system with respect to one of the parameters is totally independent of the others, which involves the complete integrability of the theory. This entails complete solutions to the Hamilton-Jacobi partial differential equations, as well as a unique solution to the characteristic equations (2.53) once the initial conditions are set. The necessary and sufficient condition that the HJ equations have to obey is given by the Frobenius' integrability condition.

Frobenius integrability conditions guarantee the existence of solutions to the characteristic equations in the phase space, as well as simultaneous complete solutions to the HJ equations. These can be expressed both in terms of the evolution parameters or in terms of their generators, but the most useful approach is to express them as follows

$$\{\mathcal{H}'_\alpha, \mathcal{H}'_\beta\} = 0. \tag{2.56}$$

If this condition is met by all the Hamiltonians of the system, the evolution with respect to any one parameter is independent from the others. This can also be expressed via the fundamental differential as

$$d\mathcal{H}'_\alpha = 0. \tag{2.57}$$

Both (2.56) and (2.57) are equivalent conditions.

Moreover, by recalling that the Poisson bracket of two invariants is an invariant, and if the algebra of $\mathcal{H}'_\alpha$ is closed [139], we can generalize (2.56) to

$$\{\mathcal{H}'_\alpha, \mathcal{H}'_\beta\} = C^\gamma_{\alpha\beta}\mathcal{H}'_\gamma. \tag{2.58}$$

It should be noted that, if $\mathcal{H}'_\alpha$ does not possess an explicit time dependence, the integrability conditions are then equivalent to the consistency conditions in the Dirac quantization method [140].

If the Frobenius integrability conditions are satisfied, then the system is in complete involution, the t^α will form an independent set of evolution parameters, and the Hamiltonians $\mathcal{H}'_\alpha$ are said to be involutive and labeled as Ω. However, for some Hamiltonians, the equation (2.56) may not be satisfied. Such Hamiltonians are called non-involutive and labeled as Λ, their presence signals the existence of more Hamiltonians. These should be identified, classified as well into involutive or non-involutive, and added to the total set of $\mathcal{H}'_\alpha$; with the correct number of Hamiltonians obtained via the nullity of the system's Hessian matrix.

The independence of this new set of Hamiltonians might be checked by first defining a matrix whose entries are the Poisson brackets between the Hamiltonians and then using its null vectors to assure independence, as will be shown in the following sections.

After their independence is assured and all Hamiltonians are classified, a fully involutive set of Hamiltonians can be obtained by defining a new bracket. The generalized bracket

$$\{A(x), B(x')\}^* = \{A(x), B(x')\} - \iint \{A(x), \Lambda^{(\mu)}(y)\}\boldsymbol{\Delta}^{(\nu)^{-1}}_{(\mu)}(y,z)\{\Lambda_{(\nu)}(z), B(x')\}d^2y d^2z. \tag{2.59}$$

Where $\Lambda^{(\mu)}$ is a vector composed of the non-involutive Hamiltonians and $\boldsymbol{\Delta}^{(\nu)}_{(\mu)}$ is a symmetric matrix whose entries are the Poisson brackets between the non-involutive Hamiltonians

$$\Lambda_{(\mu)} = \begin{pmatrix} \Lambda^1 \\ \Lambda^2 \\ \vdots \\ \Lambda^n \end{pmatrix}, \quad \mathbf{\Delta}^{(\mu)}_{(\nu)} = \begin{pmatrix} \{\Lambda^1, \Lambda^1\} & \{\Lambda^1, \Lambda^2\} & \cdots & \{\Lambda^1, \Lambda^n\} \\ \{\Lambda^2, \Lambda^1\} & \{\Lambda^2, \Lambda^2\} & \cdots & \{\Lambda^2, \Lambda^n\} \\ \vdots & \vdots & \ddots & \vdots \\ \{\Lambda^n, \Lambda^1\} & \{\Lambda^n, \Lambda^2\} & \cdots & \{\Lambda^n, \Lambda^n\} \end{pmatrix} \tag{2.60}$$

The generalized bracket takes the information from the non-involutive Hamiltonians and integrates it into the dynamics of the system. By using it, the Non-involutive Hamiltonians become essentially non-dynamical, as any bracket that involves them will be zero, which removes them from the set $\mathcal{H}'_\alpha$.

Given that we have modified the dynamics, one must re-evaluate the Frobenius integrability conditions using the generalized bracket. As one might foresee, this can turn previously involutive Hamiltonians into non-involutive, which can lead to the existence of additional Hamiltonians. As in the previous procedure, this becomes an iterative process that concludes when one has a complete set of involutive Hamiltonians $\Omega_{(\mu)}$ under the newest generalized bracket.

Once this process has ended, the dynamical information of the system is fully contained in the fundamental differential

$$dF = \{F, \mathcal{H}'_0\}^* dt^0 + \{F, \Omega_{(\mu)}\}^* dt^{(\mu)}. \tag{2.61}$$

Where we have explicitly separated the physical part, i.e. the time evolution given by the canonical Hamiltonian. The Hamiltonians $\Omega_{(\mu)}$ are not physical, since they were generated by the procedure to which the system has been subjected. The dynamical evolution described by their parameters $t^{(\mu)}$ can be understood as canonical transformations, with the involutive Hamiltonians as their corresponding generators. In order to relate these transformations to the gauge transformations, we first apply the fundamental differential to the variables

$$\delta\gamma^{(\alpha)} = \{\gamma^{(\alpha)}, \mathcal{H}'_0\}^* dt^0 + \{\gamma^{(\alpha)}, \Omega_{(\mu)}\}^* dt^{(\mu)}.$$

Where $\gamma^{(\alpha)} = (q^i, p_i, t^z, \pi_z)$. This evolution of the dynamical variables with respect to our parameters $t^{(\mu)}$, i.e. by keeping time fixed, is understood as canonical transformations

$$\delta\gamma^{(\alpha)} = \{\gamma^{(\alpha)}, \Omega_{(\mu)}\}^* dt^{(\mu)}. \tag{2.62}$$

The variation of the action under these transformations makes it possible to obtain the gauge transformations. Specifically, taking the variation as zero $\delta I = 0$, a process that is shown more explicitly thorough this work.

Chapter 3

Higher-order Maxwell-Chern-Simons

This chapter deals with the higher-order Maxwell-Chern-Simons theory. The chapter begins with a brief overview of the theory, its history, action, and equations of motion. Afterwards, the theory is examined, but not before performing a $2+1$ decomposition and taking the perturbative limit. Then, by using the Hamilton-Jacobi (HJ) framework, a complete analysis of the theory is performed. This includes the proper identification of a complete set of involutive Hamiltonians and a generalized differential, from which the symmetries of the theory and the complete dynamics are obtained. In addition, this study is completed by employing the higher order Gitman-Lyakhovich-Tyutin (GLT) framework, in which the full dynamics, gauge symmetries, constraints and Hamiltonian description are also obtained. We conclude the chapter by comparing the two frameworks. The content of this section is compiled in [141].

3.1 Theory

The higher-order Maxwell-Chern-Simons theory is, as the name suggests, a synthesis of Maxwell's theory and the higher-order Chern-Simons term; a generalization of the Chern-Simons (CS) topological term, whose properties in $2+1$ dimensions are well-known. For abelian vector fields the action is given by

$$I_{CS} = \frac{m}{2} \int \epsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma d^3x.$$

It is parity violating, of first order, metric-independent, and gauge invariant. The action leads to the locally flat field equation $F_{\alpha\beta} = 0$. Because of its origin, the CS term is found in many areas of physics, including gravitational physics, particle physics, String Theory and Loop Quantum Gravity. And wherever it is found, its presence is quite noticeable. From the particle physics point of view, the presence of the CS term leads to a gravitational anomaly-cancellation mechanism in the standard model. In cosmology, the chiral anomaly works together with inflation to amplify the production of leptons, leading to a viable model of leptogenesis [88]. Einstein's gravity, which does not have local degrees of freedom and does not propagate any physical gravitational waves, becomes a dynamical theory, with a propagating particle. The Chern-Simons term introduces a coupling between the gravitational connection and a gauge field. This modification leads to the propagation of massive gravitational excitations of helicity ± 2 [142], making the theory non-trivial. Even though the theory has a third time derivative dependence, it is ghost-free and unitary. In $3D$ gravity with a Chern-Simons term there can be black hole solutions with interesting properties, such as fractional spin and exotic statistics for excitations [143].

Returning to our subject, when added to the Maxwell action I_{MAX}, the theory becomes the so-called topologically massive electrodynamics (TME), which incorporates a topological mass term for the photon. TME excitations were found to have interesting properties, they are both massive and gauge invariant and their single, parity-violating, degrees of freedom (DOFs) are represented by scalar fields with non-vanishing spins. In general, both spin and the number of

DOFs are different in massless and massive cases. Moreover, the particle content also differs from that in conventionally massive theories. Higher derivative extensions of the Chern-Simons action have also been studied, especially, the third-order action

$$I_{ECS} = \frac{1}{2m} \int \epsilon^{\alpha\beta\gamma} \Box A_\alpha \partial_\beta A_\gamma d^3x. \tag{3.1}$$

Such a term, even if it's not present originally, is present when fermionic loop integration is performed [144]. It is an intriguing term that remains gauge-invariant and of odd parity, but it is no longer topological like the CS term. This is because of the metric dependence on the additional covariant derivative factor. Interesting properties arise from the coupling of this term to either pure Maxwell, the CS term, or to both of these. Including, but not limited to, the presence of Anyons [145], as well as the Aharonov-Bohm [146] and Hall [147] effects.

3.2 The Action

The Maxwell-Chern-Simons extended Lagrangian in $2+1$ dimensions is presented below

$$\mathcal{L} = -\frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} + \frac{\theta}{4} \epsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma} + \frac{1}{4m} \epsilon^{\alpha\beta\gamma} (\Box A_\alpha) F_{\beta\gamma}. \tag{3.2}$$

Where A_α is the gauge potential, $F_{\alpha\beta}$ is the curvature tensor, and $\epsilon^{\alpha\beta\gamma}$ is the Levi-Civita symbol. The equations of motion for a higher order field Lagrangian are

$$\frac{\partial \mathcal{L}}{\partial A_\alpha} - \partial_\beta \frac{\partial \mathcal{L}}{\partial(\partial_\beta A_\alpha)} + \partial_\gamma \partial_\beta \frac{\partial \mathcal{L}}{\partial(\partial_\gamma \partial_\beta A_\alpha)} = 0. \tag{3.3}$$

For our system, these are

$$\frac{\theta}{4} \epsilon^{\alpha\beta\gamma} F_{\beta\gamma} + \partial_\beta F^{\beta\alpha} - \frac{\theta}{2} \epsilon^{\alpha\beta\gamma} \partial_\gamma A_\beta - \frac{1}{2m} \epsilon^{\alpha\beta\gamma} \Box \partial_\gamma A_\beta + \frac{1}{4m} \partial_\mu \partial_\nu g^{\mu\nu} \epsilon^{\alpha\beta\gamma} F_{\beta\gamma} = 0. \tag{3.4}$$

These can be written more succinctly by using that

$$\epsilon^{\alpha\beta\gamma}T_\alpha F_{\beta\gamma} = \epsilon^{\alpha\beta\gamma}T_\alpha\partial_\beta A_\gamma - \epsilon^{\alpha\beta\gamma}T_\alpha\partial_\gamma A_\beta$$

$$= \epsilon^{\alpha\beta\gamma}T_\alpha\partial_\beta A_\gamma + \epsilon^{\alpha\beta\gamma}T_\alpha\partial_\beta A_\gamma.$$

That is to say

$$\epsilon^{\alpha\beta\gamma}T_\alpha F_{\beta\gamma} = 2\epsilon^{\alpha\beta\gamma}T_\alpha\partial_\beta A_\gamma. \tag{3.5}$$

Using this substitution the equations of motion become

$$\frac{\theta}{2}\epsilon^{\alpha\beta\gamma}F_{\beta\gamma} + \partial_\beta F^{\beta\alpha} + \frac{1}{2m}\epsilon^{\alpha\beta\gamma}\Box F_{\beta\gamma} = 0. \tag{3.6}$$

We continue the analysis by performing a $2+1$ decomposition

$$\mathcal{L} = -\frac{1}{4}F^{ij}F_{ij} - \frac{1}{2}F^{0i}F_{0i} + \frac{\theta}{4}\epsilon^{ij}A_0F_{ij} - \frac{\theta}{2}\epsilon^{ij}A_iF_{0j} + \frac{1}{4m}\epsilon^{ij}(\Box A_0)F_{ij} - \frac{1}{2m}\epsilon^{ij}(\Box A_i)F_{0j}. \tag{3.7}$$

Making use of (3.5) to change the third and fifth terms, in addition to using the definition of the electromagnetic tensor to modify the fourth and sixth terms

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F^{ij}F_{ij} + \frac{1}{2}\dot{A}^i\dot{A}_i + \dot{A}_i\partial^iA^0 - \frac{1}{2}\partial^iA^0\partial_iA_0 + \frac{\theta}{2}\epsilon^{ij}A_0\partial_iA_j - \frac{\theta}{2}\epsilon^{ij}A_i\partial_0A_j \\ &+ \frac{\theta}{2}\epsilon^{ij}A_i\partial_jA_0 + \frac{1}{2m}\epsilon^{ij}(\Box A_0)\partial_iA_j - \frac{1}{2m}\epsilon^{ij}(\Box A_i)\partial_0A_j + \frac{1}{2m}\epsilon^{ij}(\Box A_i)\partial_jA_0.\end{aligned} \tag{3.8}$$

Finally, splitting the d'Alembert operator in its spatial and temporal parts we obtain

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\dot{A}^i\dot{A}_i - \dot{A}_i\partial^iA_0 - \frac{1}{2}\partial^iA^0\partial_iA_0 - \frac{1}{4}F^{ij}F_{ij} + \frac{\theta}{2}\epsilon^{ij}A_0\partial_iA_j - \frac{\theta}{2}\epsilon^{ij}A_i\dot{A}_j + \frac{\theta}{2}\epsilon^{ij}A_i\partial_jA_0 \\ &- \frac{1}{2m}\epsilon^{ij}\ddot{A}_0\partial_iA_j + \frac{1}{2m}\epsilon^{ij}\nabla^2A_0\partial_iA_j + \frac{1}{2m}\epsilon^{ij}\ddot{A}_i\dot{A}_j - \frac{1}{2m}\epsilon^{ij}\nabla^2A_i\dot{A}_j - \frac{1}{2m}\epsilon^{ij}\ddot{A}_i\partial_jA_0 \\ &+ \frac{1}{2m}\epsilon^{ij}\nabla^2A_i\partial_jA_0.\end{aligned} \tag{3.9}$$

We take this Lagrangian as a starting point for our calculations.

3.3 The Gitman-Lyakhovich-Tyutin Framework

It should be noted that the Lagrangian (3.9) has dependence on the gauge potential A_α as well as on its velocity and acceleration

$$\mathcal{L} = \mathcal{L}(A_\mu, \dot{A}_\mu, \partial_i A_\mu, \partial_\nu \dot{A}_\mu, \partial_\nu \partial_i A_\mu, \ddot{A}_\mu, \partial_i \partial_j A_\mu). \tag{3.10}$$

The order of the time derivatives of the system is reduced via the introduction of the following variables

$$v_\mu = \dot{A}_\mu, \quad \beta_\mu = \dot{v}_\mu. \tag{3.11}$$

Together with their conjugate momenta π^ν, $\tilde{\pi}^\nu$; which satisfy

$$\begin{aligned} \{A_\mu, \pi^\nu\} &= \delta^\nu_\mu \delta^2(x-y), \\ \{v_\mu, \tilde{\pi}^\nu\} &= \delta^\nu_\mu \delta^2(x-y). \end{aligned} \tag{3.12}$$

An equivalent Lagrangian can be obtained by adding the changes of variables with the help of Lagrange multipliers as

$$\tilde{\mathcal{L}} = \mathcal{L} + \pi^\mu(\dot{A}_\mu - v_\mu) + \tilde{\pi}^\mu(\dot{v}_\mu - \beta_\mu). \tag{3.13}$$

The equivalence can be seen by observing the equations of motion of $\tilde{\mathcal{L}}$, which are

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta \pi^\mu} &= \dot{A}_\mu - v_\mu = 0, \\ \frac{\delta \tilde{\mathcal{L}}}{\delta \Pi^\mu} &= \dot{G}_\mu - \beta_\mu = 0, \\ \frac{\delta \tilde{\mathcal{L}}}{\delta A_\mu} &= \frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i A_\mu)} - \dot{\pi}^\mu + \partial_i \partial_j \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_j A_\mu)} = 0, \\ \frac{\delta \tilde{\mathcal{L}}}{\delta v_\mu} &= \frac{\partial \mathcal{L}}{\partial v_\mu} - \pi^\mu - \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i v_\mu)} - \dot{\Pi}^\mu = 0, \\ \frac{\delta \tilde{\mathcal{L}}}{\delta \beta_\mu} &= \frac{\partial \mathcal{L}}{\partial \beta_\mu} - \Pi^\mu = 0. \end{aligned} \tag{3.14}$$

Which can be manipulated to become

$$\frac{\delta\mathcal{L}}{\delta A_\mu} = \frac{\partial\mathcal{L}}{\partial A_\mu} - \partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0 A_\mu)} - \partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i A_\mu)} + \partial_0\partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0\partial_0 A_\mu)} + \partial_0\partial_i\frac{\partial\mathcal{L}}{\partial(\partial_0\partial_i A_\mu)} + \partial_i\partial_j\frac{\partial\mathcal{L}}{\partial(\partial_i\partial_j A_\mu)} = 0.$$

I.e., the EOM (3.3). Explicitly, the equations of motion are

$$\begin{aligned}\tilde{\mathcal{L}} = {} & \frac{1}{2}v^i v_i - v_i\partial^i A_0 - \frac{1}{2}\partial^i A^0\partial_i A_0 - \frac{1}{4}F^{ij}F_{ij} + \frac{\theta}{2}\epsilon^{ij}A_0\partial_i A_j - \frac{\theta}{2}\epsilon^{ij}A_i v_j + \frac{\theta}{2}\epsilon^{ij}A_i\partial_j A_0 \\ & - \frac{1}{2m}\epsilon^{ij}\beta_0\partial_i A_j + \frac{1}{2m}\epsilon^{ij}\nabla^2 A_0\partial_i A_j + \frac{1}{2m}\epsilon^{ij}\beta_i v_j - \frac{1}{2m}\epsilon^{ij}\nabla^2 A_i v_j - \frac{1}{2m}\epsilon^{ij}\beta_i\partial_j A_0 \\ & + \frac{1}{2m}\epsilon^{ij}\nabla^2 A_i\partial_j A_0 + \pi^\mu(\dot{A}_\mu - v_\mu) + \tilde{\pi}^\mu(\dot{v}_\mu - \beta_\mu).\end{aligned} \tag{3.15}$$

We can observe that the theory is now first order in time derivatives.

$$\mathcal{L} = \mathcal{L}(A_\mu, v_\mu, \partial_i A_\mu, \partial_\nu v_\mu, \partial_\nu\partial_i A_\mu, \beta_\mu, \partial_i\partial_j A_\mu). \tag{3.16}$$

Additionally, the canonical momenta have been introduced right at the start, which allows to identify the constraints in a more convenient manner compared to Ostrogradsky's formalism. As a matter of fact, in the GLT framework it is not necessary to introduce generalized momenta for the higher-order time derivatives of the fields, in contrast to Ostrogradsky's framework. Subsequently, the canonical Hamiltonian is given as usual, via a Legendre transformation, or, by using (3.11), as

$$\mathcal{H} = v_\mu\pi^\mu + \beta_\mu\Pi^\mu - \mathcal{L}. \tag{3.17}$$

That is

$$\begin{aligned}\mathcal{H} = {} & v_0\pi^0 + v_i\pi^i + \beta_0\tilde{\pi}^0 + \beta_i\tilde{\pi}^i - \frac{1}{2}v^i v_i + v_i\partial^i A_0 + \frac{1}{2}\partial^i A^0\partial_i A_0 + \frac{1}{4}F^{ij}F_{ij} \\ & - \frac{\theta}{2}\epsilon^{ij}A_0\partial_i A_j + \frac{\theta}{2}\epsilon^{ij}A_i v_j - \frac{\theta}{2}\epsilon^{ij}A_i\partial_j A_0 + \frac{1}{2m}\epsilon^{ij}\beta_0\partial_i A_j - \frac{1}{2m}\epsilon^{ij}\nabla^2 A_0\partial_i A_j \\ & - \frac{1}{2m}\epsilon^{ij}\beta_i v_j + \frac{1}{2m}\epsilon^{ij}\nabla^2 A_i v_j + \frac{1}{2m}\epsilon^{ij}\beta_i\partial_j A_0 - \frac{1}{2m}\epsilon^{ij}\nabla^2 A_i\partial_j A_0.\end{aligned} \tag{3.18}$$

The analysis continues as it becomes clear that we cannot solve for the velocities in terms of the conjugate momenta and the variables. So the system is constrained. A quick calculation shows that these primary constraints are

$$\begin{aligned}\phi^3 &= \frac{\partial \mathcal{L}}{\partial \beta_0} - \tilde{\pi}^0 = -\frac{1}{2m}\epsilon^{ij}\partial_i A_j - \tilde{\pi}^0 \approx 0,\\ \phi^i &= \frac{\partial \mathcal{L}}{\partial \beta_i} - \tilde{\pi}^i = \frac{1}{2m}\epsilon^{ij} v_j - \frac{1}{2m}\epsilon^{ij}\partial_j A_0 - \tilde{\pi}^i \approx 0.\end{aligned} \tag{3.19}$$

It is straightforward to see that their algebra is

$$\begin{aligned}\{\phi^3, \phi^3\} &= 0,\\ \{\phi^i, \phi^j\} &= -\frac{1}{m}\epsilon^{ij}\delta^2(x-y),\\ \{\phi^3, \phi^i\} &= 0.\end{aligned} \tag{3.20}$$

It is important to highlight the differences between the HJ and GLT frameworks. On the one hand, in the GLT formulation we must identify future constraints through consistency, classifying them into first and second class. Afterwards, Dirac's brackets can be introduced, which makes the second class constraints strongly zero. It is only at the end of the calculations that we can compare both methods. On the other hand, in the HJ scheme, the generalized brackets[1] are introduced from the beginning. After the method is completed, only the involutive Hamiltonians are left, which agree with the set of first-class constraints of the GLT formalism.

We will now make sure that these constraints are consistent over time by requiring that their time evolution is zero

$$\dot{\phi}^{(\mu)} = \{\phi^{(\mu)}, \mathcal{H}'\} = 0. \tag{3.21}$$

Where $\phi^{(\mu)} = \{\phi^3, \phi^i\}$, and $\mathcal{H}'$ is the primary Hamiltonian, made up of the canonical Hamiltonian H_c as well as the constraints

$$\mathcal{H}' = \mathcal{H} + \lambda_3\phi^3 + \lambda_i\phi^i. \tag{3.22}$$

[1] Which have a construction equivalent to the Dirac Brackets.

Where λ_3 and λ_i are Lagrange multipliers. We begin with ϕ^i. Its time evolution

$$\dot{\phi}^i = \{\phi^i, \mathcal{H}'\} \approx \{\phi^i, \mathcal{H}\} + \lambda_3\{\phi^i, \phi^3\} + \lambda_j\{\phi^i, \phi^j\} \tag{3.23}$$

Leads to a condition on the Lagrange multiplier λ_i

$$\dot{\phi}^i = -\epsilon^{ij}\lambda_j + \epsilon^{ij}\beta_j - \frac{1}{2}\epsilon^{ij}\partial_j v_0 + m\pi^i - mv^i + m\partial^i A_0 - \frac{\theta m}{2}\epsilon^{ij}A_j - \frac{1}{2}\epsilon^{ij}\nabla^2 A_j \approx 0. \tag{3.24}$$

Continuing with ϕ^3, its consistency condition

$$\dot{\phi}^3 = \{\phi^3, \mathcal{H}'\} \approx \{\phi^3, \mathcal{H}\} + \lambda_3\{\phi^3, \phi^3\} + \lambda_i\{\phi^3, \phi^i\} \tag{3.25}$$

leads to the following equation

$$\dot{\phi}^3 = -\frac{1}{2m}\epsilon^{ij}\frac{\partial}{\partial x^i}G_j(x) + \pi^0(x) = 0. \tag{3.26}$$

Given that its time evolution must be zero we are forced to introduce a new (secondary) constraint

$$\chi^0 = \pi^0 - \frac{1}{2m}\epsilon^{ij}\partial_i v_j \approx 0. \tag{3.27}$$

This newly added constraint must be treated at the same level as our previous ones, so its consistency must also be reviewed. By demanding $\dot{\chi}F^0$ to be zero we find

$$\dot{\chi}^0 = -\epsilon^{ij}\partial_i\lambda_j + \epsilon^{ij}\partial_i\beta_j - m\partial^i v_i + m\nabla^2 A^0 - \theta m\epsilon^{ij}\partial_i A_j - \epsilon^{ij}\nabla^2\partial_i A_j \approx 0. \tag{3.28}$$

Which does not lead to further constraints, but to a condition for the Lagrange multipliers. From (3.24) and (3.28), another secondary constraint can be obtained

$$\chi^1 = \partial_i\pi^i + \frac{\theta}{2}\epsilon^{ij}\partial_i A_j + \frac{1}{2m}\epsilon^{ij}\nabla^2\partial_i A_j \approx 0. \tag{3.29}$$

The consistency conditions of χ^1 are trivially null. Therefore, the complete set of GLT constraints is given by

$$\begin{aligned}
\phi^3 &= \tilde{\pi}^0 + \frac{1}{2m}\epsilon^{ij}\partial_i A_j \approx 0, \\
\phi^i &= \tilde{\pi}^i - \frac{1}{2m}\epsilon^{ij}v_j + \frac{1}{2m}\epsilon^{ij}\partial_j A_0 \approx 0, \\
\chi^0 &= \pi^0 - \frac{1}{2m}\epsilon^{ij}\partial_i v_j \approx 0, \\
\chi^1 &= \partial_i \pi^i + \frac{\theta}{2}\epsilon^{ij}\partial_i A_j + \frac{1}{2m}\epsilon^{ij}\nabla^2\partial_i A_j \approx 0.
\end{aligned} \tag{3.30}$$

To separate them into first and second class, we calculate the 5×5 matrix whose entries contain the Poisson brackets between all constraints. This is

$$A = \begin{pmatrix} -\frac{1}{m}\epsilon^{ij} & 0 & \frac{1}{m}\epsilon^{ij}\partial_j & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{m}\epsilon^{ji}\partial_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \delta^2(x-y). \tag{3.31}$$

This matrix has a rank $= 2$ and 3 null vectors; this means that there will be two second class constraints and three first class ones. The contraction of the null vectors with the constraints allows us to identify the following first-class constraints

$$\begin{aligned}
\gamma^1 &= \tilde{\pi}^0 + \frac{1}{2m}\epsilon^{ij}\partial_i A_j, \\
\gamma^2 &= \pi^0 - \partial_i \tilde{\pi}^i, \\
\gamma^3 &= \partial_i \pi^i + \frac{\theta}{2}\epsilon^{ij}\partial_i A_j + \frac{1}{2m}\epsilon^{ij}\nabla^2\partial_i A_j.
\end{aligned} \tag{3.32}$$

That is, one null vector is given by $\tilde{v} = (0, \partial_i w, w, 0)$. From the contraction with the matrix, we obtain γ^2. We observe that the constraints (3.32) coincide with the Hamiltonians obtained by means of the HJ framework in the previous section. The two second-class constraints are

$$\xi^i = \tilde{\pi}^i - \frac{1}{2m}\epsilon^{ij}v_j + \frac{1}{2m}\epsilon^{ij}\partial_j A_0. \tag{3.33}$$

These constraints are removed from the beginning in the HJ approach. In this sense, the HJ is more concise. It is worth mentioning that the constraints have been obtained consistently. It is not necessary to fix them by hand such as has been done previously in the literature [148]. With the correct identification of the constraints, the counting of physical DOFs can be carried out. There are 12 canonical variables $\{A_\mu, \pi^\nu\}$, $\{v_\mu, \tilde{\pi}^\nu\}$, three first-class constraints $(\gamma^1, \gamma^2, \gamma^3)$ and two second-class constraints (ξ^i); therefore, there are two physical DOFs, as expected.

The second-class constraints can be removed by means of the Dirac bracket

$$\{A(x), B(x')\}_D = \{A(x), B(x')\} - \int\int \{A(x), \xi^a(y)\}\Delta^{-1}_{ab}(y,z)\{\xi^b(z), B(x')\}d^2y d^2z. \quad (3.34)$$

Where Δ^{-1}_{ab} is, naturally, the inverse of Δ_{ab}, which contains in its entries the Poisson brackets among the second-class constraints $\Delta^{ab} = \{\xi^a, \xi^b\}$. That is

$$\Delta^{ij}(x,y) = m\epsilon^{ij}\delta^2(x-y). \quad (3.35)$$

Once obtained, Dirac's bracket modifies the canonical relations between the coordinates and momenta. Resulting in the following non-trivial brackets

$$\begin{aligned}
&\{A_0, \pi^0\}_D = \delta^2(x-y), && \{A_i, \pi^j\}_D = \delta_i^j\delta^2(x-y),\\
&\{\pi^0, v_i\}_D = \tfrac{1}{2}\partial_i\delta^2(x-y), && \{\pi^0, \tilde{\pi}^i\}_D = -\tfrac{1}{4m}\epsilon^{ij}\partial_j\delta^2(x-y),\\
&\{v_0, \tilde{\pi}^0\}_D = \delta^2(x-y), && \{v_i, \tilde{\pi}^j\}_D = \tfrac{1}{2}\delta_i^j\delta^2(x-y),\\
&\{v_i, v_j\}_D = m\epsilon_{ij}\delta^2(x-y), && \{\tilde{\pi}^i, \tilde{\pi}^j\}_D = \tfrac{1}{4m}\epsilon^{ij}\delta^2(x-y).
\end{aligned} \quad (3.36)$$

Using these, we see that the constraints γ^1, γ^2, and γ^3 are still first class. We will now fix the gauge in order to remove the redundant DOFs. This will also remove all first-class constraints, turning them into second class [149]. Demanding consistency of the Coulomb gauge $\gamma^4 = \partial_i A_i$ results in a new constraint

$$\dot{\gamma}^5 = \{\partial_i A^i, H'\}_D = \partial_i v^i := \gamma^6. \quad (3.37)$$

Consistency of γ^6 gives no new constraints. Below we present the nontrivial brackets among all constraints

$$
\begin{aligned}
\{\gamma^4,\gamma^3\}_D &= -\nabla^2\delta^2(x-y),\\
\{\gamma^5,\gamma^2\}_D &= \nabla^2\delta^2(x-y),\\
\{\gamma^6,\gamma^1\}_D &= \nabla^2\delta^2(x-y),\\
\{\gamma^6,\gamma^3\}_D &= \frac{\theta m}{2}\nabla^2\delta^2(x-y)+\frac{1}{2}\nabla^4\delta^2(x-y),\\
\{\gamma^5,\gamma^6\}_D &= m^2\nabla^2\delta^2(x-y).
\end{aligned}
\tag{3.38}
$$

As can be seen, all have become second class. At this point, a new Dirac bracket can be introduced. By using these brackets into a new Δ_{ab}^{-1} matrix, new Dirac's brackets, say $\{\cdot,\cdot\}_{D_2}$, can be obtained. These are

$$
\begin{aligned}
&\{A_0,v_0\}_{D_2} = m^2\tfrac{1}{\nabla^2}\delta^2(x-y), && \{A_0,v_i\}_{D_2} = m\epsilon_{ij}\tfrac{\partial^j}{\nabla^2}\delta^2(x-y),\\
&\{A_0,\pi^0\}_{D_2} = \delta^2(x-y), && \{A_0,\pi^i\}_{D_2} = -\tfrac{m}{2}\epsilon^{ij}\tfrac{\partial_j}{\nabla^2}\delta^2(x-y),\\
&\{A_0,\tilde{\pi}^i\}_{D_2} = -\tfrac{1}{2}\tfrac{\partial^i}{\nabla^2}\delta^2(x-y), && \{A_i,v_0\}_{D_2} = -m\epsilon_{ij}\tfrac{\partial^j}{\nabla^2}\delta^2(x-y),\\
&\{A_i,\tilde{\pi}^j\}_{D_2} = \tfrac{1}{2}(\delta_i^j-\tfrac{\partial^j\partial_i}{\nabla^2})\delta^2(x-y), && \{v_0,\pi^i\}_{D_2} = -(\theta m\tfrac{1}{2\nabla^2}+\tfrac{1}{2})\partial^i\delta^2(x-y),\\
&\{v_0,\tilde{\pi}^0\}_{D_2} = \tfrac{1}{2}\delta^2(x-y), && \{v_0,\tilde{\pi}^i\}_{D_2} = -\tfrac{m}{2}\epsilon^{ij}\tfrac{\partial_j}{\nabla^2}\delta^2(x-y),\\
&\{v_i,\tilde{\pi}^j\}_{D_2} = \tfrac{1}{2}(\delta_i^j-\tfrac{\partial_i\partial^j}{\nabla^2})\delta^2(x-y), && \{v_i,\pi^0\}_{D_2} = -\tfrac{1}{2}\partial_i\delta^2(x-y),\\
&\{\pi^0,\tilde{\pi}^i\}_{D_2} = -\tfrac{1}{4m}\epsilon^{ij}\partial_j\delta^2(x-y), && \{\pi^i,\tilde{\pi}^0\}_{D_2} = \tfrac{1}{4m}\epsilon^{ij}\partial_j\delta^2(x-y),\\
&\{\pi^i,\tilde{\pi}^j\}_{D_2} = \tfrac{1}{4}(\delta^{ij}-\tfrac{\partial^i\partial^j}{\nabla^2})\delta^2(x-y).
\end{aligned}
\tag{3.39}
$$

These brackets can be used for quantization of the theory by using the methods reported in [150], where a procedure of gauge fixing is developed in the path integral approach. Systems with gauge symmetries can represent different configurations of the variables with the same physical description if said states are related by means of a gauge transformation. Fixing the gauge removes this ambiguity from the system. This is the reason for which all Hamiltonians become involutive.

3.3.1 Gauge transformations

Aditionally, instead of fixing the gauge, one can obtain the gauge transformations of the system. In this section, we use Castellani's procedure [148], [151] to obtain the gauge transformations. We start this calculation with the Hamiltonian (3.18), the constraints given in (3.32), and the Dirac brackets (3.36). First, we define the gauge generator as

$$G = \int \epsilon_a \gamma^a d^2x. \tag{3.40}$$

Where ϵ_a are the gauge parameters, $a = 1, 2, 3$. This generates infinitesimal gauge transformations on the phase space variables, say F, through

$$\delta F = \int \delta\epsilon_a(y)\{F(x), \gamma^a(y)\}_D d^2y. \tag{3.41}$$

In particular, the generator obeys the following equation, called the master equation,

$$\frac{\partial}{\partial t}G + \{G, \mathcal{H}_T\}_D = 0. \tag{3.42}$$

Where $\mathcal{H}_T = \mathcal{H}_c + u_a\gamma^a$ is the total Hamiltonian. From the algebra of the constraints and the canonical Hamiltonian, we can obtain the so-called structure functions V_a^b, C_c^{ab}, given by

$$\begin{aligned} \{\mathcal{H}, \gamma^a(x)\}_D &= \int V_b^a(x, y)\gamma^b(y)d^2y, \\ \{\gamma^a(x), \gamma^b(y)\}_D &= \int C_c^{ab}(x, y, z)\gamma^c(z)d^2z. \end{aligned} \tag{3.43}$$

Using these, the master equation becomes

$$\frac{d\epsilon_a(x)}{dt} - \int \epsilon_b(y)V_a^b(x, y)d^2y - \int \epsilon_b(y)\gamma_c(z)C_a^{cb}(x, y, z)d^2yd^2z = 0. \tag{3.44}$$

Since the only nonzero structure functions are $V_2^1 = -\delta^2(x-y)$, $V_3^2 = -\delta^2(x-y)$, with all the $C_c^{ab} = 0$, we can obtain the following relations between the generators

$$\begin{aligned} \epsilon_1 &= \ddot{\epsilon}_3, \\ \epsilon_2 &= -\dot{\epsilon}_3. \end{aligned} \tag{3.45}$$

Therefore, the generator has only one parameter and can be written as

$$G = \int (\delta\ddot{\epsilon}_3\gamma^1 - \delta\dot{\epsilon}_3\gamma^2 + \delta\epsilon_3\gamma^3)d^2x. \tag{3.46}$$

using (49) the gauge transformations of the variables are

$$\begin{aligned} \delta A_0 &= \int \delta\epsilon_2(y)[\delta^2(x-y)]d^2y, \\ \delta A_i &= \int \delta\epsilon_3(y)\left[\frac{\partial}{\partial y^i}\delta^2(x-y)\right]d^2y, \\ \delta\pi^i &= \int \delta\epsilon_1(y)\left[\frac{1}{2m}\epsilon^{ij}\frac{\partial}{\partial x^j}\delta^2(x-y)\right] + \delta\epsilon_3(y)\left[-\frac{\theta}{2}\epsilon^{ij}\frac{\partial}{\partial x^j}\delta^2(x-y) - \frac{1}{2m}\epsilon^{ij}\nabla_y^2\frac{\partial}{\partial x^j}\delta^2(x-y)\right]d^2y, \\ \delta v_0 &= \int \delta\epsilon_1(y)\left[-\delta^2(x-y)\right]d^2y, \\ \delta v_i &= \int \delta\epsilon_2(y)\left[\frac{\partial}{\partial x^i}\delta^2(x-y)\right]d^2y. \end{aligned} \tag{3.47}$$

Where $\delta\pi^0 = \delta\tilde{\pi}^0 = \delta\tilde{\pi}^i = 0$. By using (54) the following gauge transformations are found

$$\begin{aligned} \delta A_\mu &= -\partial_\mu\delta\epsilon_3, \\ \delta\pi^\mu &= \epsilon^{0\mu j}\left(-\frac{\theta}{2} + \frac{1}{2m} - \frac{1}{2m}\nabla^2\right)\partial_j\delta\epsilon_3, \\ \delta v_\mu &= -\partial_\mu\delta\dot{\epsilon}_3, \\ \delta\tilde{\pi}^\mu &= 0. \end{aligned} \tag{3.48}$$

3.4 The Hamilton-Jacobi Framework

For the purpose of analysis, we will write the Lagrangian (2) in a new fashion by introducing the following variables $A_\mu \to \xi_\mu$, $\dot{A}_\mu \to v_\mu$. By doing this, the following constraints $\dot{\xi}_\mu - v_\mu = 0$ will be added to the Lagrangian by means of new nonphysical variables ψ^μ. Thus, the Lagrangian takes the form

$$\begin{aligned}\mathcal{L} = & \frac{1}{2}v^i v_i - v_i\partial^i\xi_0 - \frac{1}{2}\partial^i\xi^0\partial_i\xi_0 - \frac{1}{4}F^{ij}F_{ij} + \frac{\theta}{2}\epsilon^{ij}\xi_0\partial_i\xi_j - \frac{\theta}{2}\epsilon^{ij}\xi_i v_j + \frac{\theta}{2}\epsilon^{ij}\xi_i\partial_j\xi_0 \\ & + \frac{1}{2m}\epsilon^{ij}(-\dot{v}_0 + \nabla^2\xi_0)\partial_i\xi_j - \frac{1}{2m}\epsilon^{ij}(-\dot{v}_i + \nabla^2\xi_i)v_j + \frac{1}{2m}\epsilon^{ij}(-\dot{v}_i + \nabla^2\xi_i)\partial_j\xi_0 \\ & + \psi^0(v_0 - \dot{\xi}_0) + \psi^i(v_i - \dot{\xi}_i).\end{aligned} \tag{3.49}$$

We can observe that the theory is now linear in the temporal derivatives, and we can apply the HJ analysis. From the definition of the momenta

$$P^{(\mu)} = \frac{\partial\mathcal{L}}{\partial\dot{Q}^{(\mu)}}. \tag{3.50}$$

where $Q^{(\mu)} = \{\xi_0, \xi_i, v_0, v_i, \psi_0, \psi_i\}$ are the canonical variables and $P^{(\mu)} = \{\pi^0, \pi^i, \tilde{\pi}^0, \tilde{\pi}^i, p^0, p^i\}$ their corresponding momenta, we find the following Hamiltonians

$$\begin{aligned} &\Omega_1^0 = \pi^0 + \psi^0 = 0, && \Omega_1^i = \pi^i + \psi^i = 0, \\ &\Omega_2^0 = \tilde{\pi}^0 + \tfrac{1}{2m}\epsilon^{ij}\partial_i\xi_j = 0, && \Omega_2^i = \tilde{\pi}^i - \tfrac{1}{2m}\epsilon^{ij}(v_j - \partial_j\xi_0) = 0, \\ &\Omega_3^0 = p^0 = 0, && \Omega_3^i = p^i = 0. \end{aligned} \tag{3.51}$$

As well as the canonical Hamiltonian, obtained as usual from a Legendre transformation

$$\mathcal{H} = \dot{\xi}_\mu\pi^\mu + \dot{v}_\mu\tilde{\pi}^\mu + \dot{\psi}_\mu p^\mu - \mathcal{L} \tag{3.52}$$

This is

$$\mathcal{H} = -\frac{1}{2}v^i v_i + v_i\partial^i\xi_0 + \frac{1}{2}\partial^i\xi^0\partial_i\xi_0 + \frac{1}{4}F^{ij}F_{ij} - \frac{\theta}{2}\epsilon^{ij}\xi_0\partial_i\xi_j + \frac{\theta}{2}\epsilon^{ij}\xi_i v_j - \frac{\theta}{2}\epsilon^{ij}\xi_i\partial_j\xi_0 \\ + \tilde{\pi}^0\nabla^2\xi_0 + \tilde{\pi}^i\nabla^2\xi_i + \pi^0 v_0 + \pi^i v_i. \tag{3.53}$$

Thus, with the Hamiltonians identified, we construct the fundamental differential, which describes the evolution of any function, say F, on the phase space

$$dF = \int \left[\{F,\mathcal{H}\}dt^0 + \{F,\Omega_1^0\}d\omega_0^1 + \{F,\Omega_1^i\}d\omega_i^1 + \{F,\Omega_2^0\}d\omega_0^2 + \{F,\Omega_2^i\}d\omega_i^2 \\ + \{F,\Omega_3^0\}d\omega_0^3 + \{F,\Omega_3^i\}d\omega_i^3\right] d^2y. \tag{3.54}$$

Where ω_0^1, ω_i^1, ω_0^2, ω_i^2, ω_0^3, and ω_i^3 are parameters associated with the Hamiltonians. To this end, we separate the Hamiltonians into involutive and non-involutive. Involutive Hamiltonians are those whose Poisson brackets with all Hamiltonians, including themselves, vanish; otherwise, they are called non-involutive and labeled as Λ, respectively. The Poisson algebra between the Hamiltonians is given by

$$\begin{aligned}
\{\Omega_1^0(x),\Omega_2^i(y)\} &= -\frac{1}{2m}\epsilon^{ij}\partial_j\delta^2(x-y),\\
\{\Omega_1^0(x),\Omega_3^0(y)\} &= -\delta^2(x-y),\\
\{\Omega_1^i(x),\Omega_2^0(y)\} &= \frac{1}{2m}\epsilon^{ij}\partial_j\delta^2(x-y),\\
\{\Omega_1^i(x),\Omega_3^j(y)\} &= \eta^{ij}\delta^2(x-y),\\
\{\Omega_2^i(x),\Omega_2^j(y)\} &= -\frac{1}{m}\epsilon^{ij}\delta^2(x-y).
\end{aligned} \tag{3.55}$$

Hence, we observe that all the Hamiltonians are non-involutive, particularly those related to the nonphysical fields ψ_μ and their momenta p^μ. The matrix composed of these Poisson brackets, namely

$$\Delta_{ab} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2m}\epsilon^{jk}\partial_k & -1 & 0 \\ 0 & 0 & \frac{1}{2m}\epsilon^{ij}\partial_j & 0 & 0 & \eta^{ij} \\ 0 & -\frac{1}{2m}\epsilon^{jk}\partial_k & 0 & 0 & 0 & 0 \\ \frac{1}{2m}\epsilon^{ij}\partial_j & 0 & 0 & -\frac{1}{m}\epsilon^{ij} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\eta^{ij} & 0 & 0 & 0 & 0 \end{pmatrix} \delta^2(x-y) \tag{3.56}$$

cannot be inverted. This means that the Hamiltonians are not independent. Therefore, the null vectors ζ of the matrix Δ_{ab} are used to identify the independent ones, as is done in a pure Dirac framework.

$$\int \Delta^{\mu\nu}\zeta_\mu d^2y = 0. \tag{3.57}$$

By doing this, a null vector can be found

$$\zeta_\mu = \left(0, 0, w, 0, 0, -\frac{1}{2m}\epsilon_{lj}\partial^j w\right), \tag{3.58}$$

where w is an arbitrary function. The contraction of ζ_μ with a vector composed of the non-involutive Hamiltonians $\Omega^{(\mu)} = \{\Omega_1^0, \Omega_1^i, \Omega_2^0, \Omega_2^i, \Omega_3^0, \Omega_3^i\}$ (i.e. $\zeta_\mu\Omega^{(\mu)} = 0$) yields a new Hamiltonian, given by

$$\Omega_1 : \tilde{\pi}^0 + \frac{1}{2m}\epsilon^{ij}\partial_i\xi_j - \frac{1}{2m}\epsilon^{ij}\partial_j p_i = 0. \tag{3.59}$$

Its Poisson brackets with all other Hamiltonians can be seen to vanish. Therefore, this new Hamiltonian is an involutive one. In this manner, the complete set of non-involutive Hamiltonians is given by

$$\begin{aligned}
&\Lambda_1 = \pi^0 + \psi^0 = 0,\\
&\Lambda_2^i = \pi^i + \psi^i = 0,\\
&\Lambda_3^i = \tilde{\pi}^i - \frac{1}{2m}\epsilon^{ij}(v_j - \partial_j \xi_0) = 0,\\
&\Lambda_4 = p^0 = 0,\\
&\Lambda_5^i = p^i = 0.
\end{aligned} \tag{3.60}$$

The new Δ_{ab} matrix takes the form

$$\Delta_{ab} = \begin{pmatrix} 0 & 0 & -\frac{1}{2m}\epsilon^{jk}\partial_k & -1 & 0 \\ 0 & 0 & 0 & 0 & \eta^{ij} \\ \frac{1}{2m}\epsilon^{ij}\partial_j & 0 & -\frac{1}{m}\epsilon^{ij} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -\eta^{ij} & 0 & 0 & 0 \end{pmatrix} \delta^2(x-y). \tag{3.61}$$

This matrix is, naturally, invertible. Its inverse is found to be

$$\Delta_{ab}^{-1}(x,y) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\eta_{jl} \\ 0 & 0 & m\epsilon_{jl} & \frac{1}{2}\partial_j & 0 \\ -1 & 0 & -\frac{1}{2}\partial_l & 0 & 0 \\ 0 & \eta_{jl} & 0 & 0 & 0 \end{pmatrix} \delta^2(x-y). \tag{3.62}$$

With this at hand, we introduce the generalized brackets

$$\{A(x), B(x')\}^* = \{A(x), B(x')\} - \iint \{A(x), \Lambda^a(y)\}\Delta_{ab}^{-1}(y,z)\{\Lambda^b(z), B(x')\} d^2y d^2z. \tag{3.63}$$

Where Λ^a is a vector containing the non-involutive Hamiltonians. By using the generalized brackets, we can calculate the algebra of the phase space variables

$$\begin{aligned}
&\{\xi_\mu, \pi^\nu\}^* = \delta^\nu_\mu \delta^2(x-y), && \{\xi_\mu, \psi_\nu\}^* = -\eta_{\mu\nu}\delta^2(x-y),\\
&\{\pi^0, v_k\}^* = \tfrac{1}{2}\partial_k \delta^2(x-y), && \{\pi^0, \tilde{\pi}^k\}^* = -\tfrac{1}{4m}\epsilon^{kl}\partial_l \delta^2(x-y),\\
&\{v_0, \tilde{\pi}^0\}^* = \delta^2(x-y), && \{v_i, v_k\}^* = m\epsilon_{ik}\delta^2(x-y),\\
&\{v_i, \tilde{\pi}^k\}^* = \tfrac{1}{2}\delta^k_i \delta^2(x-y), && \{v_i, \psi_0\}^* = \tfrac{1}{2}\partial_i \delta^2(x-y),\\
&\{\tilde{\pi}^i, \tilde{\pi}^k\}^* = \tfrac{1}{4m}\epsilon^{ik}\delta^2(x-y), && \{\tilde{\pi}^i, \psi_0\}^* = -\tfrac{1}{4m}\epsilon^{ij}\partial_j \delta^2(x-y).
\end{aligned} \tag{3.64}$$

These generalized brackets will coincide with those of Dirac, which are calculated in the next section. In particular, we can observe that the generalized HJ bracket between the Hamiltonian Ω_1 with itself is trivial

$$\{\Omega_1, \Omega_1\}^* = 0 \tag{3.65}$$

Which tells us that Ω_1 is still involutive, even under the new bracket.

In this manner, the introduction of the HJ brackets removes the non-involutive Hamiltonians completely, by making them non-dynamical. This leaves us with a new fundamental differential, given by

$$dF = \int \left[\{F, \mathcal{H}(y)\}^* dt^0 + \{F, \Omega_1(y)\}^* d\sigma^1\right] d^2y. \tag{3.66}$$

By using the fundamental differential, we have removed the nonphysical degrees of freedom (DOFs) ψ_0, ψ_i, making the results in this section match those of the next section. In this regard, once the generalized brackets are introduced, we could perform the substitution of the ψ's with the π's in the action; the result would be that the HJ and GLT actions are equivalent. In other approaches, see [148], the nonphysical DOF are removed until the end of the calculations, because the separation of the constraints into first and second class allows the introduction of the Dirac brackets. In contrast, in the HJ framework, the elimination of nonphysical DOFs is more convenient. We also have to take into account the Frobenius integrability conditions, which ensure that the system is integrable. Applying these conditions to the Hamiltonian Ω_1, the following Hamiltonian arises

$$d\Omega_1(x) = \int \left[\{\Omega_1(x), \mathcal{H}(y)\}^* dt^0 + \{\Omega_1(x), \Omega_1(y)\}^* d\sigma^1\right] d^2y = \pi^0 - \partial_i \tilde{\pi}^i. \tag{3.67}$$

This entails the introduction of a new Hamiltonian, $\Omega_2 = \pi^0 - \partial_i \tilde{\pi}^i$. A quick calculation shows that

$$\{\Omega_2(x), \Omega_2(y)\}^* = \{\Omega_2(x), \Omega_1(y)\}^* = 0.$$

Therefore, Ω_2 is an involutive Hamiltonian, which has to be included in the fundamental differential. This new fundamental differential must then be used to check the involution of all Hamiltonians. The evolution of $Omega_1$ is not altered, because its bracket with $Omega_2$ is zero. But from the evolution of $Omega_2$ we see that a new Hamiltonian has been obtained.

$$d\Omega_2(x) = \int \left[\{\Omega_2(x), \mathcal{H}(y)\}^* dt^0 + \{\Omega_2(x), \Omega_1(y)\}^* d\sigma^1 + \{\Omega_2(x), \Omega_2(y)\}^* d\sigma^2\right] d^2y = \Omega_3 = 0.$$

With

$$\Omega_3 = \partial_i \pi^i + \frac{\theta}{2}\epsilon^{ij}\partial_i \xi_j + \frac{1}{2m}\epsilon^{ij}\nabla^2 \partial_i \xi_j. \tag{3.68}$$

No further Hamiltonians emerge from the inclusion of Ω_3 into the fundamental differential. As a result, the complete set of involutive Hamiltonians is given by

$$\begin{aligned} \Omega_1 &= \tilde{\pi}^0 + \frac{1}{2m}\epsilon^{ij}\partial_i \xi_j = 0, \\ \Omega_2 &= \pi^0 - \partial_i \tilde{\pi}^i = 0, \\ \Omega_3 &= \partial_i \pi^i + \frac{\theta}{2}\epsilon^{ij}\partial_i \xi_j + \frac{1}{2m}\epsilon^{ij}\nabla^2 \partial_i \xi_j = 0. \end{aligned} \tag{3.69}$$

With the fundamental differential being

$$dF = \int \left[\{F, \mathcal{H}(y)\}^* dt^0 + \{F, \Omega_1(y)\}^* d\sigma^1 + \{F, \Omega_2(y)\}^* d\sigma^2 + \{F, \Omega_3(y)\}^* d\sigma^3\right] d^2y. \tag{3.70}$$

Where σ^1, σ^2, σ^3 are parameters associated with the Hamiltonians. Therefore, we have presented an alternative for studying higher-order theories in the context of HJ theory, which is more economical than those previously reported in the literature. With the fundamental differential, we can obtain the characteristic equations and then identify the symmetries.

3.4.1 Gauge transformations

By calculating the characteristic equations from the fundamental differential, the symmetries of the theory can be obtained. Using (3.70), we find them to be

$$
\begin{aligned}
d\xi_0 &= v_0 dt - dt^2, \\
d\xi_i &= v_i dt + \partial_i dt^3, \\
d\pi^0 &= \left[\frac{1}{2}\partial_i v^i - \frac{1}{2}\nabla^2\xi_0 + \frac{3\theta}{4}\epsilon^{ij}\partial_i\xi_j - \nabla^2\tilde{\pi}^0 + \frac{1}{4m}\epsilon^{ij}\nabla^2\partial_i\xi_j + \frac{1}{2}\partial_i\pi^i\right] dt, \\
d\pi^i &= -\left[\partial_j F^{ij} + \frac{\theta}{2}\epsilon^{ij}v_j + \nabla^2\tilde{\pi}^i\right] dt - \frac{1}{2m}\epsilon^{ij}\partial_j d\sigma^1 + \left[\frac{\theta}{2}\epsilon^{ij}\partial_j + \frac{1}{2m}\epsilon^{ij}\nabla^2\partial_j\right] d\sigma^3, \\
dv_0 &= \nabla^2\xi_0 dt + d\sigma^1, \\
dv_i &= \left[\frac{1}{2}\nabla^2\xi_i + \frac{1}{2}\partial_i v_0 - m\epsilon_{ij}v^j + m\epsilon_{ij}\partial^j\xi_0 + \frac{\theta m}{2}\xi_i + m\epsilon_{ij}\pi^j\right] dt - \partial_i d\sigma^2, \\
\tilde{\pi}^0 &= -\pi^0 dt, \\
\tilde{\pi}^i &= \left[\frac{1}{2}v^i - \frac{1}{2}\partial^i\xi_0 + \frac{\theta}{4}\epsilon^{ij}\xi_j - \frac{1}{4m}\epsilon^{ij}\partial_j v_0 + \frac{1}{4m}\epsilon^{ij}\nabla^2\xi_j - \frac{1}{2}\pi^i\right] dt.
\end{aligned}
\tag{3.71}
$$

The evolution of the dynamical variables with respect to our parameters σ^i is understood as canonical transformations, with the corresponding Hamiltonians as generators. Due to Frobenius' theorem [133], the transformation with respect to one of these parameters is independent of the evolution with respect to the others. To relate these canonical transformations to the gauge ones. Setting $dt = 0$ yields

$$
\begin{aligned}
&\delta\xi^0 = -\delta\sigma^2, \\
&\delta\xi^i = \partial_i\delta\sigma^3, \\
&\delta\pi^i = -\frac{1}{2m}\epsilon^{ij}\partial_j\delta\sigma^1 + \left[\frac{\theta}{2}\epsilon^{ij}\partial_j + \frac{1}{2m}\epsilon^{ij}\nabla^2\partial_j\right]\delta\sigma^3, \\
&\delta v_0 = \delta\sigma^1, \\
&\delta v_i = -\partial_i\delta\sigma^2.
\end{aligned}
\tag{3.72}
$$

Where, naturally, $\delta\pi^i = \delta\tilde{\pi}^0 = \delta\pi^0 = 0$. In this method, in order to find the gauge transformations, it is necessary to see the specific conditions in which this equation acts on the Lagrangian. The Lagrangian must be invariant under these transformations $\delta I = 0$. This will result in relations between the parameters σ^1, σ^2, and σ^3 that will lead to the gauge transformations. The variation in the Lagrangian is

$$
\delta L = \int \left[\frac{\partial\mathcal{L}}{\partial A_\mu}\delta A_\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}\delta(\partial_\nu A_\mu) + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial^\mu A_\mu)}\delta(\partial_\nu\partial^\mu A_\mu)\right] dt d^2x. \tag{3.73}
$$

Where we used A_μ instead of ξ_μ for a more straightforward comparison of both formalisms. This, up to a total time derivative, is found to be

$$
\delta L = \int \left[\theta\epsilon^{\alpha\beta\gamma}\partial_\beta A_\gamma - \partial_\beta F^{\alpha\beta} - \frac{1}{2m}\epsilon^{\alpha\beta\gamma}(\partial_0\partial^0\partial_\beta A_\gamma) + \frac{1}{m}\epsilon^{\alpha\beta\gamma}\nabla^2\partial_\beta A_\gamma\right]\delta A_\alpha dt d^2x = 0. \tag{3.74}
$$

We can combine the first and second equations in (3.72) to write the variation as

$$
\delta A_\alpha = -\delta^0_\alpha\delta\sigma^2 + \delta^i_\alpha\partial_i\delta\sigma^3. \tag{3.75}
$$

Then, by combining the previous two equations, the variation of the action takes the form

$$
\delta L = -\int \left(\theta\epsilon^{ij}\partial_i A_j + \partial_i F^{i0} - \frac{1}{2m}\epsilon^{ij}\partial_i\ddot{A}_j + \frac{1}{m}\epsilon^{ij}\nabla^2\partial_i A_j\right)(\delta\sigma^2 + \partial_0\delta\sigma^3) dt d^2x = 0. \tag{3.76}
$$

Therefore, the theory will be invariant under the transformations (3.75) if the parameters obey

$$\delta\sigma^2 = -\partial_0 \delta\sigma^3. \tag{3.77}$$

Hence, from (3.75), the gauge transformations are given by

$$\delta A_\mu = \partial_\mu \delta\sigma^3. \tag{3.78}$$

Additionally, since $v_\mu = \dot{A}_\mu$, it can be seen that

$$\delta\sigma^1 = \partial_0 \partial_0 \delta\sigma^3. \tag{3.79}$$

By identifying $\sigma^3 = -\epsilon_3$ both formalisms agree.

Chapter 4

Weyl Gravity

Beginning with the necessity of a gravitational action endowed with conformal invariance, we derive the Weyl action along with its corresponding equations of motion. Subsequently, we delve into the study of the action utilizing the Hamilton-Jacobi (HJ) method, but not without first taking the perturbative limit of the metric around the Minkowski background. To diminish the order of time derivatives within the Lagrangian, we introduce an extrinsic curvature-like variable. The conjugate momenta are then obtained employing the conventional definition, alongside the canonical Hamiltonian. Correct identification of the Hamiltonians is facilitated through rank-nullity analysis of the Hessian matrix of the system, followed by classification of these Hamiltonians using Frobenius integrability conditions. Non-involutive Hamiltonians are seamlessly integrated into the dynamics through the introduction of generalized brackets, generating a conclusive set of involutive Hamiltonians from which the symmetries of the theory are derived. While a Hamiltonian analysis of Weyl's conformal gravity can be found in [152], employing a reduced Dirac formalism, we present an alternative analysis herein, treating all fields as dynamical. This encompasses both an extrinsic curvature-type variable and a Lagrange multiplier, the latter typically assumed as a momentum from the outset. Additionally, we provide an exhaustive Hamilton-Jacobi analysis of the gauge transformations, a facet typically addressed marginally within the Dirac method. The outcomes of this section are documented in [153], but are shown here in a more detailed manner.

4.1 Theory

The focus of this investigation revolves around Weyl gravity, a higher-order theory characterized by its covariance under diffeomorphism and invariance under conformal transformations of the metric. These transformations, preserving angles and scaling, extend the concept of Poincaré transformations and encompass scale transformations as well [154].

In 1850, physicist Mikhail Ostrogradsky introduced a theorem asserting that a non-degenerate Lagrangian, comprised of finite higher-order time derivatives, yields a Hamiltonian unbounded from below. Specifically, it was demonstrated that the Hamiltonian of such a system incorporates linearity in physical momenta, commonly termed the "Ostrogradsky ghost." This thesis delves into methods of circumventing the Ostrogradsky ghost by examining degenerate Lagrangians, thereby imposing constraints on the momenta.

Ostrogradsky's findings revealed the presence of a "ghost" within these theories, indicating the influence of an unphysical variable on the theory's dynamics. This ghost manifests through negative kinetic energy, linear physical momentum within the Hamiltonian, and unstable degrees of freedom. Consequently, the Hamiltonian lacks a lower bound, implying that a conventional one-derivative system could absorb infinite energy if coupled to an Ostrogradsky system. Essentially, the Ostrogradsky ghost, or instability, renders the theory fundamentally "sick", necessitating the avoidance of such problem for the construction of a viable physical theory. Recent research has demonstrated that degenerate second-order Lagrangians introduce constraints capable of circumventing the Ostrogradsky ghost [155]. Through the introduction of auxiliary variables and constraints, these Lagrangians can be reformulated into equivalent first-order systems. Moreover, in scenarios involving multiple fields, the presence of higher derivatives does not necessarily entail the existence of ghosts [156].

This theory encompasses six degrees of freedom, linked to ordinary massless excitations of spin 2 and 1, alongside a spin 2 ghost [157]. Moreover, the Weyl action gives rise to fourth-order differential equations and lacks any dimensional coupling constant. Furthermore, this theory finds application in conformal supergravity and twistor-string theory [158], and has spurred the development of alternative formulations, such as the Weyl-Type $f(Q, T)$ theories [159].

The Weyl action stands out for its remarkable renormalization properties, it being demonstrated as renormalizable. This property arises from the inclusion of higher-order derivatives in the Lagrangian and the presence of a dimensionless coupling constant akin to the Yang-Mills theory [122]. Consequently, it is often viewed as an ultraviolet completion of general relativity with supplementary attributes. An intriguing correlation exists between Quantum Weyl Gravity (QWG) and the $SU(3)$ Yang-Mills (YM) theory, also known as Quantum Chromodynamics (QCD). Like QCD, the gravitational interaction in QWG displays anti-screening behavior at high energies, prompting speculation regarding an analogy between R^2 gravity and QCD [160], [161]. However, QWG is generally not regarded as a feasible contender for a fundamental theory of quantum gravity due to its perturbative spectrum harboring a massless spin-2 ghost. This concern isn't exclusive to Weyl Gravity but applies to all gravitational theories featuring higher-order derivatives, where unitarity is jeopardized by the existence of states with negative kinetic energy, termed ghosts.

The theory maintains its phenomenology at solar system scales, as its solutions extend the generality of Schwarzschild's metric. However, the question of whether these solutions hold physical feasibility remains open [162]. On galactic scales, the theory offers a precise depiction of rotational velocities observed at the peripheries of galaxies, aligning closely with reported values without necessitating the inclusion of dark matter [163].

Moreover, the theory's captivating attributes extend to the cosmological scale. It aligns with the same cosmological data as the Λ-CDM model, elucidates dark energy, propels cosmic expansion, and prevents the necessity for dark matter [15]. Mannheim's work [15] demonstrated that within a universe described by the Friedmann-Robertson-Walker (FRW) metric and the energy-momentum tensor of a perfect fluid, Weyl's conformal theory excellently fits luminosity measurements from type Ia supernovae, all without requiring dark matter or dark energy. Conversely, Jawad et al. [164], employing the same metric and energy-momentum tensor considerations, explored the equilibrium perspective of thermodynamic laws for the Hubble horizon using the Weyl theory coupled to a scalar field. By employing four distinct parameterizations of the deceleration parameter, they scrutinized the behavior of the equation of state parameter, the validity of the second law of generalized thermodynamics, the thermal equilibrium condition, and the model's stability. Their findings indicate that the equation of state parameter remains sufficiently preserved in the quintessence and vacuum realms, the second law of generalized thermodynamics holds in the majority of analyzed domains, and the thermal condition is at least partially satisfied across all models.

Weyl's insights carry profound implications, offering not only the approach outlined in this dissertation but also an alternative avenue with comparably favorable outcomes. Some scholars integrate Weyl's symmetry directly into the geometry itself, leading to what's known as Weyl geometry. Here, the fundamental variables include the metric $g_{\mu\nu}$ and a Weyl vector w_μ, the latter responsible for modulating the length of vectors in parallel transport. This framework extends to generalized connections and consequently the Riemann tensor, introducing an electromagnetic-like tensor for the Weyl vector and non-metricity. Among these theories, the $f(Q, T)$ Weyl-type theory stands out, with Q representing non-metricity and T indicating the trace of the momentum-energy tensor. Studies have indicated the cosmological viability of these theories. Xu et al. [165] analyzed a Friedmann-Robertson-Walker (FRW) metric incorporating an energy-momentum tensor for a perfect fluid, demonstrating that both the Hubble and deceleration parameters mirror predictions from the Lambda-CDM model. Conversely, Gabdail et al. [166] investigated a model involving a bulk viscous fluid in a non-relativistic scenario. Their findings span the evolution of various cosmological parameters, including the Hubble parameter, density parameter ρ, Statefinder diagnostics, and Om diagnostics. They conclude that the model effectively describes the universe's acceleration in light of current observations, albeit noting that the constant deceleration parameter they derived precludes a transition from a decelerating to an accelerating phase.

In this section, we adopt a less radical interpretation of Weyl's concepts, retaining conformal invariance while omitting the unification with electromagnetism. As a result, we adhere to the standard Riemannian geometry, utilizing only the metric as the fundamental variable while imbuing gravity with additional conformal symmetry. This naturally steers us toward the domain of higher-order theories, which diverge even further from conventional frameworks. Nevertheless, it remains crucial to demonstrate that regardless of the specific formulation of Weyl's conformal ideas, they exert significant influence at the cosmological level.

The theory was conceived in 1919, when Hermann Weyl theorized [167] that Einstein's theory of general relativity should be invariant with respect to transformations of the metric tensor of the form

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(x) = e^{\epsilon\pi(x)} g_{\mu\nu}(x). \tag{4.1}$$

Which are nowadays called conformal transformations. Using this gauge principle, Weyl attempted to unify gravity and electromagnetism with a generalized Einstein-Maxwell Lagrangian. Sadly, his theory was unsuccessful, but it gave birth to the discovery of quantum-mechanical phase invariance $\Psi(x) \to e^{i\pi(x)}\Psi(x)$ [168], and to the concept of gauge symmetry, one of the most profound tenets of modern physics. Even to this day, his ideas are used in cosmological models, still in an attempt to unify gravity with the quantum theory.

During this period, Weyl developed a keen interest in the cosmological facets of general relativity, specifically focusing on the curvature characteristics of the universe. Still captivated by metric gauge transformations, Weyl sought a tensor analogous to the Riemann tensor, one that remained invariant under the aforementioned transformation. Eventually, he identified such a tensor, now recognized as the Weyl conformal tensor, only to realize much later its fundamental significance in modern comprehension of the two gravitational effects of matter: compression and tidal deformation. It's noteworthy that the Weyl tensor's derivation can be traced back to a simple requirement of invariance under (4.1), as will be shown subsequently.

Conformal transformations, such as the one in (4.1), alter the space-time interval, but do not transform the angles between vectors. Remembering the dot product between two vectors $\vec{u} \cdot \vec{v} = g(\vec{u}, \vec{v}) = g_{\mu\nu} u^\mu v^\nu$ as well as the angle between them

$$\cos\theta_{uv} = \frac{\vec{u} \cdot \vec{v}}{|u||v|} = \frac{g(\vec{u}, \vec{v})}{\sqrt{g(\vec{u}, \vec{v})}\sqrt{g(\vec{u}, \vec{v})}} \tag{4.2}$$

Applying a conformal transformation leaves this expression invariant

$$\cos\theta \to \widetilde{\cos}\theta = \frac{e^{\epsilon\pi(x)}g(\vec{u},\vec{v})}{\sqrt{e^{\epsilon\pi(x)}g(\vec{u},\vec{v})}\sqrt{e^{\epsilon\pi(x)}g(\vec{u},\vec{v})}} = \cos\theta. \tag{4.3}$$

This is not the case for the line element, which is stretched by a factor of $e^{\epsilon\pi}$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \to d\tilde{s}^2 = e^{\epsilon\pi}g_{\mu\nu}dx^\mu dx^\nu = e^{\epsilon\pi}ds^2 \tag{4.4}$$

With the exception of the light cone, where $ds^2 = 0$.

4.2 The Weyl Tensor

We now will show how to obtain the Weyl tensor in a n-dimensional manifold, working with the infinitesimal transformation up to first order. Equation (4.1) becomes

$$\tilde{g}_{\mu\nu}(x) = e^{\epsilon\pi(x)}g_{\mu\nu}(x) \simeq (1+\epsilon\pi)g_{\mu\nu}. \tag{4.5}$$

Given that the identity $g_{\mu\nu}g^{\mu\lambda} = \delta_\nu^\lambda$ must hold, the inverse metric transforms up to first order as

$$\tilde{g}^{\mu\nu}(x) \simeq (1-\epsilon\pi)g^{\mu\nu} \tag{4.6}$$

Therefore, the variation $\delta g_{\mu\nu} = \tilde{g}_{\mu\nu} - g_{\mu\nu}$ of the metric and its inverse is

$$\begin{aligned} \delta g_{\mu\nu}(x) &= \epsilon\pi g_{\mu\nu}, \\ \delta g^{\mu\nu}(x) &= -\epsilon\pi g^{\mu\nu}. \end{aligned} \tag{4.7}$$

Now we'll see how this transformation affects the Connection $\Gamma^\alpha_{\beta\mu}$, the Riemann tensor $R^\alpha{}_{\mu\beta\nu}$, Ricci's tensor $R_{\mu\nu}$, and the scalar curvature R. We begin with the connection

$$\Gamma^\alpha_{\beta\mu} = \frac{1}{2}g^{\alpha\lambda}(g_{\lambda\beta,\mu} + g_{\lambda\mu,\beta} - g_{\beta\mu,\lambda}). \tag{4.8}$$

The connection transforms as

$$\widetilde{\Gamma}^{\alpha}_{\beta\mu} = \Gamma^{\alpha}_{\beta\mu} + \frac{1}{2} g^{\alpha\sigma}[\epsilon(\pi g_{\sigma\beta})_{,\mu} + \epsilon(\pi g_{\sigma\mu})_{,\beta} - \epsilon(\pi g_{\beta\mu})_{,\sigma}] - \frac{\epsilon}{2}\pi g^{\alpha\sigma}[(g_{\sigma\beta})_{,\mu} + \partial(g_{\sigma\mu})_{,\beta} - (g_{\beta\mu})_{,\sigma}]. \quad (4.9)$$

Next is the Riemann tensor

$$R^{\rho}{}_{\mu\sigma\nu} = \Gamma^{\rho}_{\mu\nu,\sigma} - \Gamma^{\rho}_{\mu\sigma,\nu} + \Gamma^{\kappa}_{\mu\nu}\Gamma^{\rho}_{\kappa\sigma} - \Gamma^{\kappa}_{\sigma\mu}\Gamma^{\rho}_{\nu\kappa}. \quad (4.10)$$

Which transforms as

$$\widetilde{R}^{\rho}{}_{\mu\sigma\nu} = R^{\rho}{}_{\mu\sigma\nu} + \frac{\epsilon}{2}[\delta^{\rho}_{\nu}(\pi_{,\mu})_{;\sigma} - \delta^{\rho}_{\sigma}(\pi_{,\mu})_{;\nu} - g_{\mu\nu}g^{\rho\lambda}(\pi_{,\lambda})_{;\sigma} + g_{\mu\sigma}g^{\rho\lambda}(\pi_{,\lambda})_{;\nu}]. \quad (4.11)$$

By using the following tensor

$$B^{\alpha}{}_{\gamma} = \frac{\epsilon}{2g^{\alpha\beta}(\pi_{,\beta})_{;\gamma}} \quad (4.12)$$

We can express it in the following, more convenient, way

$$\widetilde{R}^{\rho}{}_{\mu\sigma\nu} = R^{\rho}_{\mu\sigma\nu} + \delta^{\rho}_{\nu}g_{\mu\lambda}B^{\lambda}{}_{\sigma} - \delta^{\rho}_{\sigma}g_{\mu\lambda}B^{\lambda}{}_{\nu} - g_{\mu\nu}B^{\rho}{}_{\sigma} + g_{\mu\sigma}B^{\rho}{}_{\nu} \quad (4.13)$$

From here, the Ricci tensor and the scalar curvature are obtained directly by contracting the indices

$$\widetilde{R}_{\mu\nu} = R_{\mu\nu} - (\mathrm{n}-2)B_{\mu\nu} - g_{\mu\nu}B, \quad (4.14)$$

$$\widetilde{R} = (1-\epsilon\pi)[R - 2(\mathrm{n}-1)B]. \quad (4.15)$$

Where $B = B^{\alpha}{}_{\alpha}$. At this point we can obtain a tensor invariant under conformal transformations by taking the product of (4.15) with $\tilde{g}_{\mu\nu}$ and working our way up to the Riemann tensor. By taking the above product we can obtain an expression for B in terms of R and $\widetilde{R}$

$$g_{\mu\nu}B = \frac{g_{\mu\nu}R - \tilde{g}_{\mu\nu}\widetilde{R}}{2(\mathrm{n}-1)} \quad (4.16)$$

Which lets us, upon substitution in (4.14), yields an expression for $B_{\mu\nu}$ in terms of Ricci's tensor, the scalar curvature and their respective transformed selves

$$B_{\mu\nu} = \frac{R_{\mu\nu} - \widetilde{R}_{\mu\nu}}{(\mathrm{n}-2)} + \frac{\tilde{g}_{\mu\nu}\widetilde{R} - g_{\mu\nu}R}{2(\mathrm{n}-2)(\mathrm{n}-1)}. \tag{4.17}$$

By lowering all indices on the Riemann tensor and inserting (4.17) into (4.13) we obtain

$$(1-\epsilon\pi)\widetilde{C}_{\kappa\mu\sigma\nu} = C_{\kappa\mu\sigma\nu}. \tag{4.18}$$

Where $C_{\kappa\mu\sigma\nu}$ is the totally covariant Weyl tensor

$$C_{\kappa\mu\sigma\nu} = R_{\kappa\mu\sigma\nu} + \frac{1}{(\mathrm{n}-2)}(g_{\nu\kappa}R_{\mu\sigma} - g_{\sigma\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\kappa\sigma} + g_{\mu\sigma}R_{\kappa\nu}) + \frac{R(-g_{\nu\kappa}g_{\mu\sigma} + g_{\sigma\kappa}g_{\mu\nu})}{(\mathrm{n}-2)(\mathrm{n}-1)}. \tag{4.19}$$

In $3+1$ dimensions the Weyl tensor reads

$$C_{\kappa\mu\sigma\nu} = R_{\kappa\mu\sigma\nu} + \frac{1}{2}(g_{\nu\kappa}R_{\mu\sigma} - g_{\sigma\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\kappa\sigma} + g_{\mu\sigma}R_{\kappa\nu}) + \frac{R}{6}(-g_{\nu\kappa}g_{\mu\sigma} + g_{\sigma\kappa}g_{\mu\nu}). \tag{4.20}$$

The Weyl tensor characterizes the distortion of the body's shape due to tidal forces along its geodesic trajectory, rather than indicating changes in volume. In contrast, the Ricci curvature, derived from the trace component of the Riemann tensor, is what describes volume changes. Consequently, the Weyl tensor constitutes the non-trace portion of the Riemann tensor and shares its symmetries

$$\begin{aligned} C_{\alpha\beta\mu\nu} &= -C_{\beta\alpha\mu\nu} = -C_{\alpha\beta\nu\mu}, \\ C_{\alpha\beta\mu\nu} &+ C_{\alpha\mu\nu\beta} + C_{\alpha\nu\beta\mu} = 0. \end{aligned} \tag{4.21}$$

But in addition to this, it is trace-free, meaning that the contraction of any pair of indices is identically zero, even in curved spaces where $R_{\alpha\beta} \neq 0$. For a manifold of dimension n

$$
\begin{aligned}
C_{\mu\nu} &= R_{\mu\nu} + \frac{1}{(\mathrm{n}-2)}(R_{\mu\nu} - \mathrm{n}R_{\mu\nu} - g_{\mu\nu}R + R_{\mu\nu}) + \frac{R}{(\mathrm{n}-2)(\mathrm{n}-1)}(\mathrm{n}g_{\mu\nu} - g_{\mu\nu}) \\
&= R_{\mu\nu} + \frac{1}{(\mathrm{n}-2)}[(2-\mathrm{n})R_{\mu\nu} - g_{\mu\nu}R] + \frac{g_{\mu\nu}R}{(\mathrm{n}-2)} \\
&= 0.
\end{aligned}
$$

It should be obvious from (4.20) that the Weyl tensor is identically zero in two dimensions, a simple calculation can also show that it is also zero in $D = 3$.

An object A is said to be of conformal weight k if $A \to e^{k(\epsilon\pi)}A$. As can be seen by (4.1), the metric tensor has weight $k = 1$, since $g^{\alpha\mu}g_{\mu\beta} = \delta^{\alpha}_{\beta}$ it follows that

$$
\tilde{g}^{\mu\nu} = e^{-\epsilon\pi}g^{\mu\nu} \tag{4.22}
$$

And thus the metric inverse has weight of $k = -1$. Additionally, given that the metric determinant is

$$
g = \varepsilon^{\alpha\beta\ldots\sigma}g_{0\alpha}g_{1\beta}\ldots g_{\mathrm{n}\sigma}. \tag{4.23}
$$

The metric determinant has weight $k = \mathrm{n}$, where n is the dimension

$$
\tilde{g} = e^{\mathrm{n}(\epsilon\pi)}g. \tag{4.24}
$$

By the same token, (4.18) shows that the totally covariant Weyl tensor has a weight of $k = 1$. A simple raising of an index via a metric tensor $C^{\alpha}{}_{\mu\beta\nu} = g^{\alpha\lambda}C_{\lambda\mu\beta\nu}$ shows that the Weyl tensor

$$
C^{\alpha}{}_{\mu\beta\nu} = R^{\alpha}{}_{\mu\beta\nu} + \frac{1}{2}(\delta^{\alpha}_{\nu}R_{\mu\beta} - \delta^{\alpha}_{\beta}R_{\mu\nu} - g_{\mu\nu}R^{\alpha}{}_{\beta} + g_{\mu\beta}R^{\alpha}{}_{\nu}) + \frac{R}{6}(-\delta^{\alpha}_{\nu}g_{\mu\beta} + \delta^{\alpha}_{\beta}g_{\mu\nu}), \tag{4.25}
$$

has a weight $k = 0$, therefore, it is conformally invariant

$$
\tilde{C}^{\alpha}{}_{\mu\beta\nu} = C^{\alpha}{}_{\mu\beta\nu}. \tag{4.26}
$$

Some authors express conformal transformations as $A \to \Omega^2 A$. In such case, the conformal weights found here can be compared by multiplying by a factor of two.

4.3 The Weyl Action and its Equation of Motion

In the pursuit of achieving a conformally invariant gravitational action, a logical approach involves investigating contractions and squares of the Weyl tensor. Given that the Weyl tensor is traceless, the simplest conformal action that can be constructed from the metric and the Weyl tensor is that of the square of the Weyl tensor

$$I = \int \sqrt{-g} C_{\alpha\beta\mu\nu} C^{\alpha\beta\mu\nu} d^4x. \tag{4.27}$$

This action was originally proposed by Rudolf Bach in 1920. In 1977, Kaku, Nieuwenhuizen, and Townsend showed that Bach's theory of gravity may be regarded as the gauge theory of the conformal group [169]. We can see that this action is a conformal invariant by expressing it in the following way

$$\sqrt{-g} C^{\alpha}{}_{\beta\mu\nu} C^{\sigma}{}_{\rho\lambda\gamma} g_{\alpha\sigma} g^{\beta\rho} g^{\mu\lambda} g^{\nu\gamma}.$$

By adding the conformal weights of each term $k = \frac{n}{2} + 0 + 0 + 1 - 3 = \frac{n}{2} - 2$, we can see that the Weyl action is conformally invariant exclusively in four dimensions.

Expressed in terms of the Riemann tensor, Ricci's tensor and the scalar curvature, this action becomes

$$I = \int \sqrt{-g} \left(R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2 \right) d^4x. \tag{4.28}$$

Before continuing with the equations of motion, it is perhaps worth recalling the work of Cornelius Lanczos [170]; which, following Weyl's lead, managed to describe the Riemannian quadratic term $R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}$ in terms of $R_{\mu\nu} R^{\mu\nu}$ and R^2 without affecting the equations of motion. Thus decreasing the abundance of conformal invariants that can be added to the action, simplifying the mathematical difficulty of the theory and improving its logical attractiveness.

Avoiding, for example, once the Lanczos trick is applied, potential conflicts like those present on Riemann-curved but Ricci-flat spaces, such as with the Schwarzschild metric.

Lanczos' trick is performed by subtracting I_g from our original action, with

$$I_g = \int \sqrt{-g}(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + AR_{\mu\nu}R^{\mu\nu} + BR^2)d^4x.$$

With the judicious choice of $A = -3$ and $B = 2/3$ it can be shown that the contribution of this action to the equations of motion is proportional to a Bianchi identity. This leaves the EOM unchanged, but removes the troublesome $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ from the Lagrangian [1]. Therefore, the simplified Weyl action becomes

$$I = \int \sqrt{-g}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) d^4x. \tag{4.29}$$

Its equations of motion can be obtained via the variation of the action, which can be initially split into a variation with respect to the metric tensor and a variation of Ricci's tensor

$$\delta I = \int \sqrt{-g}\left[\frac{1}{2}g_{\mu\nu}\left(\frac{1}{3}R^2 - R_{\alpha\beta}R^{\alpha\beta}\right) + 2R_{\mu\lambda}R_{\nu}{}^{\lambda} - \frac{2}{3}RR_{\mu\nu}\right]\delta(g^{\mu\nu}) + \left[2R^{\mu\nu} - \frac{2}{3}Rg^{\mu\nu}\right]\delta(R_{\mu\nu})d^4x.$$

Then, by means of Palatini's contracted identity $\delta R_{\mu\nu} = (\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda} - (\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu}$, this can be turned into

$$\delta I = \int \sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{6}g^{\mu\nu}R^2 - 2R^{\mu}{}_{\alpha}R^{\nu\alpha} + \frac{2}{3}RR^{\mu\nu} + 2R^{\mu\alpha;\nu}_{;\alpha} - R^{\mu\nu;\alpha}_{;\alpha} - \frac{2}{3}R^{;\mu;\nu}\right.$$
$$\left.+\frac{2}{3}g^{\mu\nu}R_{;\alpha}{}^{;\alpha} - g^{\mu\nu}R^{\alpha\beta}{}_{;\alpha;\beta}\right]\delta g_{\mu\nu}d^4x.$$

Finally, using the contracted Bianchi identity $R^{\alpha\beta}{}_{;\beta} = \frac{1}{2}R^{;\alpha}$ the last term can be transformed into $\frac{1}{2}g^{\mu\nu}R_{;\alpha}{}^{;\alpha}$, yielding

$$\frac{1}{2}g^{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{6}g^{\mu\nu}R^2 - 2R^{\mu}{}_{\alpha}R^{\alpha\nu} + \frac{2}{3}RR^{\mu\nu} + 2R^{\mu\alpha}{}_{;\alpha}{}^{\nu} - R^{\mu\nu}{}_{;\alpha}{}^{;\alpha} - \frac{2}{3}R^{;\mu;\nu} + \frac{1}{6}g^{\mu\nu}R_{;\alpha}{}^{;\alpha} = 0. \tag{4.30}$$

[1]For a more detailed calculation see [171]

This equation can be further manipulated to yield

$$\frac{1}{2}g^{\mu\nu}\left(R_{\alpha\beta}R^{\alpha\beta}-\frac{1}{3}R^2\right)-\Box\left(R^{\mu\nu}-\frac{1}{6}g^{\mu\nu}R\right)-2R^{\mu}{}_{\alpha}R^{\nu\alpha}+\frac{2}{3}RR^{\mu\nu}+2R^{\mu\alpha}{}_{;\alpha}{}^{;\nu}-\frac{2}{3}R^{;\mu;\nu}=0. \tag{4.31}$$

A version more commonly found in the literature.

4.4 Linearization

In this section we will perform a perturbative analysis of the Weyl action, starting from the simplified Weyl action (4.29)

$$I=\int\sqrt{-g}\left(R_{\mu\nu}R^{\mu\nu}-\frac{1}{3}R^2\right)d^4x.$$

In particular, we will perform a perturbative expansion of the metric tensor up to first order

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}+\mathcal{O}^2(h). \tag{4.32}$$

Where $\eta_{\mu\nu}$ is a flat background metric, i.e. the Minkowski metric, and $h_{\mu\nu}$ is a perturbative field. The inverse of the metric tensor, up to first order can be obtained via the following property

$$g_{\mu\alpha}g^{\alpha\nu}=\delta^{\nu}_{\mu},$$

and reads as

$$g^{\mu\nu}=\eta^{\mu\nu}-h^{\mu\nu}.$$

Additionally, as a quick calculation can show, the determinant of the metric expands as

$$\sqrt{-g}\simeq\sqrt{-\eta}\left(1+\frac{1}{2}h\right).$$

In the following, we will summarize the first order expressions of the relevant tensors. The first-order expression of the Christoffel symbols is shown for completeness

$$\Gamma^{\alpha}_{\mu\nu} \simeq \frac{1}{2}\eta^{\alpha\beta}(h_{\nu\beta,\mu} + h_{\beta\mu,\nu} - h_{\mu\nu,\beta}). \tag{4.33}$$

The first-order expressions for the Riemann tensor and its contractions are shown below

$$\begin{aligned} R^{\alpha}{}_{\mu\beta\nu} &\simeq \frac{1}{2}\eta^{\alpha\lambda}(h_{\nu\lambda,\mu,\beta} - h_{\mu\nu,\lambda,\beta} - h_{\beta\lambda,\mu,\nu} + h_{\mu\beta,\lambda,\nu}), \\ R_{\mu\nu} &\simeq \frac{1}{2}\eta^{\alpha\beta}(h_{\mu\alpha,\nu,\beta} - h_{\mu\nu,\alpha,\beta} - h_{\alpha\beta,\mu,\nu} + h_{\nu\beta,\mu,\alpha}), \\ R &\simeq h_{\alpha\beta}{}^{,\alpha,\beta} - h_{,\alpha}{}^{,\alpha}. \end{aligned} \tag{4.34}$$

We now focus on the terms appearing in the simplified Weyl action. After a simple, but somewhat lengthy, calculation, the Ricci squared term can be written as

$$\begin{aligned} R_{\mu\nu}R^{\mu\nu} = \frac{1}{4}(&2h_{\mu}{}^{\alpha}{}_{,\alpha,\nu}h^{\mu}{}_{\beta}{}^{,\beta,\nu} - 4h_{\mu}{}^{\alpha}{}_{,\alpha,\nu}h^{\mu\nu}{}_{,\beta}{}^{,\beta} - 4h_{\mu}{}^{\alpha}{}_{,\alpha,\nu}h^{,\mu,\nu} + 2h_{\mu}{}^{\alpha}{}_{,\alpha,\nu}h^{\nu\beta}{}_{,\beta}{}^{,\mu} \\ &+ h_{\mu\nu}{}^{,\alpha}{}_{,\alpha}h^{\mu\nu}{}_{,\beta}{}^{,\beta} + 2h_{\mu\nu}{}^{,\alpha}{}_{,\alpha}h^{,\mu,\nu} + h_{,\mu,\nu}h^{,\mu,\nu}). \end{aligned} \tag{4.35}$$

Thereafter, the scalar quadratic curvature term can be calculated in a straightforward way

$$R^2 = h_{\alpha\beta}{}^{,\alpha,\beta}h_{\mu\nu}{}^{,\mu,\nu} - 2h_{,\alpha}{}^{,\alpha}h_{\mu\nu}{}^{,\mu,\nu} + h_{,\alpha}{}^{,\alpha}h_{,\beta}{}^{,\beta}. \tag{4.36}$$

After an initial substitution, Weyl's action reads as follows

$$\begin{aligned} I = \int \sqrt{-\eta}\Big[&\frac{1}{2}h_{\mu}{}^{\alpha}{}_{,\alpha,\nu}h^{\mu}{}_{\beta}{}^{,\beta,\nu} - h_{\mu}{}^{\alpha}{}_{,\alpha,\nu}h^{\mu\nu}{}_{,\beta}{}^{,\beta} - h_{\mu}{}^{\alpha}{}_{,\alpha,\nu}h^{,\mu,\nu} + \frac{1}{2}h_{\mu}{}^{\alpha}{}_{,\alpha,\nu}h^{\nu\beta}{}_{,\beta}{}^{,\mu} \\ &+ \frac{1}{4}h_{\mu\nu}{}^{,\alpha}{}_{,\alpha}h^{\mu\nu}{}_{,\beta}{}^{,\beta} + \frac{1}{2}h_{\mu\nu}{}^{,\alpha}{}_{,\alpha}h^{,\mu,\nu} + \frac{1}{4}h_{,\mu,\nu}h^{,\mu,\nu} - \frac{1}{3}h_{\alpha\beta}{}^{,\alpha,\beta}h_{\mu\nu}{}^{,\mu,\nu} \\ &+ \frac{2}{3}h_{,\alpha}{}^{,\alpha}h_{\mu\nu}{}^{,\mu,\nu} - \frac{1}{3}h_{,\alpha}{}^{,\alpha}h_{,\beta}{}^{,\beta}\Big]d^4x. \end{aligned}$$

By performing a number of integration by parts this becomes

$$I = \int \sqrt{-\eta}\left(-\frac{1}{2}\partial_\alpha\partial_\mu h_\nu{}^\alpha\Box h^{\mu\nu} + \frac{1}{2}\partial_\alpha\partial_\mu h_\nu{}^\alpha\partial_\beta\partial^\nu h^{\mu\beta} + \frac{1}{4}\Box h_{\mu\nu}\Box h^{\mu\nu} \right.$$
$$\left. -\frac{1}{12}\partial_\nu\partial_\mu h\partial^\mu\partial^\nu h - \frac{1}{3}\partial^\alpha\partial^\beta h_{\alpha\beta}\partial^\mu\partial^\nu h_{\mu\nu} + \frac{1}{6}\partial^\mu\partial^\nu h_{\mu\nu}\Box h\right) d^4x. \tag{4.37}$$

The action above is invariant under both linearized diffeomorphisms and the linearized version of the Weyl rescalings. Its EOM can be obtained either through variation of the action or via the Euler-Lagrange equations, or even by linearizing (4.30)

$$-\frac{1}{2}\partial^\alpha\partial_\mu\Box h^{\mu\beta} - \frac{1}{2}\partial^\beta\partial_\mu\Box h^{\alpha\mu} + \frac{1}{3}\partial^\alpha\partial^\beta\partial_\mu\partial_\nu h^{\mu\nu} + \frac{1}{2}\Box^2 h^{\alpha\beta} - \frac{1}{6}g^{\alpha\beta}\Box^2 h$$
$$+\frac{1}{6}\partial^\alpha\partial^\beta\Box h + \frac{1}{6}g^{\alpha\beta}\Box\partial^\mu\partial^\nu h_{\mu\nu} = 0. \tag{4.38}$$

This equation is of fourth order in the derivatives; a property particular to higher-order theories, that causes the appearance of additional degrees of freedom and ghosts. After linearizing the system we proceed with the Hamilton-Jacobi method, but not before performing a space-time decomposition, as well as a change of variable.

4.5 The Hamilton-Jacobi Framework

In this section we will use the Hamilton-Jacobi method to obtain a consistent Hamiltonian with dynamics equivalent to that of the original Lagrangian (4.28). This requires the identification of the conjugate momenta, which in turn requires a more explicit identification of the time derivatives. This is why we will first perform a $3 + 1$ decomposition. Starting from (4.37) and performing a decomposition, we arrive at

$$\frac{1}{4}\mathcal{L}_3 = \frac{1}{4}\dot{h}_{0i,j}\dot{h}^{0i,j} + \frac{1}{2}\dot{h}_{0i,j}\ddot{h}^{ij} - \frac{2}{3}\nabla^2 h_{00}\dot{h}^{0j}{}_{,j} + \frac{5}{12}\dot{h}_{0i}{}^{,i}\dot{h}^{0j}{}_{,j} - \frac{1}{8}\ddot{h}_{ij}\ddot{h}^{ij} - \frac{1}{4}\ddot{h}_{ij}h^{00,i,j} - \frac{1}{6}\nabla^2 h_{00}\nabla^2 h^{00}$$
$$+ \frac{1}{4}\nabla^2 h_{0j}\nabla^2 h^{0j} - \frac{1}{2}\nabla^2 h_{0m}\dot{h}^{m}{}_{j}{}^{,j} - \frac{3}{4}h_{0i}{}^{,i}{}_{,m}h^{0j}{}_{,j}{}^{,m} + \frac{1}{2}\dot{h}_{ij,k}h^{0k,i,j} - \frac{1}{2}\dot{h}_{ij,k}\dot{h}^{ij,k} + \frac{3}{4}\dot{h}_{m}{}^{i}{}_{,i}\dot{h}^{m}{}_{j}{}^{,j}$$
$$+ \frac{1}{8}\nabla^2 h_{mn}\nabla^2 h^{mn} - \frac{3}{4}h_{mi}{}^{,i}{}_{,n}h^{mj}{}_{,j}{}^{,n} + \frac{1}{12}h_{ij}{}^{,i,j}h^{mn}{}_{,m,n} - \frac{1}{6}\ddot{h}_{i}{}^{i}\dot{h}^{0}{}_{j}{}^{,j} - \frac{1}{12}\ddot{h}_{i}{}^{i}\nabla^2 h^{00} - \frac{1}{24}\ddot{h}_{i}{}^{i}\ddot{h}_{j}{}^{j}$$
$$- \frac{1}{2}\dot{h}_{i}{}^{i}{}_{,m}\dot{h}^{m}{}_{j}{}^{,j} + \frac{1}{2}h_{0i}{}^{,i}{}_{,m}\nabla^2 h^{0m} + \frac{1}{4}\dot{h}_{i}{}^{i}{}_{,m}\dot{h}^{j}{}_{j}{}^{,m} + \frac{1}{2}h_{m}{}^{i}{}_{,i,n}\nabla^2 h^{mn} + \frac{1}{12}\nabla^2 h_{i}{}^{i}h^{mn}{}_{,m,n}$$
$$- \frac{1}{24}\nabla^2 h_{i}{}^{i}\nabla^2 h_{j}{}^{j} + \frac{1}{3}\dot{h}_{i}{}^{0,i}h_{nm}{}^{,n,m} - \frac{1}{3}\dot{h}_{i}{}^{0,i}\nabla^2 h_{n}{}^{n} + \frac{1}{6}\ddot{h}^{i}{}_{i}h_{nm}{}^{,n,m} - \frac{1}{6}\ddot{h}^{i}{}_{i}\nabla^2 h_{n}{}^{n}$$
$$- \frac{1}{6}\nabla^2 h_{0}{}^{0}h_{nm}{}^{,n,m} + \frac{1}{6}\nabla^2 h_{0}{}^{0}\nabla^2 h_{n}{}^{n}.$$

A large expression. Fortunately, there is a more compact way to write this action, which has the added benefit of performing a change of variable. Using the following extrinsic curvature-like tensor

$$K_{ij} = \frac{1}{2}(\dot{h}_{ij} - \partial_i h_{0j} - \partial_j h_{0i}). \tag{4.39}$$

We can obtain the following expression

$$\mathcal{L} = \frac{1}{2}(\dot{K}_{ij}\dot{K}^{ij} + \tilde{R}_{ij}\tilde{R}^{ij} + \dot{K}_{ij}\partial^i\partial^j h^{00} - 2\dot{K}_{ij}\tilde{R}^{ij} - \partial_i\partial_j h_{00}\tilde{R}^{ij}) - \frac{1}{6}(\dot{K}^2 + \tilde{R}^2$$
$$+ \dot{K}\nabla^2 h_{00} - 2\dot{K}\tilde{R} - \tilde{R}\nabla^2 h_{00}) + \frac{1}{12}\nabla^2 h_{00}\nabla^2 h_{00} - 2\partial_j K_{ik}\partial^j K^{ik} + 3\partial^i K_{ij}\partial_k K^{kj} \tag{4.40}$$
$$- 2\partial_i K\partial_j K^{ij} + \partial_i K\partial^i K.$$

As mentioned in previous chapters, in order to apply the Hamilton-Jacobi (HJ) method to higher order systems, it is necessary to perform changes of variables. This is done in order to reduce the order of the time derivatives. By introducing the extrinsic curvature-like tensor K_{ij} not only have we simplified the expression, but we also have taken care of this step. Now, instead of having time derivatives in the perturbation of order two $\ddot{h}_{ij}$, we will at most have first derivatives of the extrinsic curvature $\dot{K}_{ij}$

$$\mathcal{L} = \frac{1}{2}(\dot{K}_{ij}\dot{K}^{ij} + \tilde{R}_{ij}\tilde{R}^{ij} + \dot{K}_{ij}\partial^i\partial^j h^{00} - 2\dot{K}_{ij}\tilde{R}^{ij} - \partial_i\partial_j h_{00}\tilde{R}^{ij}) - \frac{1}{6}(\dot{K}^2 + \tilde{R}^2$$
$$+ \dot{K}\nabla^2 h_{00} - 2\dot{K}\tilde{R} - \tilde{R}\nabla^2 h_{00}) + \frac{1}{12}\nabla^2 h_{00}\nabla^2 h_{00} - 2\partial_j K_{ik}\partial^j K^{ik} + 3\partial^i K_{ij}\partial_k K^{kj} \tag{4.41}$$
$$- 2\partial_i K\partial_j K^{ij} + \partial_i K\partial^i K - \lambda^{ij}\Big[K_{ij} - \frac{1}{2}(\dot{h}_{ij} - h_{0j,i} - h_{0i,j})\Big].$$

The last term includes the change of variables which, by means of a Lagrange multiplier, is included in the Lagrangian to maintain consistency. This procedure has additional consequences. As we can see, the number of variables has increased. To the initial set of variables $\{h_{00}, h_{0i}, h_{ij}\}$ we have added the newly introduced K_{ij}. Not only that, but we will also add the Lagrange multiplier λ^{ij}, as per the method. Therefore, the set of variables becomes

$$Q^{(\mu)} = \{K_{ij}, h_{ij}, h_{0i}, h_{00}, \lambda^{ij}\}. \tag{4.42}$$

Their conjugated momenta are presented in orderly fashion as follows

$$P^{(\mu)} = \{\Pi^{ij}, \pi^{ij}, \pi^{0i}, \pi^{00}, \Upsilon_{ij}\}. \tag{4.43}$$

These can be readily acquired using the standard definition

$$P^{(\mu)} = \frac{\partial \mathcal{L}}{\partial \dot{Q}^{(\mu)}}. \tag{4.44}$$

Only two of them are non-zero

$$
\begin{aligned}
\Pi^{ab} &= \dot{K}^{ab} + \frac{1}{2}h^{00,a,b} - \tilde{R}^{ab} - \frac{1}{3}\eta^{ab}\dot{K} - \frac{1}{6}\eta^{ab}\nabla^2 h_{00} + \frac{1}{3}\eta^{ab}\tilde{R}, \\
\pi^{ab} &= \frac{1}{2}\lambda^{ba}, \\
\pi^{0a} &= \frac{\partial \mathcal{L}}{\partial \dot{h}_{0a}} = 0, \\
\pi^{00} &= \frac{\partial \mathcal{L}}{\partial \dot{h}_{00}} = 0, \\
\Upsilon_{ab} &= \frac{\partial \mathcal{L}}{\partial \dot{\lambda}^{ab}} = 0.
\end{aligned}
\tag{4.45}
$$

Fulfilling the required canonical relationships

$$
\begin{aligned}
\{K_{ij}(x), \Pi^{ab}(y)\} &= \frac{1}{2}(\delta_i^a\delta_j^b + \delta_j^a\delta_i^b)\delta^3(x-y), \\
\{h_{ij}(x), \pi^{ab}(y)\} &= \frac{1}{2}(\delta_i^a\delta_j^b + \delta_j^a\delta_i^b)\delta^3(x-y), \\
\{h_{0i}(x), \pi^{0a}(y)\} &= \delta_i^a\delta^3(x-y), \\
\{h_{00}(x), \pi^{00}(y)\} &= \delta^3(x-y), \\
\{\lambda^{ij}(x), \Upsilon_{ab}(y)\} &= \frac{1}{2}(\delta_a^i\delta_b^j + \delta_b^i\delta_a^j)\delta^3(x-y).
\end{aligned}
$$

Having obtained the canonical momenta we can proceed to obtain the canonical Hamiltonian $\mathcal{H}_c$ through a Legendre transformation of all the variables

$$
\mathcal{H}_c = \dot{K}_{ij}\Pi^{ij} + \dot{h}_{ij}\pi^{ij} + \dot{h}_{0i}\pi^{0i} + \dot{h}_{00}\pi^{00} + \dot{\lambda}_{ij}\Upsilon^{ij} - \mathcal{L}. \tag{4.46}
$$

A fist substitution yields

$$
\begin{aligned}
\mathcal{H}_c = {} & \dot{K}_{ij}\Pi^{ij} + \dot{h}_{ij}\pi^{ij} + \dot{h}_{0i}\pi^{0i} + \dot{h}_{00}\pi^{00} + \dot{\lambda}_{ij}\Upsilon^{ij} - \frac{1}{2}\dot{K}_{ij}\dot{K}^{ij} - \frac{1}{2}\tilde{R}_{ij}\tilde{R}^{ij} - \frac{1}{2}\dot{K}_{ij}\partial^i\partial^j h^{00} \\
& + \dot{K}_{ij}\tilde{R}^{ij} + \frac{1}{2}\partial_i\partial_j h_{00}\tilde{R}^{ij} + \frac{1}{6}\dot{K}^2 + \frac{1}{6}\tilde{R}^2 + \frac{1}{6}\dot{K}\nabla^2 h_{00} - \frac{1}{3}\dot{K}\tilde{R} - \frac{1}{6}\tilde{R}\nabla^2 h_{00} \\
& - \frac{1}{12}\nabla^2 h_{00}\nabla^2 h_{00} + 2\partial_j K_{ik}\partial^j K^{ik} - 3\partial^i K_{ij}\partial_k K^{kj} + 2\partial_i K\partial_j K^{ij} - \partial_i K\partial^i K \\
& + \lambda^{ij}(K_{ij} - \frac{1}{2}\dot{h}_{ij} + \frac{1}{2}h_{0j,i} + \frac{1}{2}h_{0i,j}).
\end{aligned}
$$

An expression that can be simplified using the definitions of momentum, leading to

$$\begin{aligned}\mathcal{H}_c = \frac{1}{2}\Pi_{ij}\Pi^{ij} - \frac{1}{2}\Pi_{ij}\partial_i\partial_j h^{00} + \Pi_{ij}\tilde{R}^{ij} + 2\partial_j K_{ik}\partial^j K^{ik} - 3\partial^i K_{ij}\partial_k K^{kj} \\ + 2\partial_i K\partial_j K^{ij} - \partial_i K\partial^i K + 2\pi^{ij}(K_{ij} + h_{0i,j}).\end{aligned} \tag{4.47}$$

Notice that we cannot solve for any velocities from the definitions of momenta, which signals the existence of both constraints and gauge symmetries. To identify the correct number of constraints, we turn our attention to the Hessian matrix

$$\frac{\partial\mathcal{L}}{\partial\dot{Q}^{(\mu)}\partial\dot{Q}^{(\nu)}} = \begin{pmatrix} \frac{\partial^2\mathcal{L}}{\partial\dot{K}_{ab}\partial\dot{K}^{ij}} & \frac{\partial^2\mathcal{L}}{\partial\dot{K}_{ab}\partial\dot{h}^{00}} & \frac{\partial^2\mathcal{L}}{\partial\dot{K}_{ab}\partial\dot{h}^{0i}} & \frac{\partial^2\mathcal{L}}{\partial\dot{K}_{ab}\partial\dot{h}^{ij}} & \frac{\partial^2\mathcal{L}}{\partial\dot{K}_{ab}\partial\dot{\lambda}^{ij}} \\ \frac{\partial^2\mathcal{L}}{\partial\dot{h}_{00}\partial\dot{K}^{ij}} & \frac{\partial^2\mathcal{L}}{\partial\dot{h}_{00}\partial\dot{h}^{00}} & \frac{\partial^2\mathcal{L}}{\partial\dot{h}_{00}\partial\dot{h}^{0i}} & \frac{\partial^2\mathcal{L}}{\partial\dot{h}_{00}\partial\dot{h}^{ij}} & \frac{\partial^2\mathcal{L}}{\partial\dot{h}_{00}\partial\dot{\lambda}^{ij}} \\ \frac{\partial^2\mathcal{L}}{\partial\dot{h}_{0a}\partial\dot{K}^{ij}} & \frac{\partial^2\mathcal{L}}{\partial\dot{h}_{0a}\partial\dot{h}^{00}} & \frac{\partial^2\mathcal{L}}{\partial\dot{h}_{0a}\partial\dot{h}^{0i}} & \frac{\partial^2\mathcal{L}}{\partial\dot{h}_{0a}\partial\dot{h}^{ij}} & \frac{\partial^2\mathcal{L}}{\partial\dot{h}_{0a}\partial\dot{\lambda}^{ij}} \\ \frac{\partial^2\mathcal{L}}{\partial\dot{h}_{ab}\partial\dot{K}^{ij}} & \frac{\partial^2\mathcal{L}}{\partial\dot{h}_{ab}\partial\dot{h}^{00}} & \frac{\partial^2\mathcal{L}}{\partial\dot{h}_{ab}\partial\dot{h}^{0i}} & \frac{\partial^2\mathcal{L}}{\partial\dot{h}_{ab}\partial\dot{h}^{ij}} & \frac{\partial^2\mathcal{L}}{\partial\dot{h}_{ab}\partial\dot{\lambda}^{ij}} \\ \frac{\partial^2\mathcal{L}}{\partial\dot{\lambda}_{ab}\partial\dot{K}^{ij}} & \frac{\partial^2\mathcal{L}}{\partial\dot{\lambda}_{ab}\partial\dot{h}^{00}} & \frac{\partial^2\mathcal{L}}{\partial\dot{\lambda}_{ab}\partial\dot{h}^{0i}} & \frac{\partial^2\mathcal{L}}{\partial\dot{\lambda}_{ab}\partial\dot{h}^{ij}} & \frac{\partial^2\mathcal{L}}{\partial\dot{\lambda}_{ab}\partial\dot{\lambda}^{ij}} \end{pmatrix} \tag{4.48}$$

Since our variables are all symmetric[2] this matrix is of dimension 22. The analysis is greatly simplified by taking into account that only one element is non-zero

$$\frac{\partial\mathcal{L}}{\partial\dot{Q}^{\mu}\partial\dot{Q}^{\nu}} = \begin{pmatrix} \frac{\partial^2\mathcal{L}}{\partial\dot{K}_{ab}\partial\dot{K}^{ij}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.49}$$

Where

$$\frac{\partial^2\mathcal{L}}{\partial\dot{K}^{ij}\partial\dot{K}_{ab}} = \frac{1}{2}\delta_i{}^a\delta_j{}^b + \frac{1}{2}\delta_j{}^a\delta_i{}^b - \frac{1}{3}\eta^{ab}\eta_{ij}. \tag{4.50}$$

[2]Notice that the actual number of *independent* variables is reduced due to the symmetries. Decreasing the number of variables below 36.

This is a 6×6 matrix, since the rest of the matrix is trivial (rank 0 and nullity 16) our attention will center only on this component. Its explicit form is shown below

$$
\begin{array}{c}
\\
\frac{\partial}{\partial \dot{K}^{11}} \\[6pt]
\frac{\partial}{\partial \dot{K}^{12}} \\[6pt]
\frac{\partial}{\partial \dot{K}^{13}} \\[6pt]
\frac{\partial}{\partial \dot{K}^{22}} \\[6pt]
\frac{\partial}{\partial \dot{K}^{23}} \\[6pt]
\frac{\partial}{\partial \dot{K}^{33}}
\end{array}
\begin{array}{c}
\begin{array}{cccccc}
\frac{\partial \mathcal{L}}{\partial \dot{K}_{11}} & \frac{\partial \mathcal{L}}{\partial \dot{K}_{12}} & \frac{\partial \mathcal{L}}{\partial \dot{K}_{13}} & \frac{\partial \mathcal{L}}{\partial \dot{K}_{22}} & \frac{\partial \mathcal{L}}{\partial \dot{K}_{23}} & \frac{\partial \mathcal{L}}{\partial \dot{K}_{33}}
\end{array} \\
\begin{pmatrix}
\frac{2}{3} & 0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
-\frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 & -\frac{1}{3} \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
-\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 & \frac{2}{3}
\end{pmatrix}
\end{array}
$$

The rank of this submatrix is 5 and its nullity is 1, which means that the rank and nullity of the Hessian matrix are 5 and 17, respectively. As a consequence, we should have 17 constraints. The momenta we have account for sixteen of these. The missing one can be obtained by analyzing the submatrix in detail. Since the first column can be constructed from the fourth and the sixth $C_1 = -(C_4 + C_6)$ then

$$\frac{\partial \Pi^{11}}{\partial \dot{K}^{ab}} + \frac{\partial \Pi^{22}}{\partial \dot{K}^{ab}} = -\frac{\partial \Pi^{33}}{\partial \dot{K}^{ab}} \longrightarrow \frac{\partial tr(\Pi)}{\partial \dot{K}^{ab}} = 0.$$

Which is equivalent to

$$\Pi = 0. \tag{4.51}$$

This is our missing element. Therefore, the total set of constraints is

$$\begin{aligned}
\Omega_{(0)} &= \Pi \simeq 0, \\
\Omega_{(1)} &= \pi^{00} \simeq 0, \\
\Omega_{(2)}^{0i} &= \pi^{0i} \simeq 0, \\
\Omega_{(3)}^{ij} &= -2\pi^{ij} + \lambda^{ij} \simeq 0, \\
\Omega_{ij}^{(4)} &= \Upsilon_{ij} \simeq 0.
\end{aligned} \tag{4.52}$$

The HJ method identifies these as Hamiltonians and classifies them in accordance with their Poisson brackets. We expect the Hamiltonians to be involutive, i.e., for any two $\Omega_{(a)}$ and $\Omega_{(b)}$ their Poisson bracket must satisfy $\{\Omega_{(a)}, \Omega_{(b)}\} \simeq 0$. But we will most likely also find non-involutive Hamiltonians, which will have at least one non-zero bracket[3]. In such a case, it is necessary to generalize to the Poisson bracket to eliminate the non-involutive Hamiltonians. To examine the evolution of the Hamiltonians, we will first obtain their algebra. The only non-zero bracket is the following

$$\{\Omega_{(3)}^{ij}, \Omega_{ab}^{(4)}\} = \frac{1}{2}(\delta_a^i \delta_b^j + \delta_b^i \delta_a^j). \tag{4.53}$$

Therefore, Hamiltonians $\Omega_{(3)}^{ij}$ and $\Omega_{ij}^{(4)}$ are non-involutive. To differentiate these from the ones that are in involution, we will use the symbol Λ. Our set of Hamiltonians becomes

$$\begin{aligned}
\Omega_{(0)} &= \Pi \simeq 0, \\
\Omega_{(1)} &= \pi^{00} \simeq 0, \\
\Omega_{(2)}^{0i} &= \pi^{0i} \simeq 0, \\
\Lambda_{(3)}^{ij} &= -2\pi^{ij} + \lambda^{ij} \simeq 0, \\
\Lambda_{ij}^{(4)} &= \Upsilon_{ij} \simeq 0.
\end{aligned} \tag{4.54}$$

[3] As will be the case for $\Omega_{(3)}^{ij}$ and $\Omega_{ab}^{(4)}$

Now that we have a complete set of Hamiltonians and the non-involutive ones have been identified, we can eliminate the latter using the generalized bracket. To be precise, we will modify the Poisson bracket in such a way that the brackets involving the Λ's are always zero, thus making the non-involutive Hamiltonians non-dynamical. The new "generalized bracket" is defined in the following way

$$\{A(x), B(x')\}^* = \{A(x), B(x')\} - \iint \{A(x), \mathbf{\Lambda}^{(\mu)}(y)\} \mathbf{\Delta}_{(\mu)}^{(\nu)^{-1}}(y,z) \{\mathbf{\Lambda}_{(\nu)}(z), B(x')\} d^2y d^2z. \tag{4.55}$$

Where $\mathbf{\Delta}_{(\mu)}^{(\nu)}$ is a matrix whose entries are made of the Poisson bracket among the non-involutive Hamiltonians

$$\mathbf{\Delta}_{(\nu)}^{(\mu)} = \begin{pmatrix} \{\Lambda^1, \Lambda^1\} & \{\Lambda^1, \Lambda^2\} & \cdots & \{\Lambda^1, \Lambda^n\} \\ \{\Lambda^2, \Lambda^1\} & \{\Lambda^2, \Lambda^2\} & \cdots & \{\Lambda^2, \Lambda^n\} \\ \vdots & \vdots & \ddots & \vdots \\ \{\Lambda^n, \Lambda^1\} & \{\Lambda^n, \Lambda^2\} & \cdots & \{\Lambda^n, \Lambda^n\} \end{pmatrix}. \tag{4.56}$$

And $\Lambda^{(\mu)}$, $\Lambda_{(\nu)}$ are vectors composed of the same Λ's

$$\mathbf{\Lambda}^{(\mu)} = \begin{pmatrix} \Lambda^1 & \Lambda^2 & \ldots & \Lambda^m \end{pmatrix} \quad , \quad \mathbf{\Lambda}_{(\nu)} = \begin{pmatrix} \Lambda^1 \\ \Lambda^2 \\ \vdots \\ \Lambda^n \end{pmatrix} \tag{4.57}$$

The matrix at hand is relatively simple as we have only four elements

$$\mathbf{\Delta}^{ij}_{ab}(x,y) = \begin{pmatrix} \{\Lambda^{ij}_{(3)}(x), \Lambda^{ab}_{(3)}(y)\} & \{\Lambda^{ij}_{(3)}(x), \Lambda^{(4)}_{ab}(y)\} \\ \{\Lambda^{(4)}_{ij}(x), \Lambda^{ab}_{(3)}(y)\} & \{\Lambda^{(4)}_{ij}(x), \Lambda^{(4)}_{ab}(y)\} \end{pmatrix} = \{\Lambda^{ij}_{(3)}(x), \Lambda^{(4)}_{ab}(y)\} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

That is

$$\mathbf{\Delta}^{ij}_{ab}(x,y) = \frac{1}{2}(\delta^i_a \delta^j_b + \delta^i_b \delta^j_a) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \delta^3(x-y). \tag{4.58}$$

To obtain its inverse we use the property that a matrix times its inverse gives the identity matrix

$$\int \mathbf{\Delta}_{ab}^{ij}(x,y)\mathbf{\Delta}_{jc}^{bk^{-1}}(y,z)d^3y = \frac{1}{2}(\delta_a^i\delta_c^k + \delta_c^i\delta_a^k)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\delta^2(x-z). \tag{4.59}$$

A trivial calculation shows that the inverse is

$$\mathbf{\Delta}_{cj}^{bk^{-1}}(y,z) = \delta_c^b\delta_j^k\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\delta^2(y-z). \tag{4.60}$$

Inserting this into (4.54) yields the Generalized bracket, with the help of which we can obtain the generalized algebra of coordinates and momenta. To avoid unnecessary work, note that only the brackets involving h_{ij}, Υ_{ij} and λ^{ij} are modified. The resulting generalized brackets for our coordinates and momenta are

$$\begin{aligned}
&\{K_{ij}(x), \Pi^{kl}(x')\}^* = \frac{1}{2}(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k)\delta^3(x-x'),\\
&\{h_{ij}(x), \pi^{kl}(x')\}^* = \frac{1}{2}(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k)\delta^3(x-x'),\\
&\{h_{0i}(x), \pi^{0k}(x')\}^* = \delta_i^k\delta^3(x-x'),\\
&\{h_{00}(x), \pi^{00}(x')\}^* = \delta^3(x-x'),\\
&\{\Upsilon_{ij}(x), \lambda^{kl}(x')\}^* = 0,\\
&\{h_{ij}(x), \lambda^{kl}(x')\}^* = (\delta_i^k\delta_j^l + \delta_i^l\delta_j^k)\delta^3(x-x').
\end{aligned} \tag{4.61}$$

There are only two brackets that differ from Poisson's and their differences are directly related to the non-involutive Hamiltonians. The second to last bracket tells us that Υ_{ij} is no longer a dynamical variable, since its bracket with its conjugate variable λ^{kl} is now zero. The last one tells us that λ^{kl} now behaves as a conjugate momentum to h_{ij}, a role that shares with π^{ij}. This is the same information that is contained in the non-involutive Hamiltonians, therefore we have now transformed the weak equalities $\Lambda_{(3)}^{ij} \simeq 0$, $\Lambda_{ij}^{(4)} \simeq 0$ into strong ones. Therefore, the set of Hamiltonians is now reduced to

$$\begin{aligned}
\Omega_{(0)} &= \Pi, \\
\Omega_{(1)} &= \pi^{00}, \\
\Omega^{0i}_{(2)} &= \pi^{0i}.
\end{aligned} \tag{4.62}$$

Now, the introduction of a new bracket naturally modifies the dynamics of the system. Therefore, we must ensure that the Hamiltonians are still in involution. As we have not modified the brackets of $\Pi^{ij}, \pi^{ij}, \pi^{0i}$, the equation (4.62) is an involutive set, at least with one another. All that remains to be seen is whether they are in involution with the canonical Hamiltonian. A quick calculation shows us that we got three more Hamiltonians

$$\begin{aligned}
\int \{\Omega_{(0)}(x), \mathcal{H}_c(y)\}^* d^3y &= -2\tilde{\pi}, \\
\int \{\Omega_{(1)}(x), \mathcal{H}_c(y)\}^* d^3y &= \frac{1}{2}\partial_i\partial_j\Pi^{ij}, \\
\int \{\Omega^{0i}_{(2)}(x), \mathcal{H}_c(y)\}^* d^3y &= 2\partial_j\pi^{ij}.
\end{aligned} \tag{4.63}$$

These Hamiltonians depend only on the momenta. Given that we have not modified the brackets relating to the momenta they are in involution

$$\begin{aligned}
\Omega_{(0)} &= \Pi, \\
\Omega_{(1)} &= \pi^{00}, \\
\Omega^{0i}_{(2)} &= \pi^{0i}, \\
\Omega_{(6)} &= \tilde{\pi}, \\
\Omega_{(7)} &= \partial_i\partial_j\Pi^{ij}, \\
\Omega^{0i}_{(8)} &= \partial_j\pi^{ij}.
\end{aligned} \tag{4.64}$$

Once more, we need just to check the generalized bracket with the canonical Hamiltonian

$$\begin{aligned}
&\int \{\Omega_{(6)}(x), \mathcal{H}_c(y)\}^* d^3y = \frac{1}{2}\nabla^2\Omega_{(0)} + \frac{1}{2}\Omega_{(6)} \simeq 0, \\
&\int \{\Omega_{(7)}(x), \mathcal{H}_c(y)\}^* d^3y = 2\partial_i\Omega^{0i}_{(7)} \simeq 0, \\
&\int \{\Omega^{0i}_{(7)}(x), \mathcal{H}_c(y)\}^* d^3y = 0.
\end{aligned} \tag{4.65}$$

These do not lead to any more Hamiltonians, so the set (4.64) is involutive. These, together with the canonical Hamiltonian, fully describe the system's dynamics. We do this by means of the fundamental differential

$$\begin{aligned}
dF = \int \Big[&\{F, \mathcal{H}\}^* dt + \{F, \Omega_{(0)}\}^* d\xi^{(0)} + \{F, \Omega_{(1)}\}^* d\xi^{(1)} + \{F, \Omega^{0i}_{(2)}\}^* d\xi^{(2)}_{0i} \\
&+\{F, \Omega_{(6)}\}^* d\xi^{(6)} + \{F, \Omega_{(7)}\}^* d\xi^{(7)} + \{F, \Omega^{0i}_{(8)}\}^* d\xi^{(8)}_{0i}\Big] d^3y.
\end{aligned} \tag{4.66}$$

The parameters $\xi^{(\mu)}$ correspond to the evolution parameters of each Hamiltonian $\Omega_{(\mu)}$. According to the Frobenius integrability conditions, the evolution concerning one of these parameters, including time, remains independent of the others. As a result, equation (4.66) encapsulates all the system's dynamical information, encompassing both the characteristic equations and the gauge symmetries.

Applying the fundamental differential to the variables allows us to derive the characteristic equations mentioned earlier

$$
\begin{aligned}
dh_{ij} &= (2K_{ij} + \partial_i h_{0j} + \partial_j h_{0i})dt + \eta_{ij} d\xi^{(6)} - \frac{1}{2}(\partial_j d\xi^{(8)}_{0i} + \partial_i d\xi^{(8)}_{0j}),\\
dh_{0i} &= d\xi^{(2)}_{0i},\\
dh_{00} &= d\xi^{(1)},\\
d\lambda^{ij} &= -\left[\partial^i\partial_a\Pi^{aj} - \nabla^2\Pi^{ij} + \partial^j\partial_a\Pi^{ai}\right]dt,\\
dK_{ij} &= (\Pi_{ij} - \frac{1}{2}\partial_i\partial_j h_{00} + \tilde{R}_{ij})dt + \eta_{ij} d\xi^{(0)} + \partial_i\partial_j d\xi^{(7)}\\
d\pi^{ij} &= -\frac{1}{2}\left[\partial^i\partial_a\Pi^{aj} - \nabla^2\Pi^{ij} + \partial^j\partial_a\Pi^{ai}\right]dt,\\
d\Pi^{ij} &= \Big[4\nabla^2 K^{ij} - 3\partial^i\partial_a K^{aj} - 3\partial^j\partial_a K^{ai} + 2\partial^i\partial^j K + 2\eta^{ij}\partial_a\partial_b K^{ab}\\
&\quad - 2\eta^{ij}\nabla^2 K - 2\pi^{ij}\Big]dt.
\end{aligned}
\tag{4.67}
$$

Additionally, $d\Upsilon = d\pi^{00} = d\pi^{0i} = 0$, which tells us that Υ_{ij}, π^{00}, and π^{0i} are non-dynamical. On further inspection, it can be seen that $d\lambda^{ij}$ and $d\pi^{ij}$'s equations contain the same information. Furthermore, by considering only the evolution with respect to time, we are left with

$$
\begin{aligned}
dh_{ij} &= \big(2K_{ij} + \partial_i h_{0j} + \partial_j h_{0i}\big)dt,\\
dK_{ij} &= \left(\Pi_{ij} - \frac{1}{2}\partial_i\partial_j h_{00} + \tilde{R}_{ij}\right)dt,\\
d\pi^{ij} &= -\frac{1}{2}\big(\partial^i\partial_m\Pi^{mj} - \nabla^2\Pi^{ij} + \partial^j\partial_m\Pi^{mi}\big)dt,\\
d\Pi^{ij} &= \big(4\nabla^2 K^{ij} - 3\partial^i\partial_a K^{aj} - 3\partial^j\partial_a K^{ai} + 2\partial^i\partial^j K + 2\eta^{ij}\partial_a\partial_b K^{ab}\\
&\quad - 2\eta^{ij}\nabla^2 K - 2\pi^{ij}\big)dt.
\end{aligned}
\tag{4.68}
$$

The first two equations yield the equation of motion for h_{ij}, effectively reducing the number of dynamical variables to 18. Subsequently, we can proceed to count the degrees of freedom. Excluding $\Omega_{(1)}$ and $\Omega^{0i}_{(2)}$, given that π^{00}, π^{0i} are not dynamical, we encounter a total of six constraints. Consequently, the number of degrees of freedom amounts to six.

The canonical transformations can be obtained by taking $dt = 0$ in the characteristic equations and relating the parameters $\xi^{(\mu)}$.

$$\begin{aligned} dh_{ij} &= \eta_{ij} d\xi^{(6)} - \frac{1}{2}\big(\partial_j d\xi^{(8)}_{0i} + \partial_i d\xi^{(8)}_{0j}\big), \\ dh_{0i} &= d\xi^{(2)}_{0i}, \\ dh_{00} &= d\xi^{(1)}, \\ dK_{ij} &= \eta_{ij} d\xi^{(0)} + \partial_i \partial_j d\xi^{(7)}. \end{aligned} \tag{4.69}$$

To obtain the gauge transformation we must first seek a covariant expression for the metric perturbation by combining the characteristic equations for each component

$$\delta h_{\mu\nu} = \delta^0_\mu \delta^0_\nu \delta\xi^{(1)} + \frac{1}{2}(\delta^0_\mu \delta^i_\nu + \delta^i_\mu \delta^0_\nu)\delta\xi^{(2)}_i + \delta^i_\mu \delta^j_\nu (\eta_{ij}\delta\xi^{(6)} - \frac{1}{2}\partial_j \delta\xi^{(8)}_{0i} - \frac{1}{2}\partial_i \delta\xi^{(8)}_{0j}). \tag{4.70}$$

The gauge transformations are those that leave the action unaltered. Therefore, we take the variation of (4.37), as given by (4.70)[4]. This will guide us to the relationships that the parameters $\xi^{(\mu)}$ must adhere to for the transformation to truly qualify as gauge transformations

$$\begin{aligned} \delta S = \int \Big[&- \Box \partial^\mu \partial_\sigma h^{\sigma\nu} + \frac{1}{3}\partial^\mu \partial^\nu \partial_\sigma \partial_\delta h^{\sigma\delta} + \frac{1}{2}\Box\Box h^{\mu\nu} \\ &- \frac{1}{6}\eta^{\mu\nu}\Box\Box h_\delta{}^\delta + \frac{1}{6}\Box\partial^\mu\partial^\nu h_\delta{}^\delta + \frac{1}{6}\eta^{\mu\nu}\Box\partial^\alpha\partial^\beta h_{\alpha\beta}\Big]\delta h_{\mu\nu} dt d^3x \end{aligned} \tag{4.71}$$

After substituting equation (4.70) this transforms to

$$\begin{aligned} \delta S = \int \Big[&- \Box \partial^\mu \partial_\sigma h^{\sigma\nu} + \frac{1}{3}\partial^\mu \partial^\nu \partial_\sigma \partial_\delta h^{\sigma\delta} + \frac{1}{2}\Box\Box h^{\mu\nu} - \frac{1}{6}\eta^{\mu\nu}\Box\Box h_\delta{}^\delta \\ &+ \frac{1}{6}\Box\partial^\mu\partial^\nu h_\delta{}^\delta + \frac{1}{6}\eta^{\mu\nu}\Box\partial^\alpha\partial^\beta h_{\alpha\beta}\Big]\Big[\delta^0_\mu \delta^0_\nu \delta\xi^{(1)} + \frac{1}{2}(\delta^0_\mu \delta^i_\nu + \delta^i_\mu \delta^0_\nu)\delta\xi^{(2)}_i \\ &+ \delta^i_\mu \delta^j_\nu (\eta_{ij}\delta\xi^{(6)} - \frac{1}{2}\partial_j \delta\xi^{(8)}_{0i} - \frac{1}{2}\partial_i \delta\xi^{(8)}_{0j})\Big] dt d^3x \end{aligned}$$

Which can be further manipulated into

$$\delta S = \int \Big(\Xi_{(1)}\delta\xi^{(1)} + \Xi^i_{(2)}\delta\xi^{(2)}_i - \Xi_{(6)}\delta\xi^{(6)} + \Xi^{ij}_{(8)}\partial_j\delta\xi^{(8)}_{0i} + \Xi^{ij}_{(8)}\partial_i\delta\xi^{(8)}_{0j}\Big) dt d^3x \tag{4.72}$$

[4] That is, omitting time.

Where

$$
\begin{aligned}
\Xi_{(1)} &= \frac{2}{3}\nabla^2\partial_0\partial_i h^{0i} - \frac{1}{2}\partial_0\partial^0\partial_i\partial_j h^{ij} + \frac{1}{3}\nabla^2\nabla^2 h^{00} + \frac{1}{6}\nabla^2\partial_0\partial^0\tilde{h} + \frac{1}{6}\nabla^2\nabla^2\tilde{h} - \frac{1}{6}\nabla^2\partial_i\partial_j h^{ij}, \\
\Xi^i_{(2)} &= \frac{1}{2}\partial_0\partial^0\partial_0\partial_j h^{ij} + \frac{1}{2}\nabla^2\partial_0\partial_j h^{ij} + \frac{1}{6}\partial^0\partial^i\partial_0\partial_j h^{0j} + \frac{1}{3}\partial^0\partial^i\partial_j\partial_k h^{jk} + \frac{1}{2}\nabla^2\partial_0\partial^0 h^{0i} \\
&\quad + \frac{1}{2}\nabla^2\nabla^2 h^{0i} + \frac{1}{6}\partial^0\partial^i\partial_0\partial^0\tilde{h} + \frac{1}{3}\nabla^2\partial^0\partial^i h^{00} + \frac{1}{6}\nabla^2\partial^0\partial^i\tilde{h} - \frac{1}{2}\nabla^2\partial^i\partial_j h^{j0}, \\
\Xi_{(6)} &= -\frac{1}{2}\partial_0\partial^0\partial_i\partial_j h^{ij} - \frac{1}{6}\nabla^2\partial_i\partial_j h^{ij} + \frac{2}{3}\nabla^2\partial_0\partial_i h^{0i} + \frac{1}{3}\nabla^2\nabla^2 h^{00} + \frac{1}{6}\nabla^2\partial_0\partial^0\tilde{h} + \frac{1}{6}\nabla^2\nabla^2\tilde{h}, \\
\Xi^{ij}_{(8)} &= -\frac{1}{2}\partial^0\partial^i\partial_0\partial^0 h^{0j} + \frac{1}{2}\partial_0\partial^0\partial^i\partial_k h^{kj} - \frac{1}{2}\nabla^2\partial^0\partial^i h^{0j} + \frac{1}{2}\nabla^2\partial^i\partial_k h^{kj} + \frac{1}{4}\partial_0\partial^0\partial^i\partial^j h^{00} \\
&\quad - \frac{1}{3}\partial^i\partial^j\partial_0\partial_k h^{0k} - \frac{1}{6}\partial^i\partial^j\partial_a\partial_b h^{ab} - \frac{1}{4}\partial_0\partial^0\partial_0\partial^0 h^{ij} - \frac{1}{2}\nabla^2\partial_0\partial^0 h^{ij} - \frac{1}{4}\nabla^2\nabla^2 h^{ij} \\
&\quad + \frac{1}{12}\eta^{ij}\partial_0\partial^0\partial_0\partial^0\tilde{h} + \frac{1}{12}\eta^{ij}\nabla^2\partial_0\partial_0 h^{00} + \frac{1}{6}\eta^{ij}\nabla^2\partial_0\partial^0\tilde{h} - \frac{1}{12}\eta^{ij}\nabla^2\nabla^2 h^{00} + \frac{1}{12}\eta^{ij}\nabla^2\nabla^2\tilde{h} \\
&\quad - \frac{1}{12}\partial_0\partial^0\partial^i\partial^j\tilde{h} + \frac{1}{12}\nabla^2\partial^i\partial^j h^{00} - \frac{1}{12}\nabla^2\partial^i\partial^j\tilde{h} - \frac{1}{6}\eta^{ij}\partial_0\partial^0\partial_0\partial_k h^{0k} - \frac{1}{12}\eta^{ij}\partial_0\partial^0\partial_a\partial_b h^{ab} \\
&\quad - \frac{1}{6}\eta^{ij}\nabla^2\partial_0\partial_k h^{0k} - \frac{1}{12}\eta^{ij}\nabla^2\partial_a\partial_b h^{ab}.
\end{aligned}
\tag{4.73}
$$

We now proceed to relate the parameters $\xi^{(\mu)}$ in order to find the gauge transformations. A brief inspection shows that equations $\Xi_{(1)} = \Xi_{(6)}$, therefore we rename both of them as Ξ. This establishes a connection between the corresponding parameters $\xi^{(1)}$ and $\xi^{(6)}$. Following substitution and subsequent partial integration, we are presented with

$$
\delta S = \int \left[\Xi\left(\delta\xi^{(1)} + \delta\xi^{(6)}\right) + \Xi^{0i}_{(2)}\delta\xi^{(2)}_{0i} - \left(\partial_j\Xi^{ij}_{(8)} + \partial_j\Xi^{ji}_{(8)}\right)\delta\xi^{(8)}_{0i}\right] dt d^3x \tag{4.74}
$$

With

$$
\begin{aligned}
\partial_j\Xi^{ij}_{(8)} + \partial_j\Xi^{ji}_{(8)} &= -\frac{1}{2}\nabla^2\partial^0\partial_0\partial^0 h^{0i} - \frac{1}{2}\nabla^2\nabla^2\partial^0 h^{0i} - \frac{1}{3}\nabla^2\partial^i\partial_0\partial_0 h^{00} - \frac{1}{2}\nabla^2\partial^i\partial_0\partial_k h^{0k} \\
&\quad - \frac{1}{2}\partial_0\partial^0\partial_0\partial^0\partial_j h^{ij} + \frac{1}{6}\partial_0\partial^0\partial_0\partial^0\partial^i\tilde{h} + \frac{1}{6}\nabla^2\partial_0\partial^0\partial^i\tilde{h} + \frac{1}{6}\partial_0\partial^0\partial^i\partial_0\partial_k h^{0k} \\
&\quad + \frac{1}{3}\partial_0\partial^0\partial^i\partial_k\partial_l h^{kl} - \frac{1}{2}\nabla^2\partial_0\partial^0\partial_j h^{ij}.
\end{aligned}
\tag{4.75}
$$

This expression corresponds to a time derivative of $\Xi^i_{(2)}$

$$\Xi^i \equiv \partial_0 \Xi^i_{(2)} = \partial_j \Xi^{ij}_{(8)} + \partial_j \Xi^{ji}_{(8)}. \tag{4.76}$$

Which will relate $\xi^{(2)}_{0i}$ with $\xi^{0i}_{(8)}$. By reformulating the evolution parameter, we incorporate the time derivative

$$\delta\xi^{(2)}_{0i} = \partial_0 \delta\xi^{(2)}_i + \partial_i \delta\xi^{(2)}_0 \tag{4.77}$$

Upon insertion into the variation of the action and subsequent partial integrations

$$\delta S = \int \left[\Xi \left(\delta\xi^{(1)} + \delta\xi^{(6)} \right) - \partial_i \Xi^i_{(2)} \delta\xi^{(2)}_0 - \Xi^i \left(\delta\xi^{(2)}_i + \delta\xi^{(8)}_{0i} \right) \right] dt d^3x. \tag{4.78}$$

And by using $\partial_i \Xi^i_{(2)} \delta\xi^{(2)}_0 = \partial^0 \Xi \delta\xi^{(2)}_0$. This ends up as

$$\delta S = \int \left[\Xi \left(\delta\xi^{(1)} + \delta\xi^{(6)} + \partial^0 \delta\xi^{(2)}_0 \right) - \Xi^i \left(\delta\xi^{(2)}_i + \delta\xi^{(8)}_{0i} \right) \right] dt d^3x. \tag{4.79}$$

In order for this variation to be zero, the parameters have to be related in the following way

$$\begin{aligned} \delta\xi^{(1)} &= \delta\xi^{(6)} = \delta\xi, \\ \partial^0 \delta\xi^{(2)}_0 &= \partial^0 \delta\xi_0 = -2\delta\xi, \\ \delta\xi^{(2)}_i &= \delta\xi_i = -\delta\xi^{(8)}_{0i}. \end{aligned} \tag{4.80}$$

Using this in (4.69b) reveals the gauge symmetries as

$$\delta h_{\mu\nu} = \eta_{\mu\nu}\delta\xi + \frac{1}{2}\left(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu\right). \tag{4.81}$$

At this point, a canonical Hamiltonian, whose dynamics are equivalent to the original higher-order Lagrangian, has been found. Additionally, it is possible to provide a comprehensive description of the system's dynamics with the help of the fundamental differential, the generalized bracket and the involutive Hamiltonians. Furthermore, the theory's gauge symmetries have been found to be those of conformal gravity [152].

Chapter 5

Conclusions

In this thesis work, the extended Maxwell-Chern-Simons theory was analyzed via two different methods. Regarding the Hamilton-Jacobi study, the theory was written as a first-order Lagrangian using auxiliary fields. Then, with help of the null vectors, all Hamiltonians were correctly identified. Following this, substituting the Poisson brackets for the generalized brackets allowed the elimination of all non-physical fields. Afterwards, a fundamental differential was constructed, given in terms of both the generalized brackets and the remaining, fully involutive, Hamiltonians. As a consequence, the correct identification of the characteristic equations could be performed, after which the equations of motion, as well as the gauge transformations, were found. In this way, it is demonstrated that the Hamilton-Jacobi approach is an outstanding method for the analysis of higher-order systems.

Moreover, using the Gitman-Lyakhovich-Tyutin (GLT) method we reported the full structure of the constraints. Using this method, the constraints were obtained in a consistent way, without having to set them manually, as is commonly done in some published studies. The analysis was concluded by fixing the gauge, which allowed the complete structure of the Dirac brackets to be observed. This analysis, originally presented in (reference) and shown in an extended form here, extends previous results. This analysis serves as proof that this type of study can be carried out for higher order theories, as was done in chapter 4 with Weyl theory.

Following this preliminary analysis, the Hamilton-Jacobi method was used to analyze Weyl gravity. Following the same guidelines, supplementary variables were added to reduce the Lagrangian system to a first order system, which allowed us to obtain the canonical Hamiltonian in a natural way. As a result of this process, an intricate relationship between the variables of the system was generated, leading to the existence of Hamiltonians, which, using the null vectors, were classified. Subsequently, with the help of the generalized bracket, additional degrees of freedom and non-involutive Hamiltonians were eliminated. The physical degrees of freedom were counted and the fundamental differential was constructed, which allowed the identification of the characteristic equations and, through a quick analysis, the gauge transformations.

These methods presented here outline a more detailed type of analysis than has been seen before, at least concerning higher order theories. It provides a better handling of the Hamiltonians, allowing gauge symmetries to be seen more easily. In general, it is desirable to have at hand different methods to analyze higher order theories, in order to take advantage of the benefits of each one. Specifically regarding the Hamilton-Jacobi method, it offers the possibility to obtain gauge symmetries without having to manually fix the gauge, as is common in other methods. It should be noted that at the end of the procedure the generalized bracket brackets of the HJ method coincide with those of Dirac. Nevertheless, opting for the HJ method avoids the laborious classification of the constraints, since the involutive Hamiltonians are obtained by means of the Frobenius inegrability conditions. This also avoids the Dirac conjecture altogether. With respect to the gauge symmetries, these can be obtained using the HJ method by demanding the action's invariance under the canonical transformations obtained from the characteristic equations.

As concluding remarks, these methods have previously only been applied to second order theories. The generalization of these techniques to higher order theories may provide valuable information, especially when quantization is being pursued. Gravitational theories, in particular, could benefit, as renormalization is a still an unresolved issue. By revealing properties such as Ostrogradsky instabilities, as well as providing a Hamiltonian description, these formalisms can help identify potentially viable higher-order theories. Furthermore, the methods leave the theories such that a first quantization can be done in a straightforward manner.

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